6.300 Signal Processing

Week 6, Lecture B:

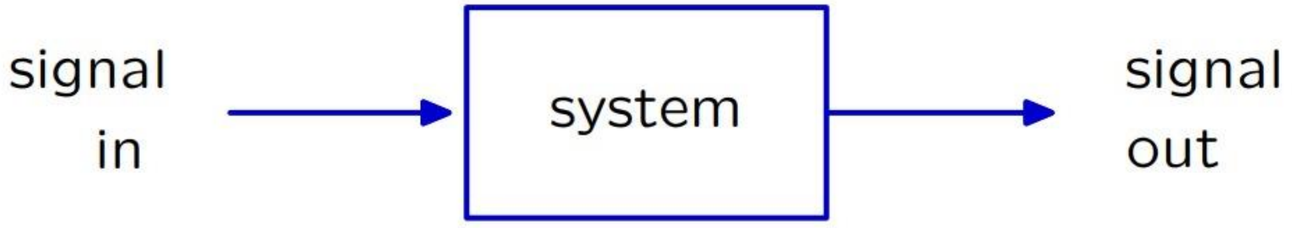
System Abstraction (II): Impulse Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

Last Time: The System Abstraction

Represent a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

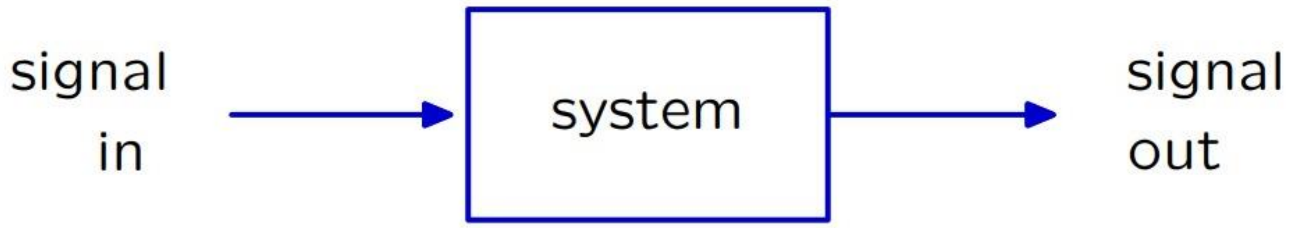


In this class, we will focus primarily on LTI (Linear, Time-Invariant) systems:

- **Linearity** (additivity and homogeneity)
- **Time invariance**

Such systems are both prevalent and mathematically tractable.

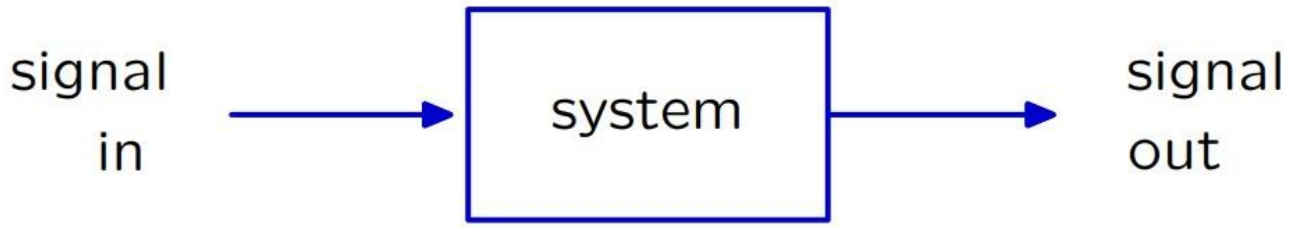
Multiple Representation of Systems



We can represent a system in the following three ways:

- Difference (Differential) Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system by its amplification or attenuation of frequency components

Today: Representing a System by its Unit-Sample Response



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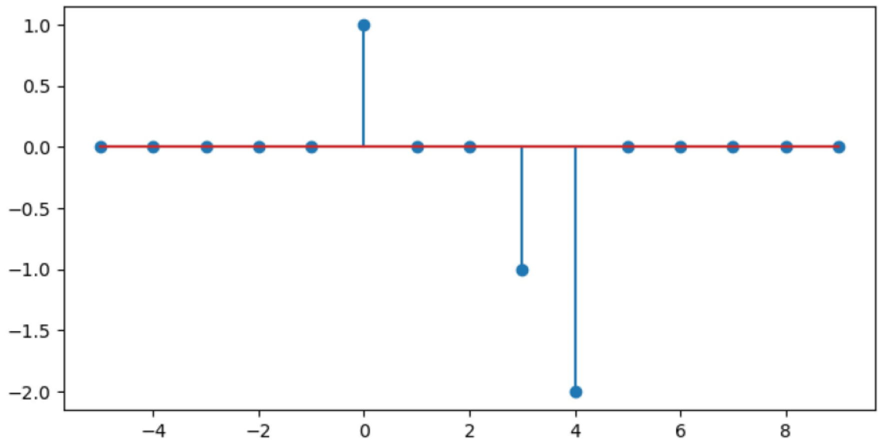
Representing a DT Signal as Sums of Delta's

Consider the following signal:

$$x[n] = \begin{cases} 1, & if\ n = 0 \\ -1, & if\ n = 3 \\ -2, & if\ n = 4 \\ 0, & otherwise \end{cases}$$



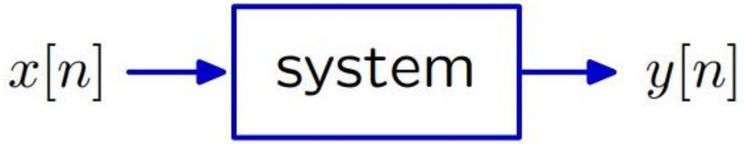
This signal can be represented as:

Participation question for Lecture

In general, we can represent a DT signal as a sum of scaled, shifted delta's:

Unit Sample Response

If a system is linear and time-invariant, its input-output relation is completely specified by the system's unit sample response h[n].



The unit sample response h[n] is the output of the system when the input is the unit sample signal δ[n].

The output for more complicated inputs can be computed by superposition of the unit sample response.

