6.300 Signal Processing

Week 6, Lecture A:
Systems

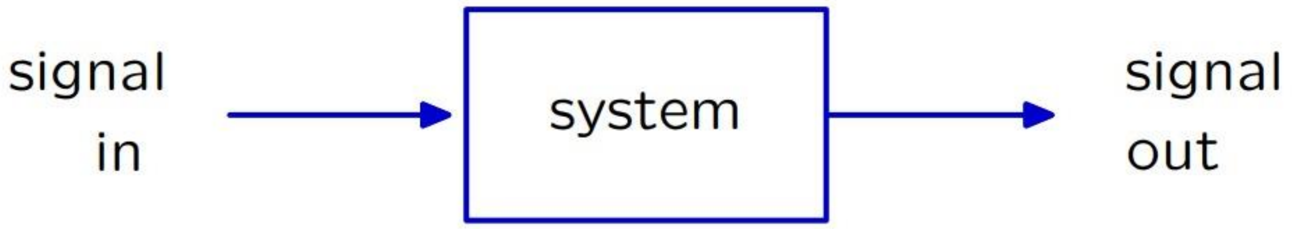
- System Abstraction
- Linearity and Time Invariance

Lecture slides are available on CATSOOP:

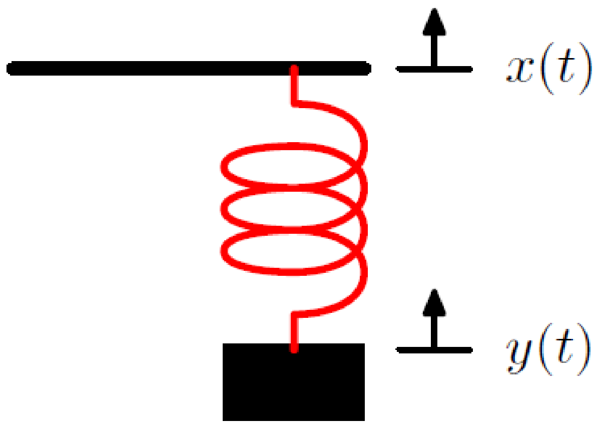
https://sigproc.mit.edu/fall24

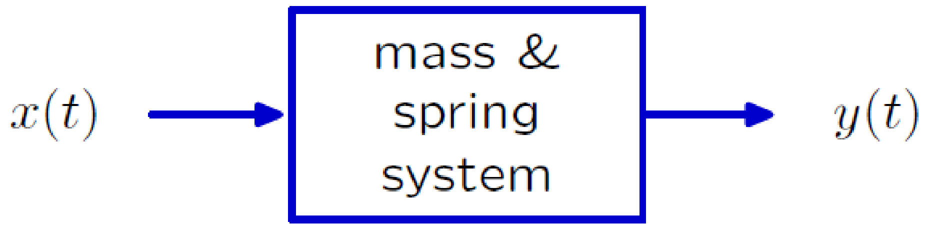
From Signal to Systems: The System Abstraction

