6.300 Signal Processing

Week 6, Lecture A:
Systems

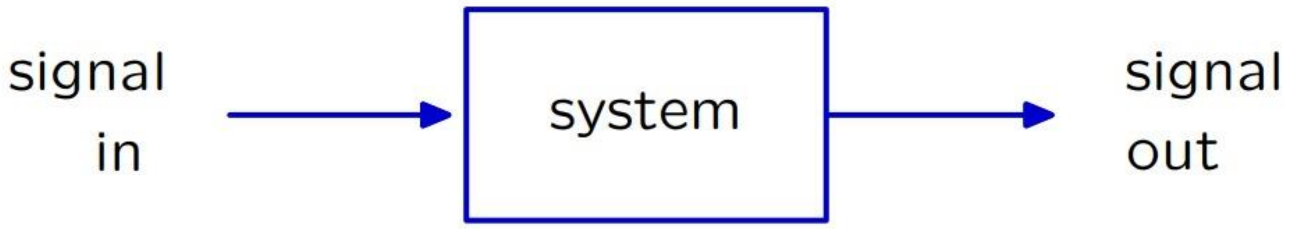
- System Abstraction
- Linearity and Time Invariance

Lecture slides are available on CATSOOP:

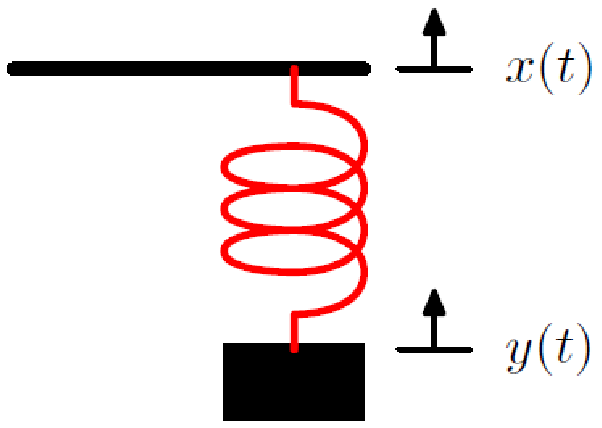
https://sigproc.mit.edu/fall24

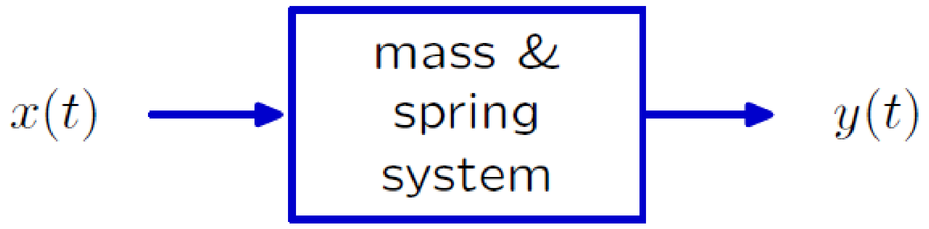
From Signal to Systems: The System Abstraction