Superposition

In general, we can represent a signal as a sum of scaled, shifted delta's:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$$

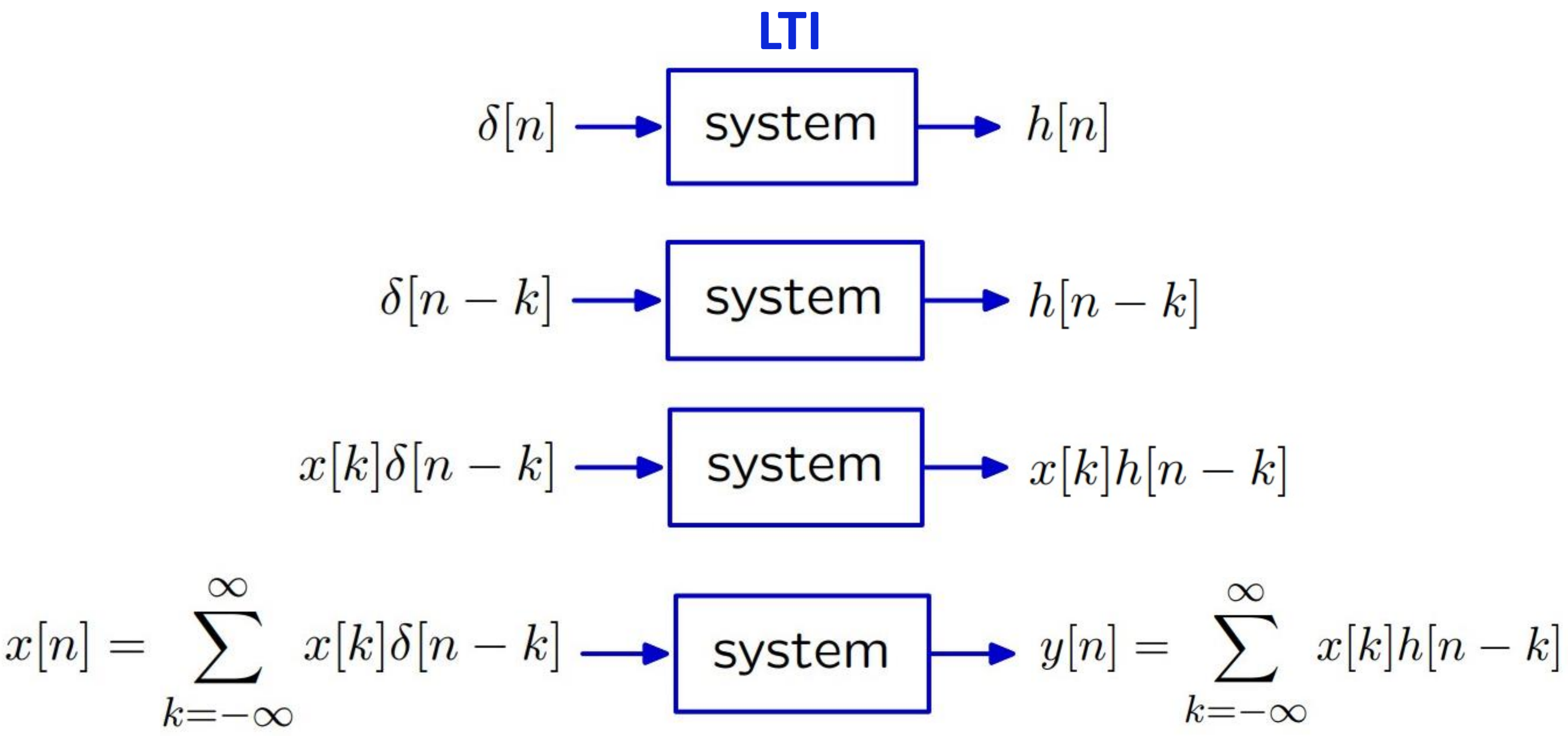
$$= \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

If h[·] is the unit sample response of an LTI system, then the output of that system in response to this arbitrary input x[·] can be viewed as a sum of scaled, shifted unit sample responses:

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]\,h[n-m]$$

$$= \cdots + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

Structure of Superposition



If a system is linear and time-invariant (LTI) then its output is the sum of weighted and shifted unit sample responses.

Convolution

Response of an LTI system to an arbitrary input:

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

and the system’s unit sample response is h[·],

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \;\equiv (x * h)[n]$$

This operation is called convolution (verb form: convolve).

Definition:

$$\boxed{(x * h)[n] \equiv \sum_{m=-\infty}^{\infty} x[m]\,h[n-m]}$$

Note: It is customary (but confusing) to abbreviate this notation: $(x * h)[n] = x[n] * h[n]$

we avoid using that in 6.300.

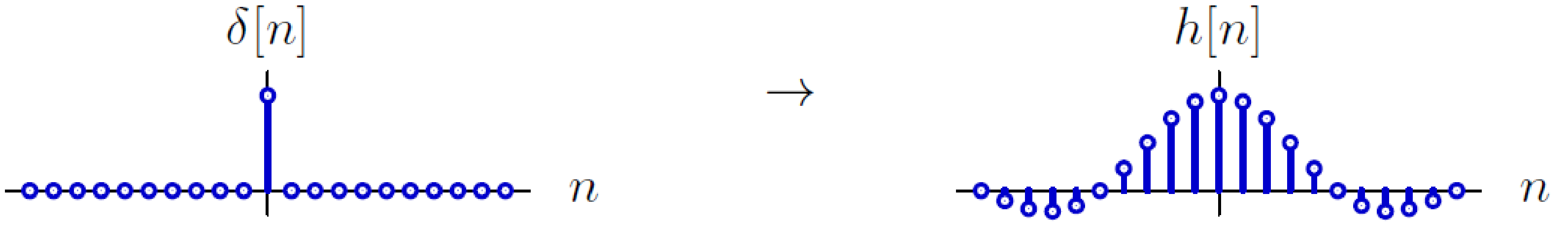
Convolution operates on signals, not samples. The symbols x and h represent DT signals. Convolving x with h generates a new DT signal $x * h$.

Unit Sample Response

The unit-sample response is a complete description of an LTI system.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

It can be used to determine the response to any other input.



Given h[·] one can compute the response y[·] to any input x[·] :

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Ways to Compute Convolution Result

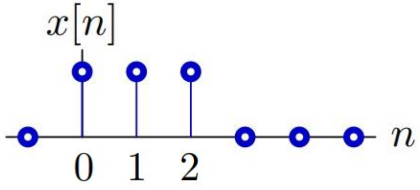
Knowing the unit sample response h[·] to a LTI system, there are different ways to compute the output y[·] to an arbitrary input x[·] :

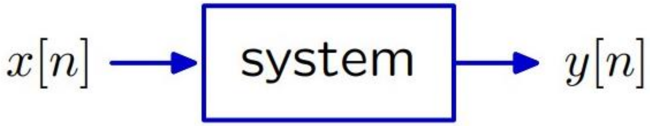
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

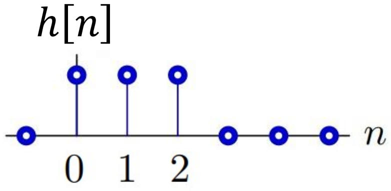
1. Directly compute the superposition result (sum the scaled and shifted responses)
2. Compute the convolution result at each particular n: flip and shift

Example

The unit sample response to a LTI system is $h[n] = \delta[n] + \delta[n-1] + \delta[n-2]$, find the output y[n] to this system with an input $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$:







1. Directly compute the superposition result (sum the scaled and shifted responses):

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Example

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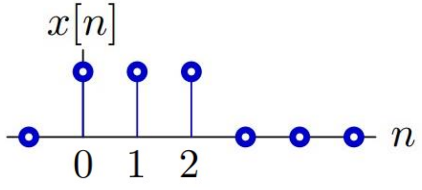
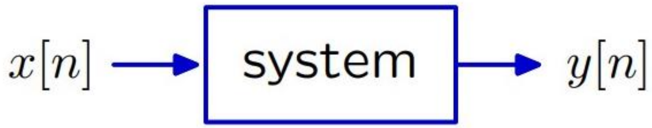
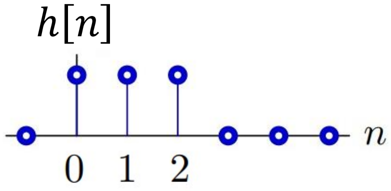




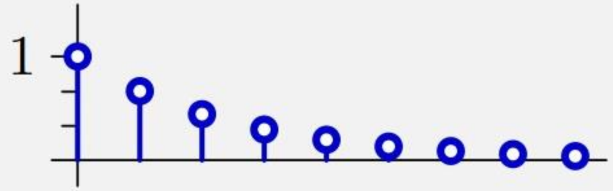


2. Compute the convolution result at each particular n: flip and shift

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Check yourself

Which plot shows the result of the convolution above?

1.


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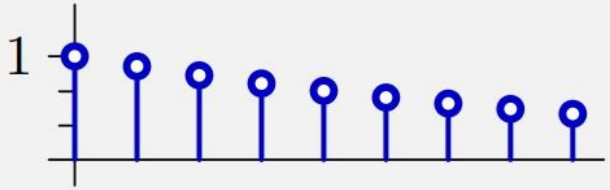

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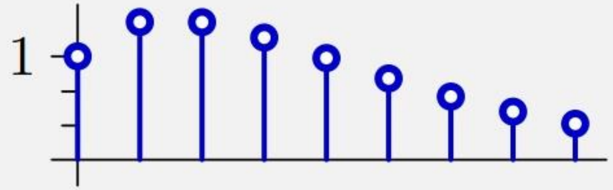

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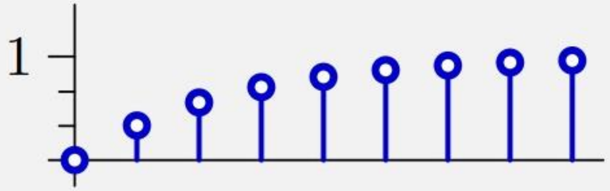
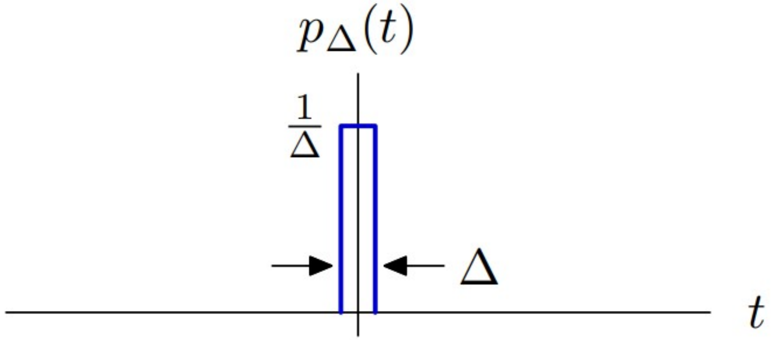


Continuous-Time LTI Systems

Superposition and convolution are of equal importance for CT systems.

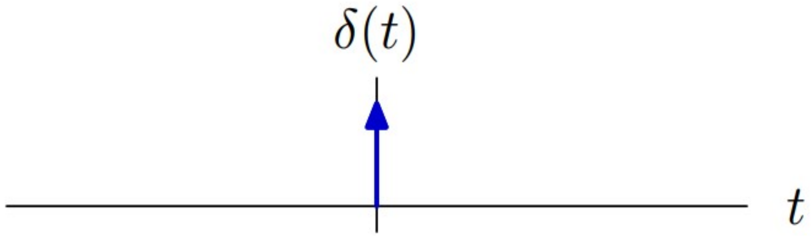
A CT LTI system is completely characterized by its impulse response, much as a DT LTI system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function δ(t) previously. It's defined in a limit as follows. Let $p_\Delta(t)$ represent a pulse of width Δ and height 1/Δ so that its area is 1.



Then

$$\delta(t) = \lim_{\Delta \to 0} p_\Delta(t)$$

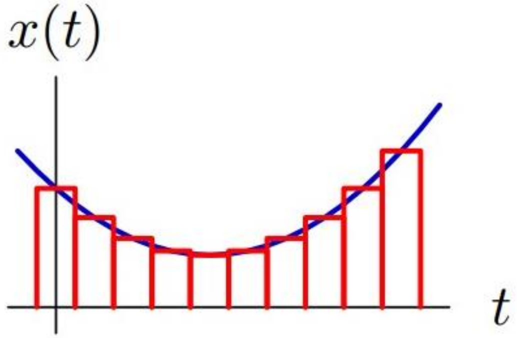


The impulse function can be used to break an arbitrary input x(t) into time-based components, much as δ[k] is used for discrete-time signals.

Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal x(t) (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta$$

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\,\delta[n-m]$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate x(t).

$$\lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is completely specified by the system's impulse response h(t).

1. One can always find the impulse response of a LTI system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

2. Time invariance implies that shifting the input simply shifts the output.

$$\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow h(t-\tau)$$

3. Homogeneity implies that scaling the input simply scales the output.

$$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{system}} \longrightarrow x(\tau)h(t-\tau)$$

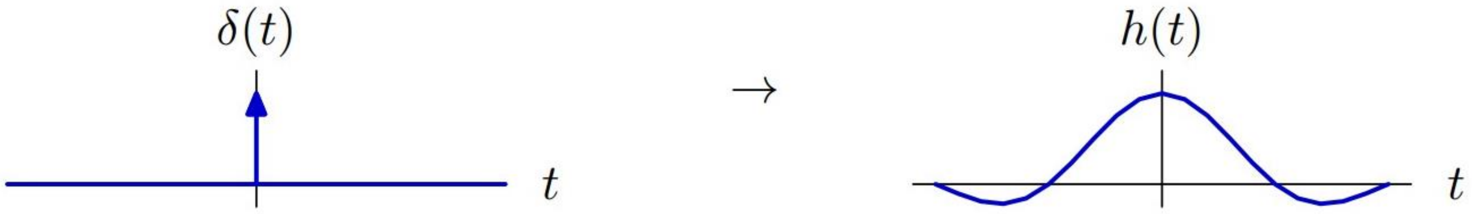
4. Additivity implies that the response to a sum is the sum of responses.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t)$$

The output of an LTI system can always be found by convolving: (x∗h)(t).