Represent a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

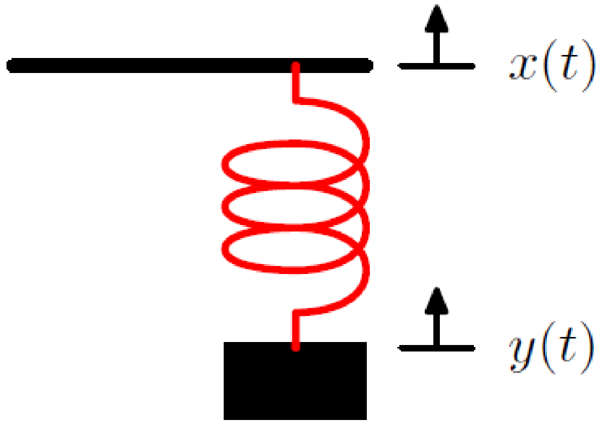


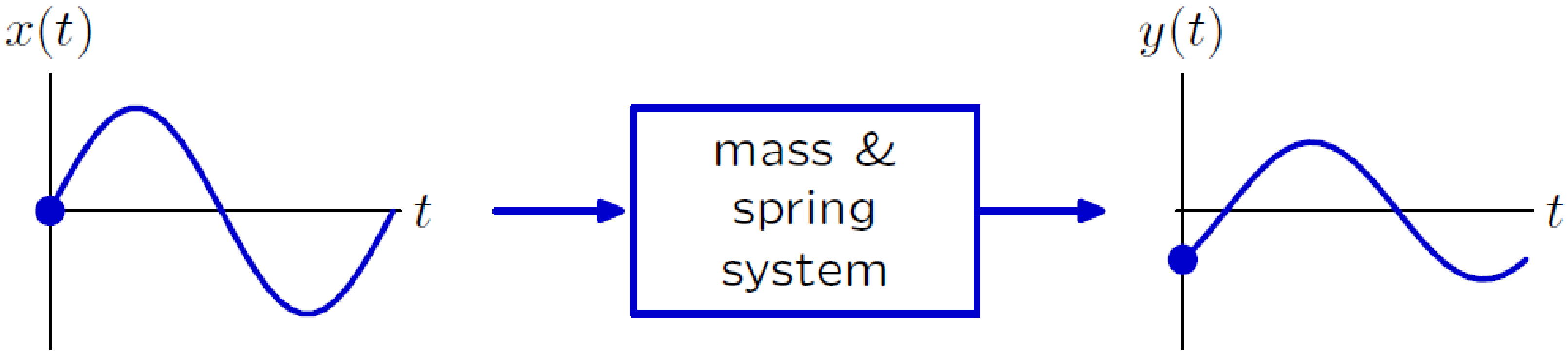
Example: Mass and Spring



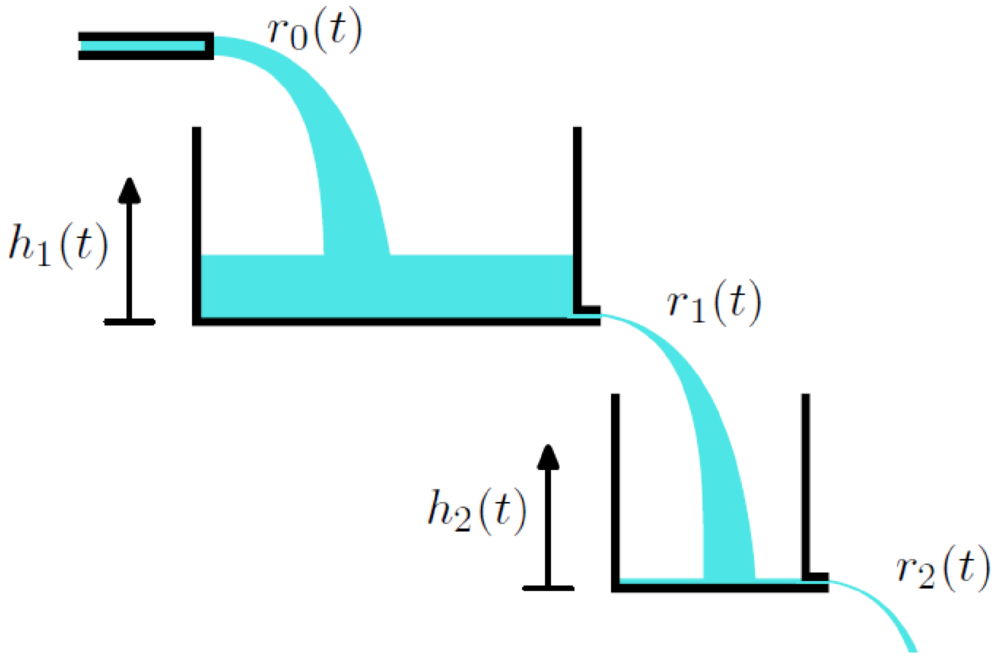


Example: Mass and Spring

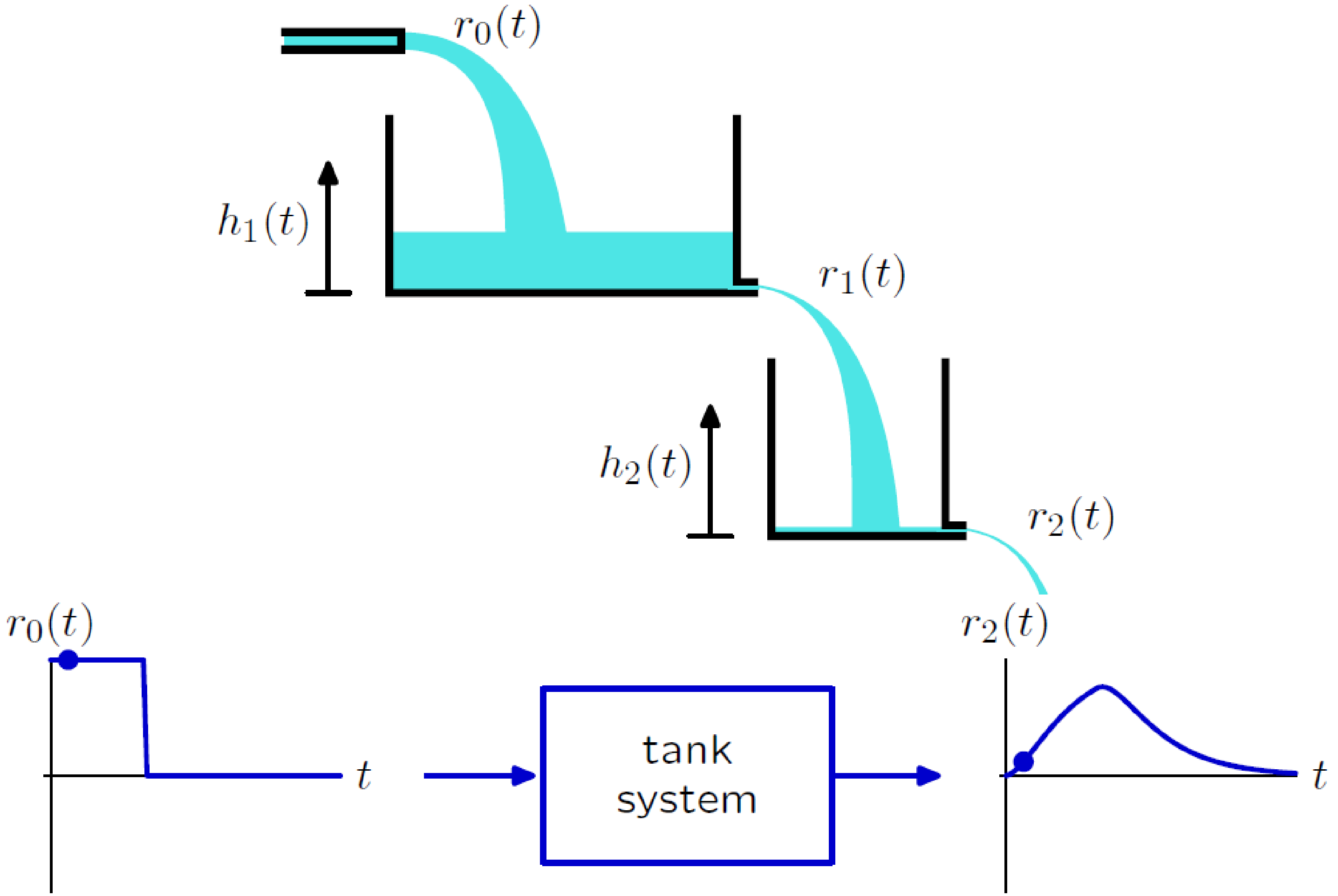




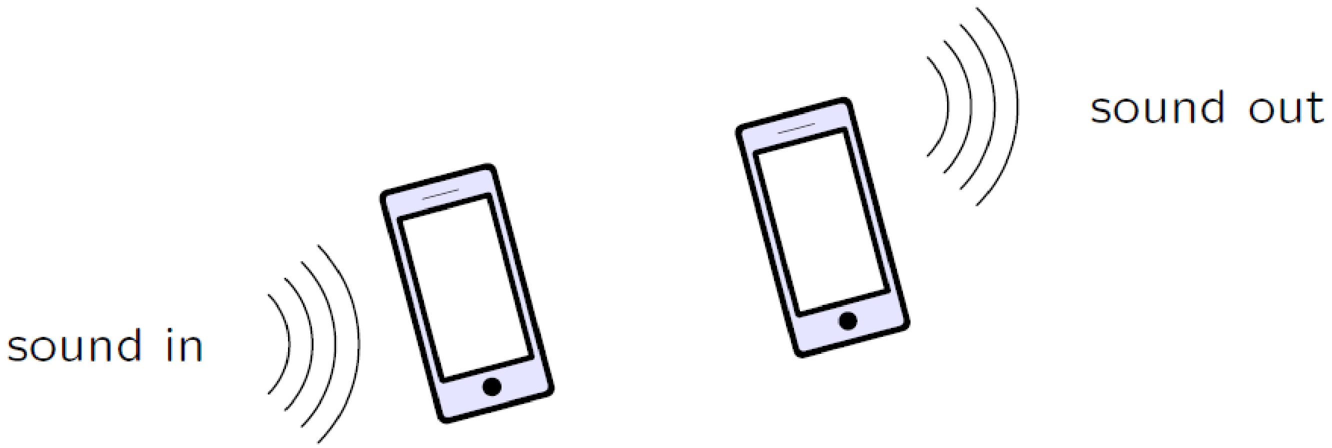