Represent a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

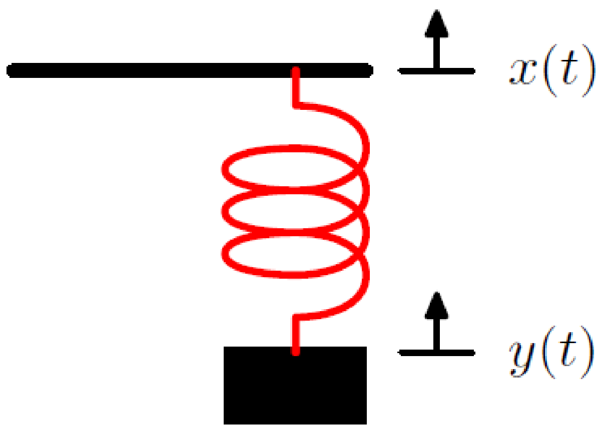


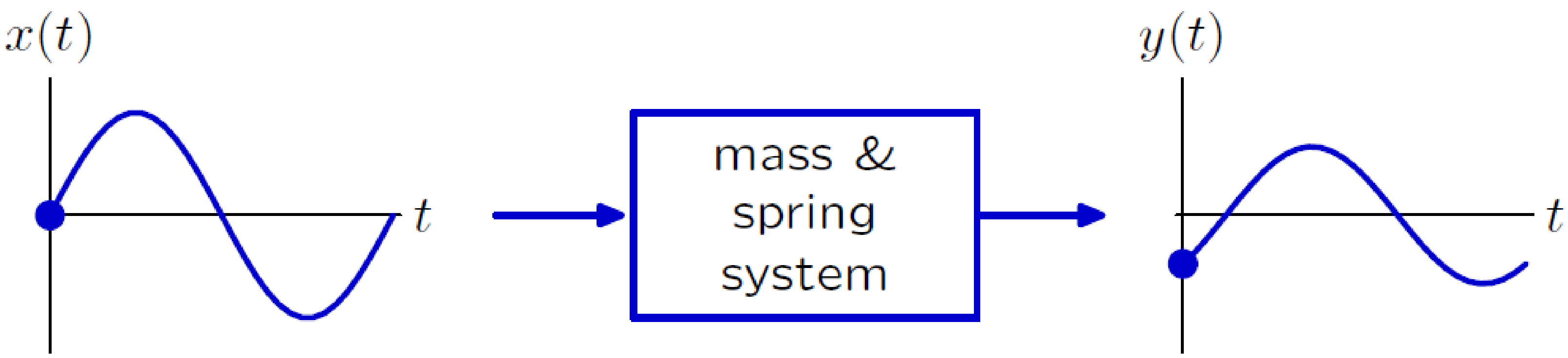
Example: Mass and Spring



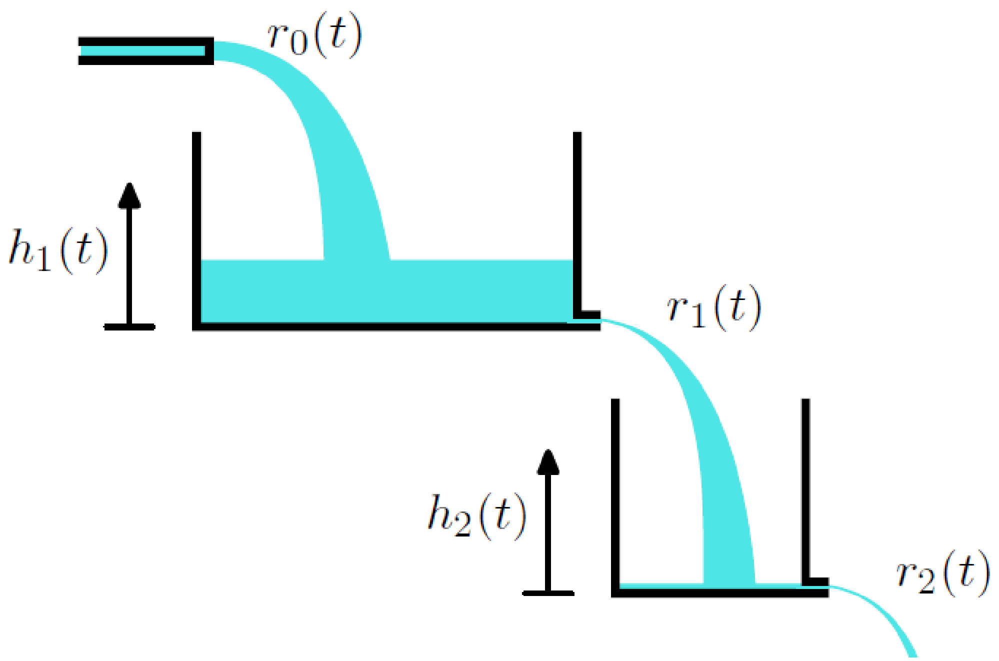