Impulse Response

The impulse response is a complete description of a CT LTI system.



Given h(t) one can compute the response to **any** arbitrary input signal x(t).

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Comparison of CT and DT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Properties of Convolution(I) Commutativity:

$$(g * h)(t) = (h * g)(t)$$

$$(g * h)(t) \equiv \int_{-\infty}^{\infty} g(\tau) \cdot h(t-\tau)\, d\tau$$

$$let\ \lambda = t - \tau, then\ \ \tau = t - \lambda, d\tau = -d\lambda$$

for τ $goes\ from\ -\infty\ to\ \infty, \lambda\ goes\ from\ \infty\ to\ -\infty$

$$(g * h)(t) = \int_{\infty}^{-\infty} g(t-\lambda) \cdot h(\lambda)\,(-d\lambda)$$

$$= \int_{-\infty}^{\infty} g(t-\lambda) \cdot h(\lambda)\, d\lambda$$

$$= (h * g)(t)$$

$$(g * h)[n] = (h * g)[n]$$

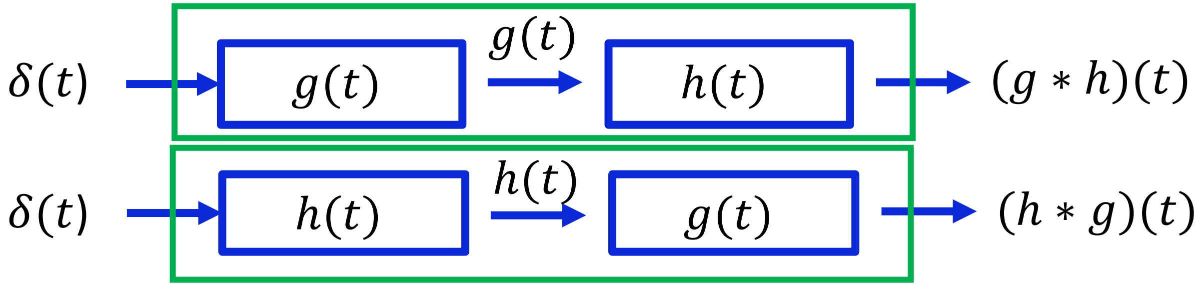
$$(g * h)[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$$

$$let\ m = n - k, then\ \ k = n - m,$$

for k $goes\ from\ -\infty\ to\ \infty, m\ goes\ from\ \infty\ to\ -\infty$

$$(g * h)[n] = \sum_{m=\infty}^{-\infty} g[n-m]h[m]$$

$$= \sum_{m=-\infty}^{\infty} g[n-m]h[m] = (h * g)[n]$$



Properties of Convolution(II) Associativity:

$$\big((x * g) * h\big)(t) = (x * (g * h))(t)$$

$$((x * g) * h)(t) \equiv \int_{-\infty}^{\infty} h(t-\lambda)d\lambda \int_{-\infty}^{\infty} x(\tau)\cdot g(\lambda-\tau)\,d\tau$$

$let\ \mu = \lambda - \tau, then\ \ \lambda = \mu + \tau, d\lambda = d\mu$

for λ $goes\ from\ -\infty\ to\ \infty, \mu\ goes\ from\ -\infty\ to\ \infty$

$$((x * g) * h)(t) = \int_{-\infty}^{\infty} x(\tau)d\tau \int_{-\infty}^{\infty} h(t-\tau-\mu)\cdot g(\mu)\,d\mu$$

$$= \int_{-\infty}^{\infty} x(\tau)(g * h)(t-\tau)d\tau$$

$$= (x * (g * h))(t)$$

$$\big((x * g) * h\big)[n] \ = (x * (g * h))[n]$$

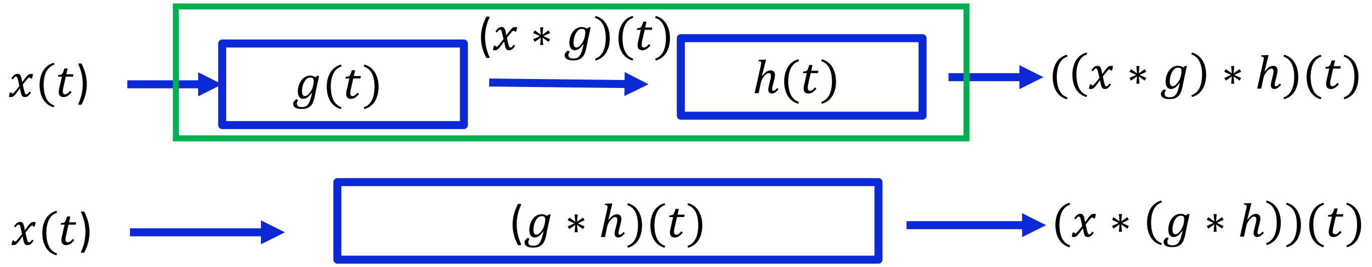
$$((x * g) * h)[n] = \sum_{m=-\infty}^{\infty} h[n-m] \sum_{k=-\infty}^{\infty} x[k]g[m-k]$$

$let\ l = m - k, then\ \ m = k + l,$

for m $goes\ from\ -\infty\ to\ \infty, l\ goes\ from\ -\infty\ to\ \infty$

$$((x * g) * h)[n] = \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h[n-k-l]g[l]$$

$$= (x * (g * h))[n]$$



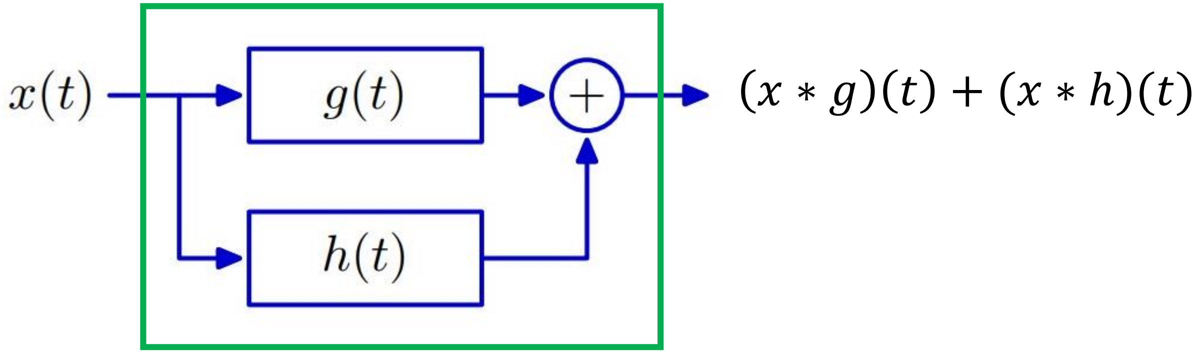
Properties of Convolution(III) Distributivity over addition