Example: Tanks

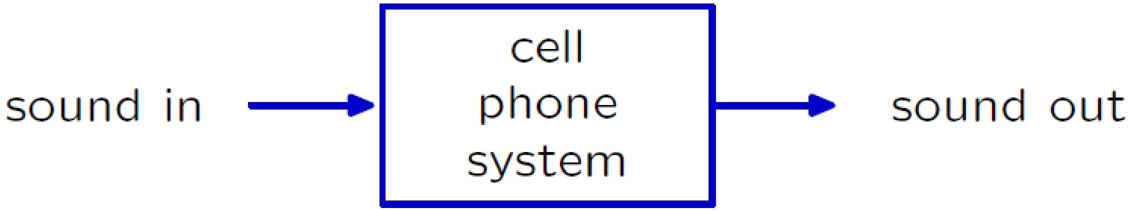


Example: Tanks

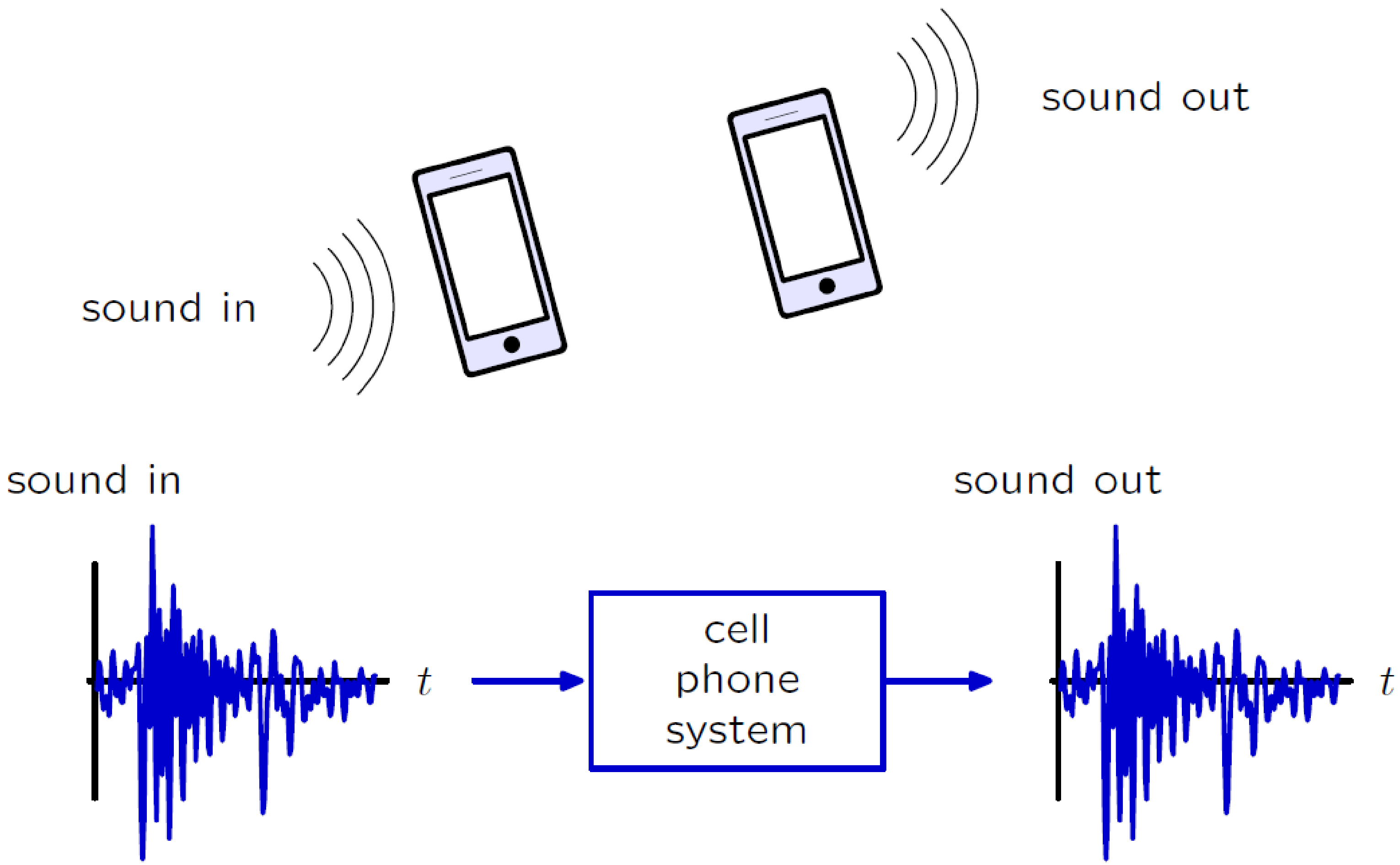


Example: Cell Phone System



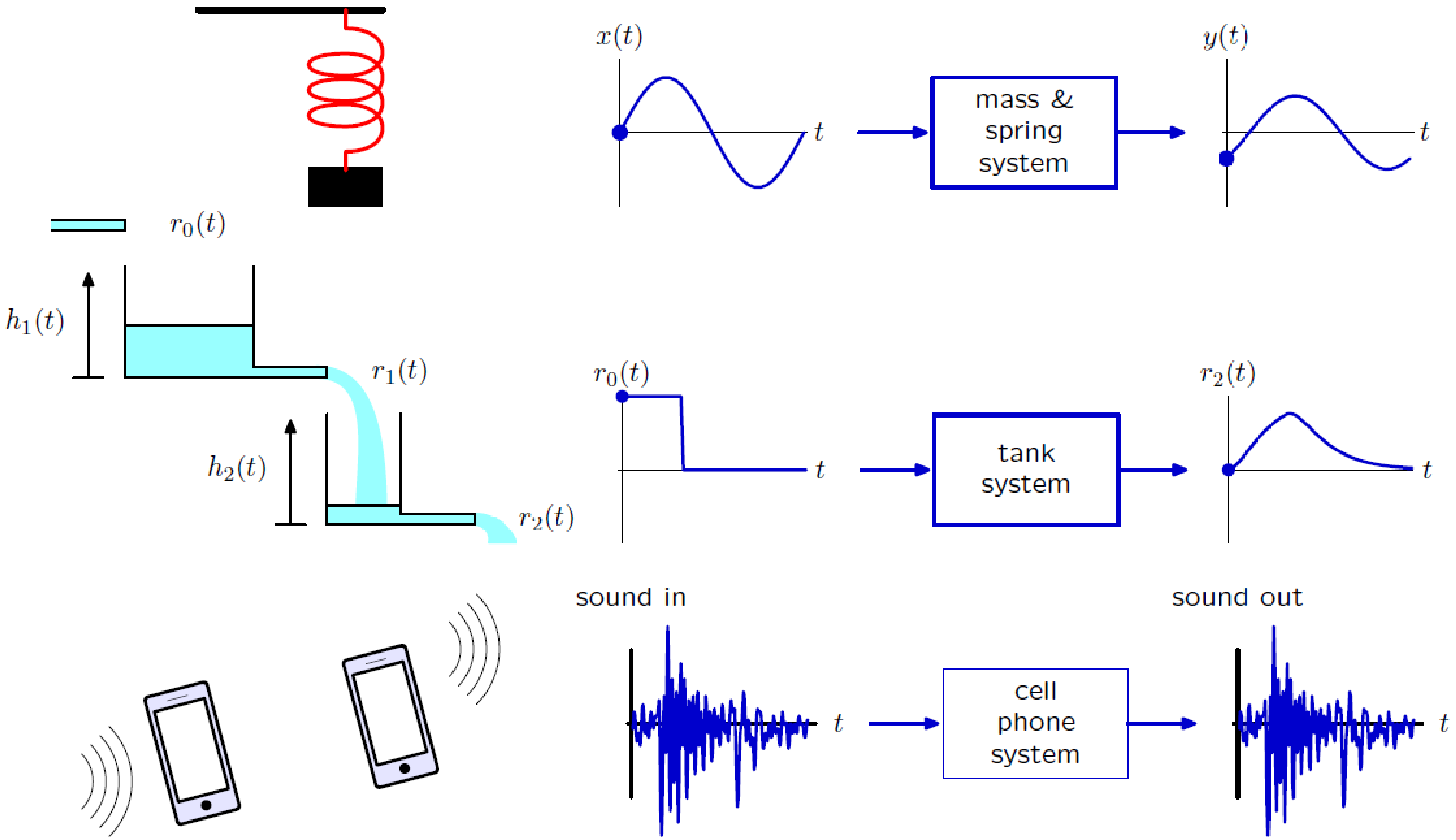


Example: Cell Phone System



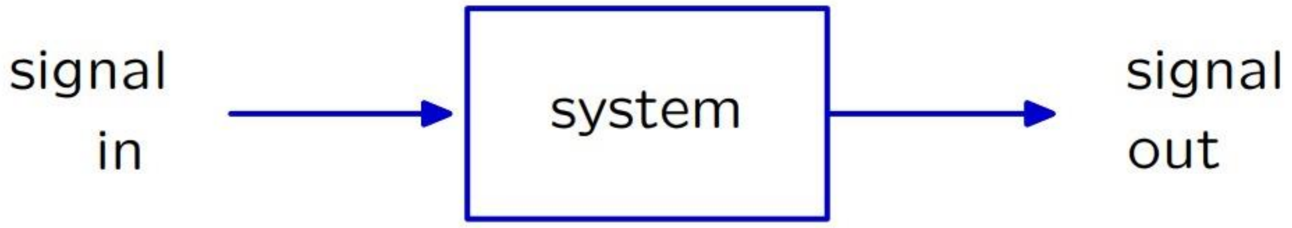
Signals and Systems: Widely Applicable