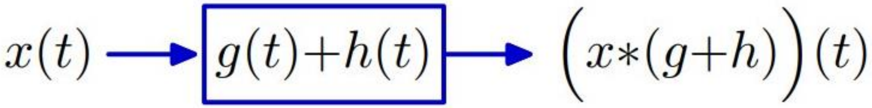
$$(x * (g + h))(t) = (x * g)(t) + (x * h)(t)$$

$$\begin{aligned}
&(x * (g + h))(t) \\
&= \int_{-\infty}^{\infty} x(\tau) \cdot (g(t-\tau) + h(t-\tau))\, d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau + \int_{\infty}^{-\infty} x(\tau) h(t-\tau)\, d\tau \\
&= (x * g)(t) + (x * h)(t)
\end{aligned}$$

$$(x * (g + h))[n] = (x * g)[n] + (x * h)[n]$$

$$\begin{aligned}
&(x * (g + h))[n] \\
&= \sum_{k=-\infty}^{\infty} x[k] \cdot (g[n-k] + h[n-k]) \\
&= \sum_{k=-\infty}^{\infty} x[k] \cdot g[n-k] + \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \\
&= (x * g)[n] + (x * h)[n]
\end{aligned}$$



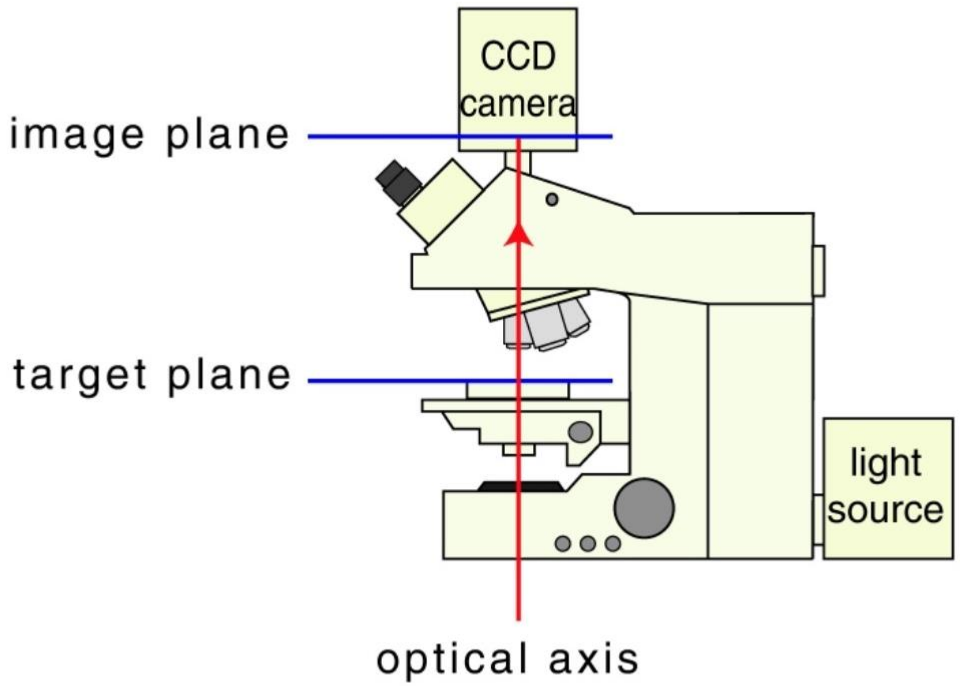


Applications of Convolution

Convolution is an important conceptual tool: it provides an important new way to think about the behaviors of systems.

Example systems: microscopes and telescopes.

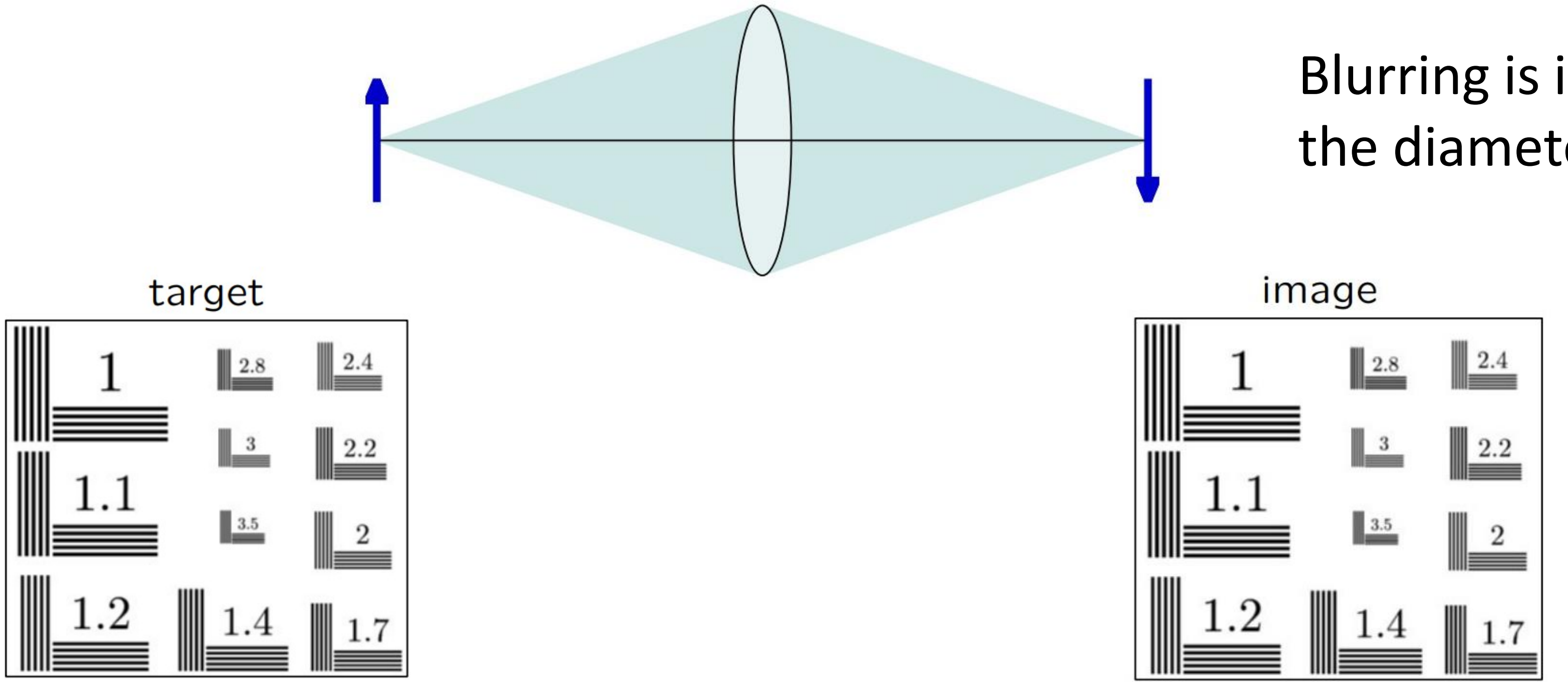
Microscopes:



Images from even the best microscopes are blurred.

Microscope

A perfect lens transforms a spherical wave of light from the target into a spherical wave that converges to the image.



Blurring is inversely related to the diameter of the lens.

Microscope

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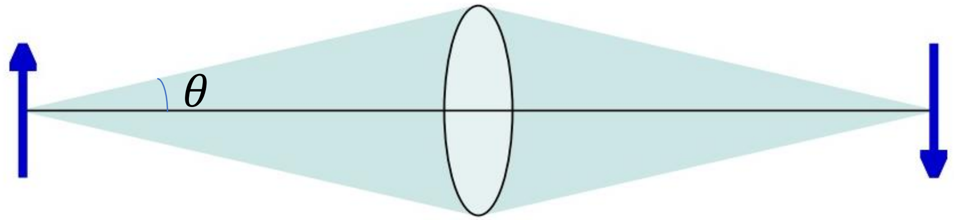

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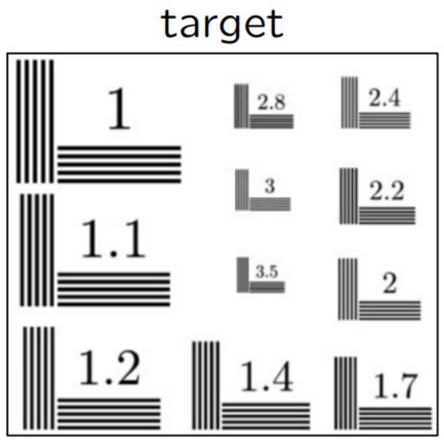
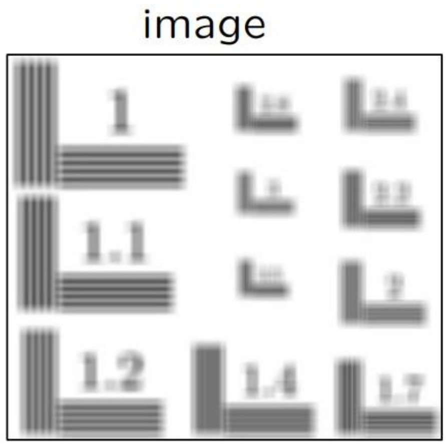
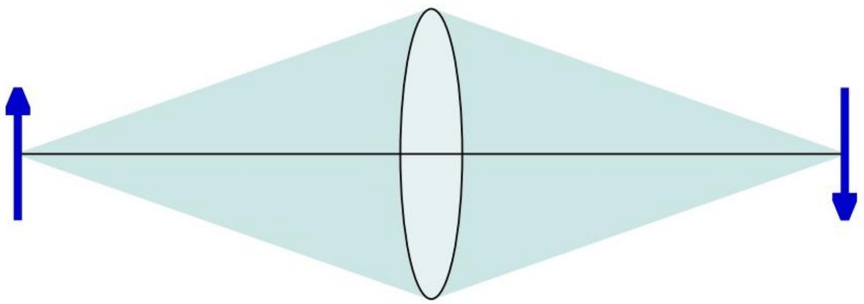




Numerical Aperture: $NA = n sin\theta$. n is refractive index

Microscope

Blurring can be represented by convolving the image with the optical "point-spread-function" (3D impulse response).



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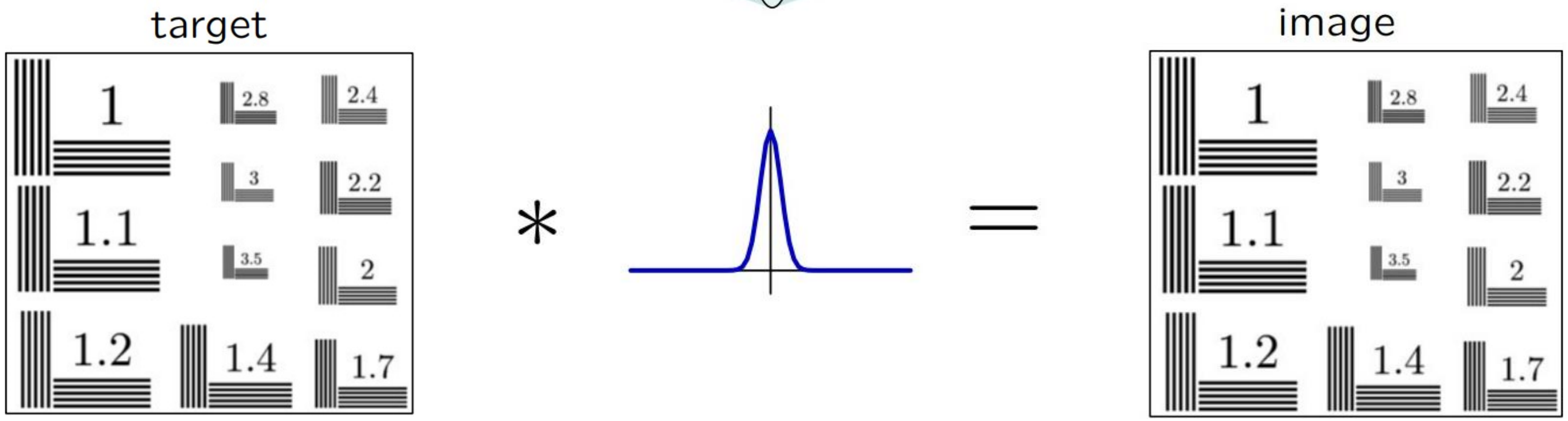

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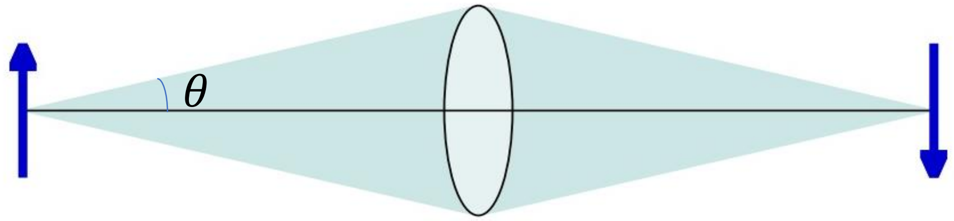