The Signals and Systems approach has broad applications: electrical, mechanical, optical, acoustic, biological, financial, …



The System Abstraction

Many applications of signal processing can be formulated as systems that convert an input signal into an output signal.



Examples:

• **audio**: equalization, noise reduction, reverberation reduction, echo cancellation, pitch shift (auto-tune)

• **image:** smoothing, edge enhancement, unsharp masking, feature detection

• **video:** image stabilization, motion magnification

Example

Audio: Vocal removal

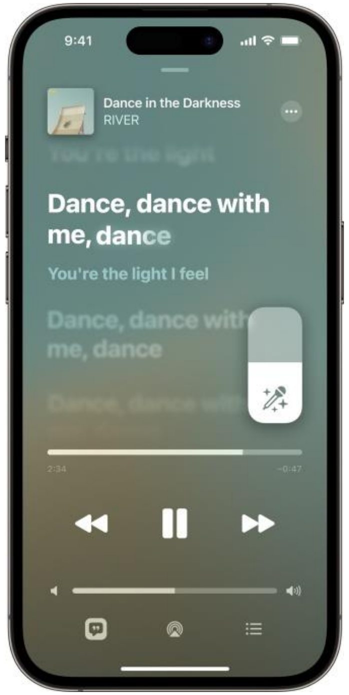
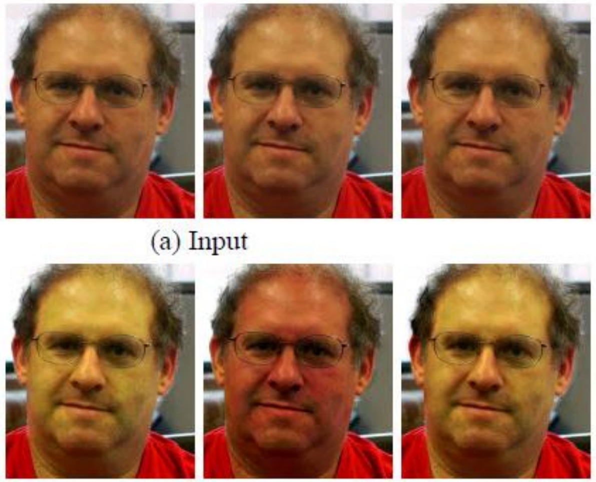