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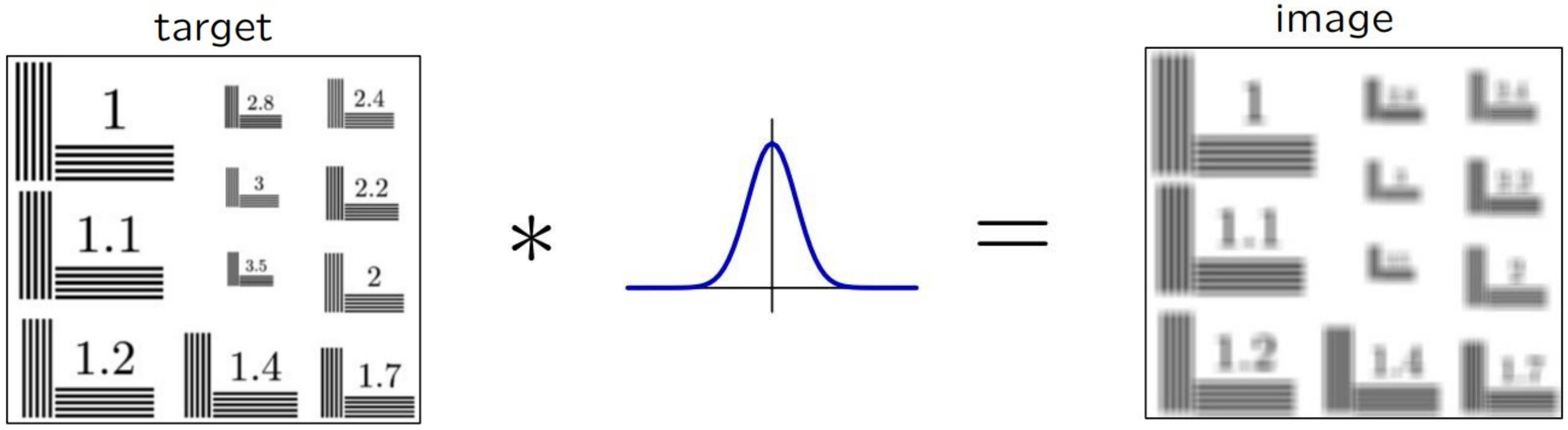


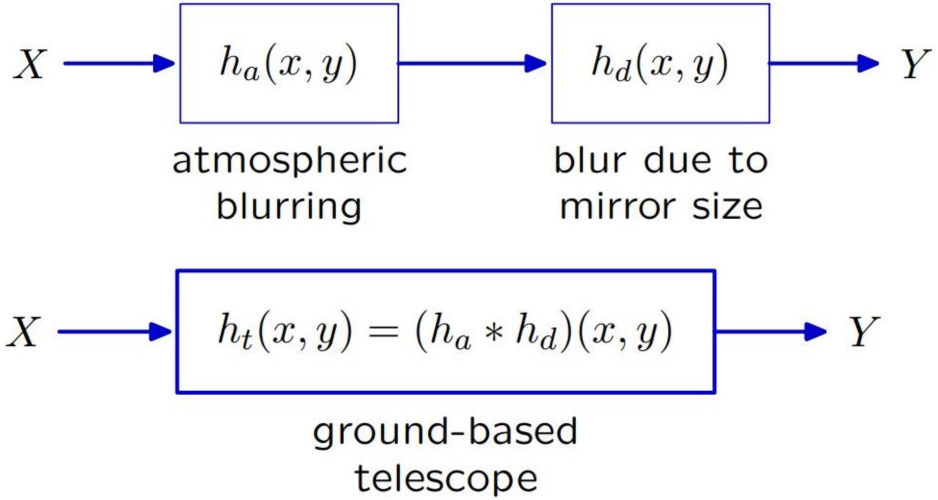
Hubble Space Telescope

Hubble Space Telescope (1990-)

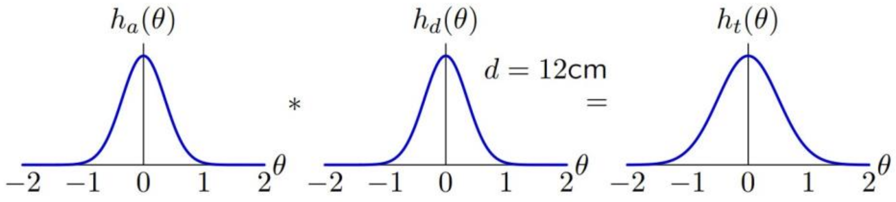
https://hubblesite.org/

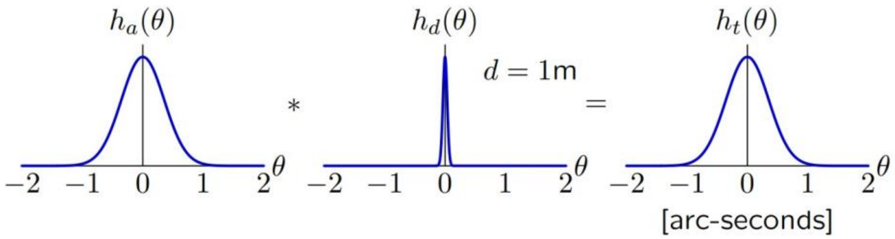
Why build a space telescope?

Telescope images are blurred by the telescope lenses AND by atmospheric turbulence.



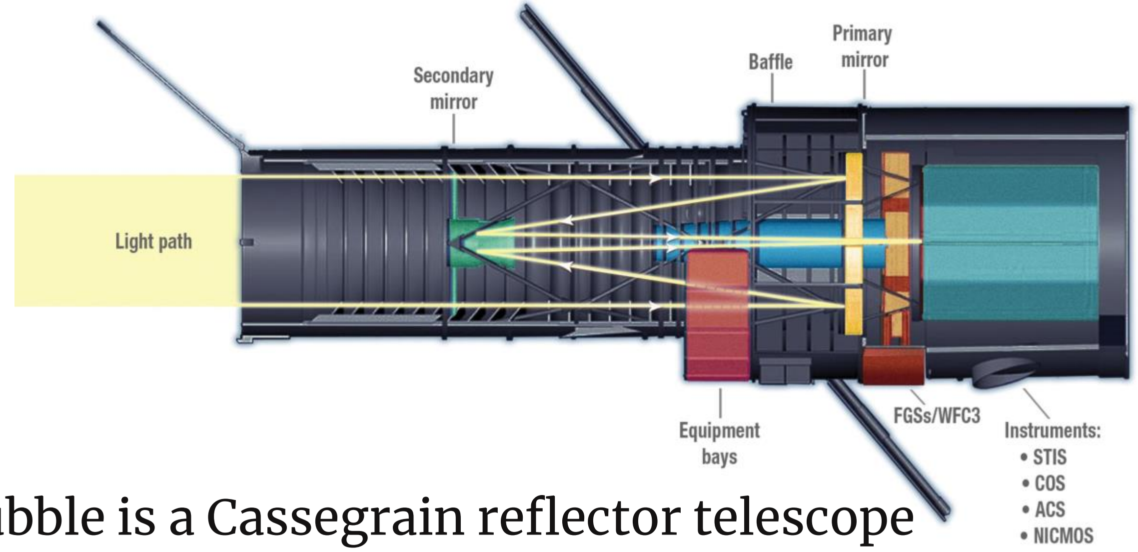
Telescope blur can be represented by the convolution of blur due to atmospheric turbulence and blur due to mirror size.