Image: Denoising

Video: Motion magnification

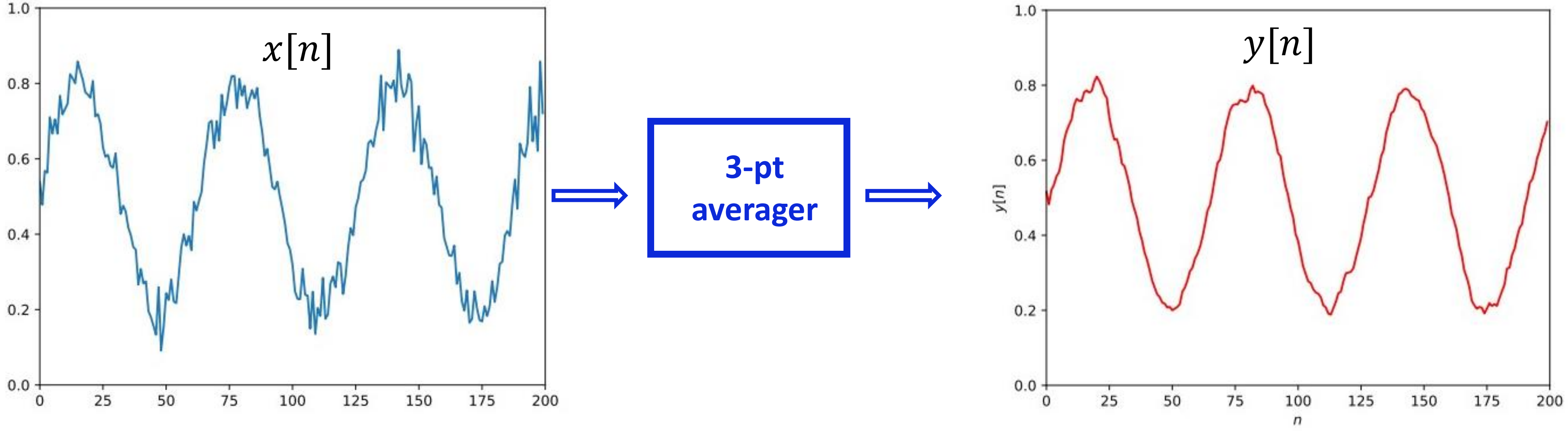


(a) Input

(b) Magnified

Example: Running Average

Noisy sensor data can be "smoothed" to reduce the impact of noise on the signal. For example, consider the following data on the left, consisting of a sinusoid corrupted with noise:



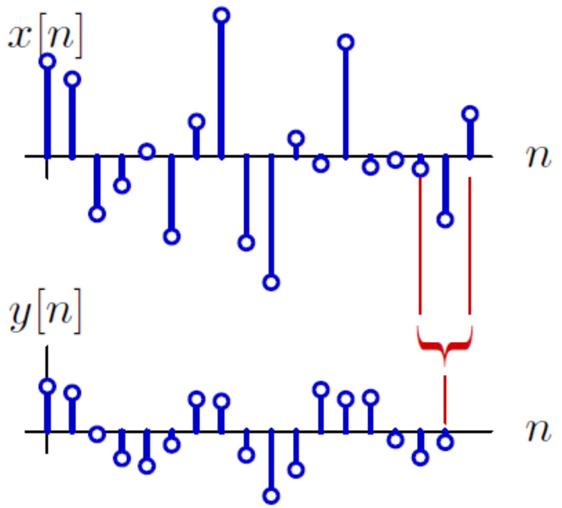
Consider the case where this signal is the input to a system described as "three point averager", whose output at time n is the average of three consecutive input samples:

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$

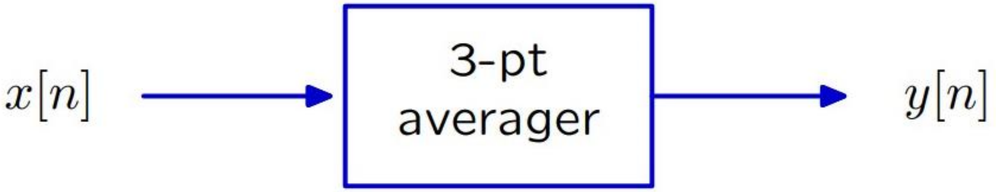
Example System: Three-point Averaging

The output at time *n* is average of inputs at times *n*-1, *n*, and *n*+1.

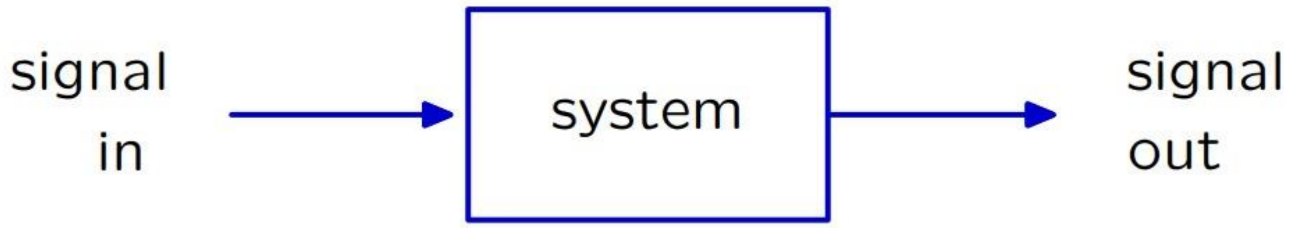
$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Think of this process as a system with input $x[n]$ and output $y[n]$.



Multiple Representation of Systems



We can represent a system in the following three ways:

- Difference Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system by its amplification or attenuation of frequency components

Linear, Time-Invariant(LTI) System

Arbitrary systems are arbitrarily difficult to describe.

Fortunately, many useful systems have two important properties:

- **Linearity** (additivity and homogeneity)
- **Time invariance**

In 6.300, we will focus on systems that have both of these properties, which are called **LTI systems**.

Additivity

A system is additive if its response to a sum of inputs is equal to the sum of its responses to each input taken one at a time.

Given $$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

And $$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the system is additive if

$$x_1[n] + x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] + y_2[n]$$

is true for all possible inputs.

Homogeneity

A system is homogeneous if multiplying its input by a constant multiplies its output by the same constant.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

the system is homogeneous if

$$\alpha x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n]$$

is true for all α and all possible inputs.

Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum (i.e. superposition) of its responses to each of the inputs.

Given $$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

And $$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the system is linear if

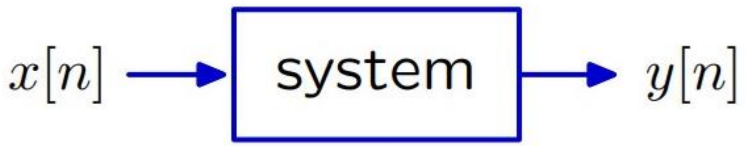
$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β and all possible inputs.

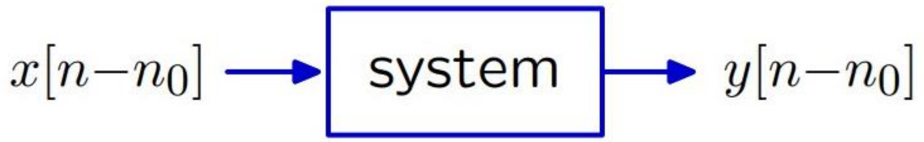
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given



the system is time-invariant if



is true for all n_0 and for all possible inputs.

Check yourself (I)

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1] \qquad \text{(for all n)}$$

is this system linear?

If $\quad x_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n] \quad \Longrightarrow \quad y_1[n] = x_1[n] + x_1[n-1]$

and $\quad x_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n] \quad \Longrightarrow \quad y_2[n] = x_2[n] + x_2[n-1]$

Let $\quad x[n] = \alpha x_1[n] + \beta x_2[n] \quad$ then

$$\begin{aligned} y[n] &= x[n] + x[n-1] \\ &= (\alpha x_1[n] + \beta x_2[n]) + (\alpha x_1[n-1] + \beta x_2[n-1]) \\ &= \alpha(x_1[n] + x_1[n-1]) + \beta(x_2[n] + x_2[n-1]) \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

$\Longrightarrow$ The system is linear.

Check yourself (II)

Consider a system represented by the following difference equation:

$$y[n] = x[n] + 1 \qquad \text{(for all n)}$$

is this system linear?

If $x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] \quad \Longrightarrow \quad y_1[n] = x_1[n] + 1$

and $x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n] \quad \Longrightarrow \quad y_2[n] = x_2[n] + 1$

Let $x[n] = \alpha x_1[n] + \beta x_2[n]$ then

$$\begin{aligned} y[n] &= x[n] + 1 \\ &= (\alpha x_1[n] + \beta x_2[n]) + 1 \\ &\neq \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

$$\alpha y_1[n] + \beta y_2[n] = \alpha(x_1[n] + 1) + \beta(x_2[n] + 1)$$

$\Longrightarrow$ The system is NOT linear.

Check yourself (III)

Consider a system represented by the following difference equation:

$$y[n] = x[n] \times x[n-1] \qquad \text{(for all n)}$$

is this system linear?

Participation question for Lecture

If $\quad x_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n] \quad \Longrightarrow \quad y_1[n] = x_1[n] \times x_1[n-1]$

and $\quad x_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n] \quad \Longrightarrow \quad y_2[n] = x_2[n] \times x_2[n-1]$

Let $\quad x[n] = \alpha x_1[n] + \beta x_2[n] \quad$ then

$$\begin{aligned} y[n] &= x[n] \times x[n-1] \\ &= (\alpha x_1[n] + \beta x_2[n]) \times (\alpha x_1[n-1] + \beta x_2[n-1]) \\ &\neq \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

$$\alpha y_1[n] + \beta y_2[n] = \alpha(x_1[n] \times x_1[n-1]) + \beta(x_2[n] \times x_2[n-1])$$

$\Longrightarrow$ The system is NOT linear.