Hubble Space Telescope

The main optical components of the Hubble Space Telescope are two mirrors.

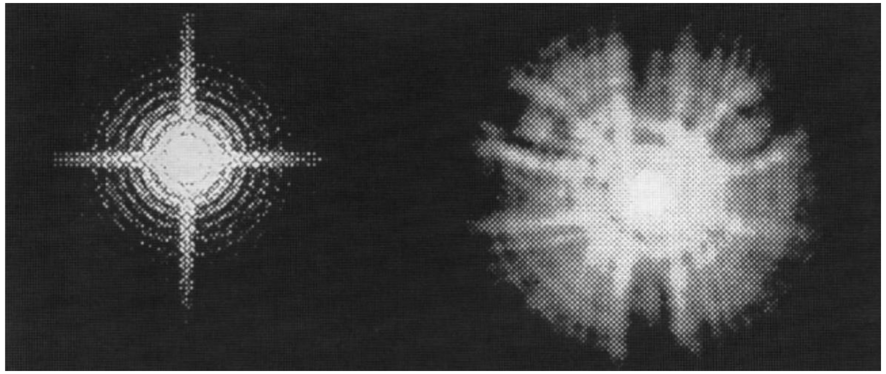


Hubble is a Cassegrain reflector telescope

The diameter of the primary mirror is 2.4 meters.

the outer edge of the mirror was ground too flat by a depth of 4 microns!

Hubble's first pictures of distant stars (May 20, 1990) were more blurred than expected.



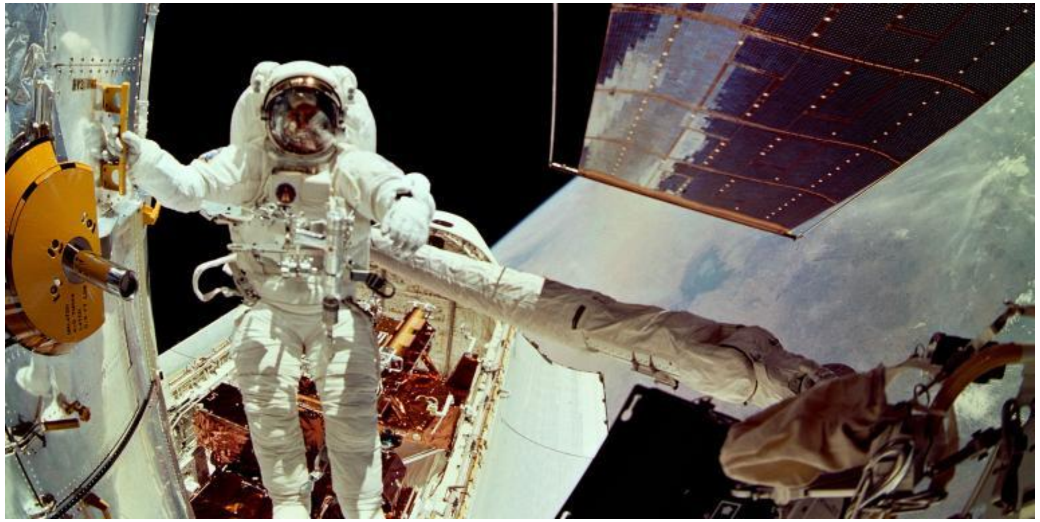
expected point-spread function   early Hubble image of distant star

Scientists immediately began experimenting with algorithms to resolve and make use of Hubble's data and imagery, pushing forward image processing technology.

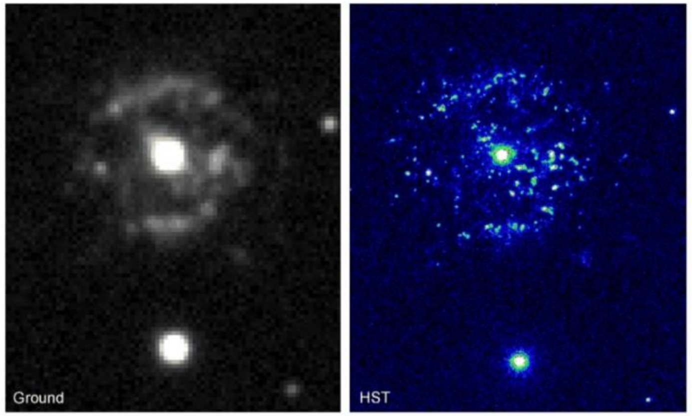
https://hubblesite.org/

Hubble Space Telescope

Corrective Optics Space Telescope Axial Replacement (COSTAR): eyeglasses for Hubble!

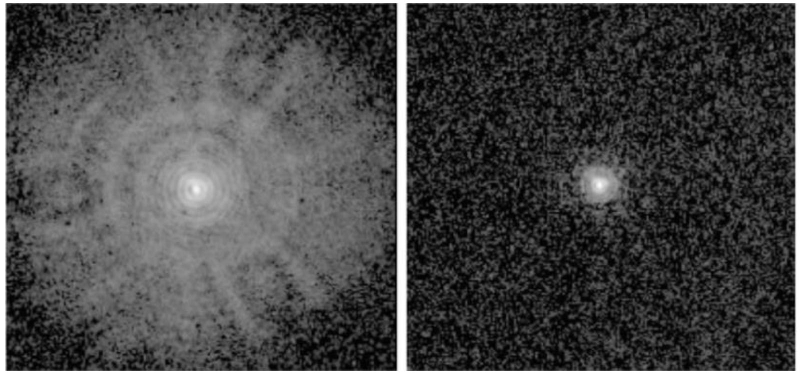


Astronauts completed major upgrades to the Hubble Space Telescope during a ten-day mission December 1993.



Images from ground-based telescope and Hubble.

Hubble images before and after COSTAR.



before after

https://hubblesite.org/

Summary

Convolution

DT: $$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

CT: $$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The unit-sample/impulse response is a complete description of an LTI system.

$$\delta(t) \longrightarrow \boxed{\text{system}} \longrightarrow h(t)$$

One can find the response to an arbitrary input signal by convolving the input signal with the impulse response.

The impulse response is an especially useful description of some types of systems, e.g., optical systems, where blurring is an important figure of merit.