Check yourself (IV)

Consider a system represented by the following difference equation:

$y[n] = nx[n]$ (for all n)

is this system linear?

If $x_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n]$ $\Longrightarrow$ $y_1[n] = nx_1[n]$

and $x_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n]$ $\Longrightarrow$ $y_2[n] = nx_2[n]$

Let $x[n] = \alpha x_1[n] + \beta x_2[n]$ then

$$\begin{aligned} y[n] &= nx[n] \\ &= n(\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha \cdot nx_1[n] + \beta \cdot nx_2[n] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

$\Longrightarrow$ The system is linear.

Check yourself (V)

Consider a system represented by the following difference equation:

$$y[n] = nx[n] \qquad \text{(for all n)}$$

is this system time-invariant?

If $\quad x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] \quad \Longrightarrow \quad y_1[n] = nx_1[n]$

Let $\quad x_1[n] = x[n - n_0] \quad \Longrightarrow \quad y_1[n] = nx[n - n_0]$

But based on $\quad y[n] = nx[n] \qquad y[n - n_0] = (n - n_0) \cdot x[n - n_0]$

$$y[n - n_0] \neq y_1[n]$$

$\Longrightarrow$ The system is NOT time-invariant.

Represent a LTI system with Difference Equations

A system is linear and time-invariant if it can be expressed in terms of a linear difference equation with constant coefficients of the following form:

$$\sum_m C_m \, y[n-m] = \sum_k d_k \, x[n-k]$$

e.g. 3-pt averager:

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$

Linearity: weighted sum of outputs equal to weighted sum of inputs

$$\sum_m C_m \left(\alpha y_1[n-m] + \beta y_2[n-m]\right) = \sum_k d_k \cdot \alpha x_1\,[n-k] + \sum_k d_k \cdot \beta x_2\,[n-k]$$

Time invariance: delaying an input delays its output

$$\sum_m C_m \, y[n-n_0-m] = \sum_k d_k \, x[n-n_0-k]$$

The three-point averager is a linear, time-invariant system, but the system $y[n] = x[n] + 1$ is not a LTI system.

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients as follows, then the system is LTI.

$$\sum_l c_l \frac{d^l}{dt^l} y(t) = \sum_m d_m \frac{d^m}{dt^m} x(t)$$

Additivity: output of sum is sum of outputs.

$$\sum_l c_l \frac{d^l}{dt^l} (y_1(t) + y_2(t)) = \sum_m d_m \frac{d^m}{dt^m} (x_1(t) + x_2(t))$$

Homogeneity: scaling an input scales its output.

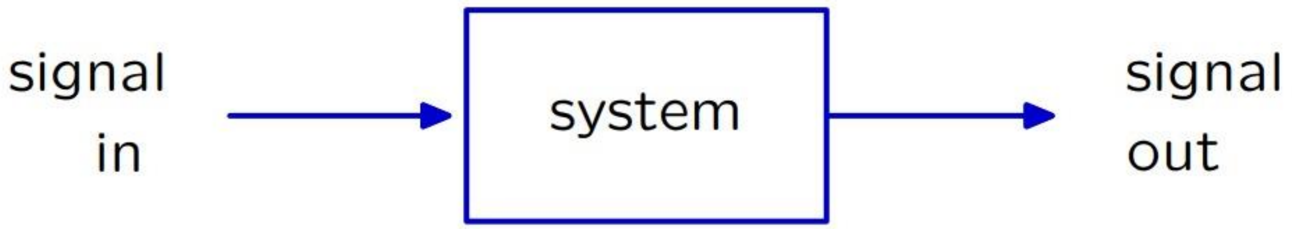
$$\sum_l c_l \frac{d^l}{dt^l} (\alpha y(t)) = \sum_m d_m \frac{d^m}{dt^m} (\alpha x(t))$$

Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l}{dt^l} y(t - t_0) = \sum_m d_m \frac{d^m}{dt^m} x(t - t_0)$$

Multiple Representation of LTI Systems

Next: Representing a system by its unit-sample response

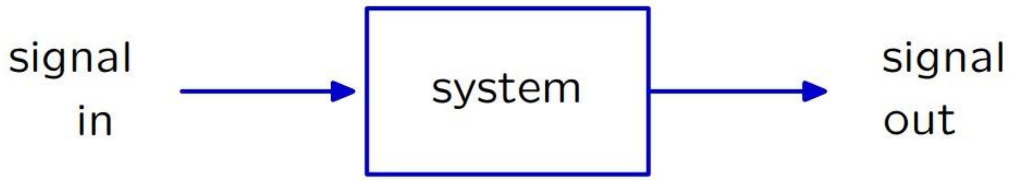


✓ Difference (differential) Equation: represent system by algebraic constraints on samples

- **Convolution: represent a system by its unit-sample response**
- Filter: represent a system as by its amplification or attenuation of frequency components

Summary

The concept of "system" to represent the process/method to manipulate signals:



Linear, Time-Invariant Systems

Three ways of representing a LTI system:

- Difference (differential) Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system as by its frequency response

We will now go to 4-370 for recitation & common hour