Example: Mass and Spring

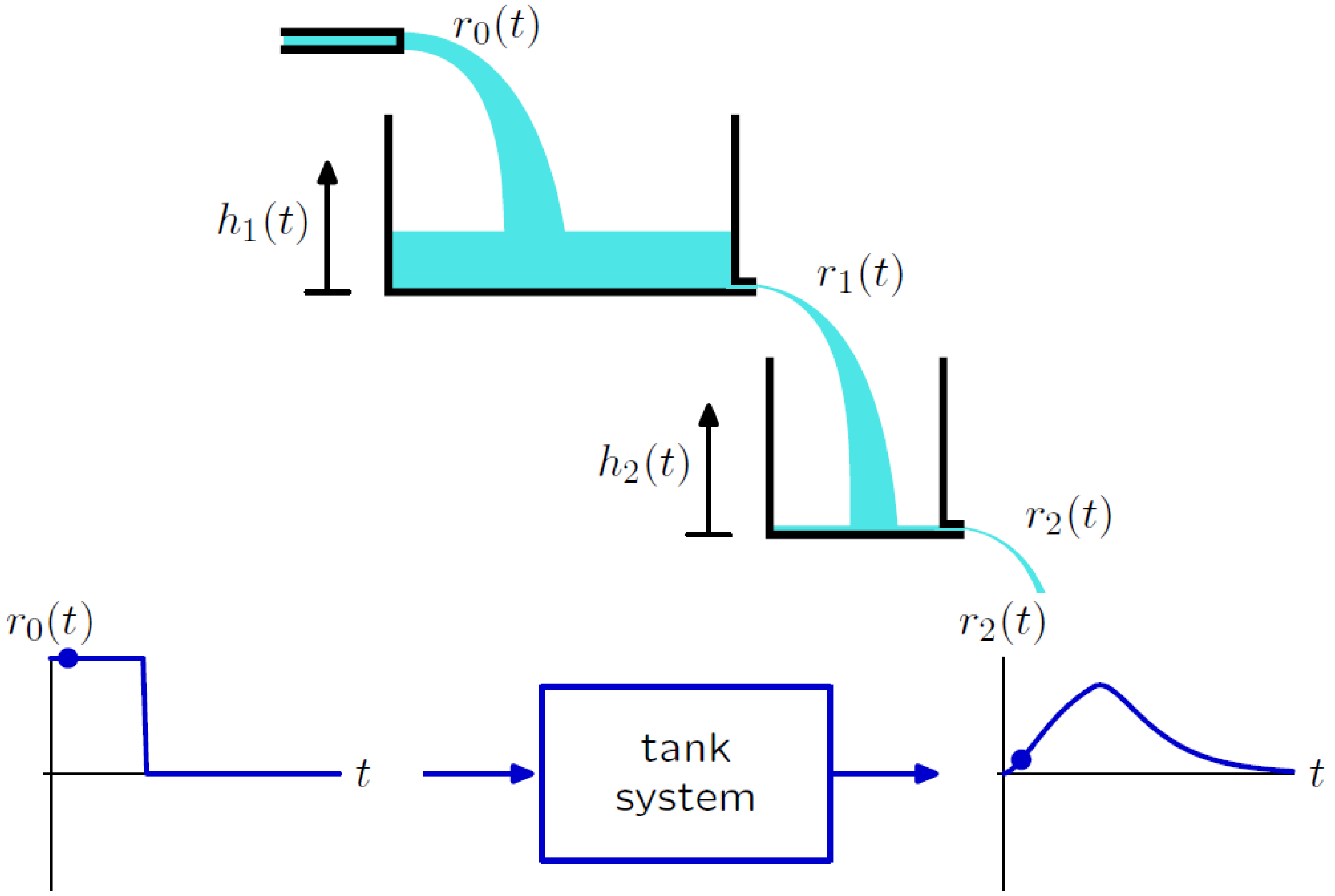




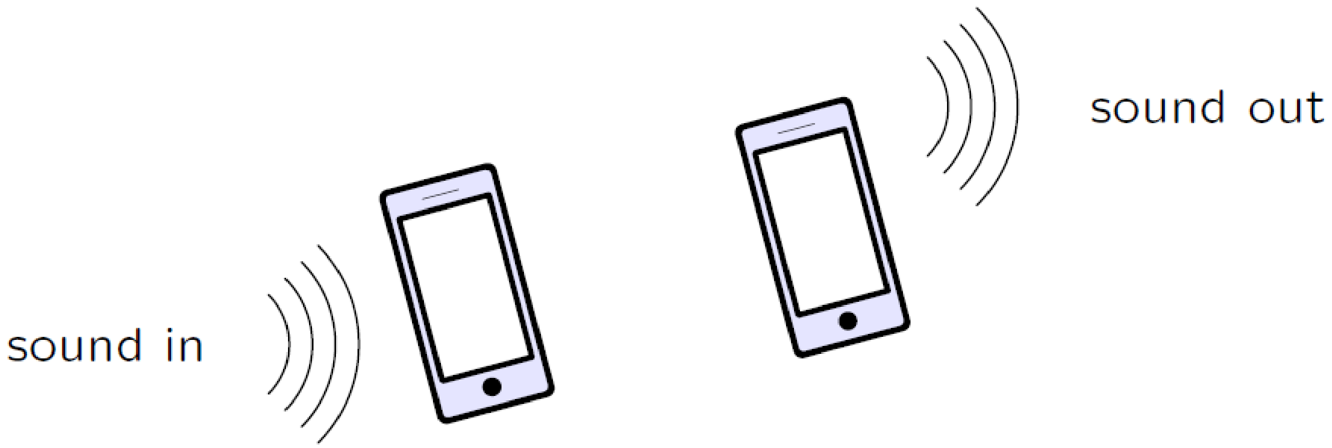
Example: Tanks

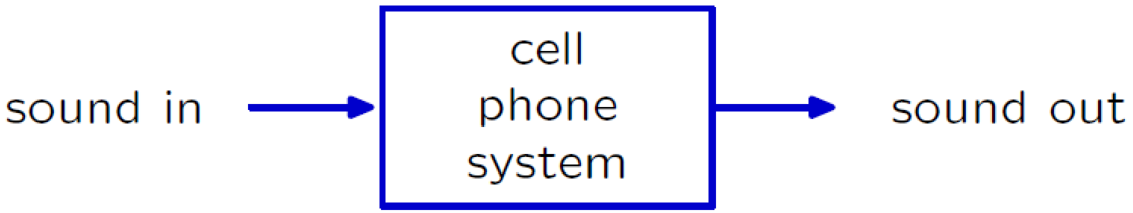


Example: Tanks

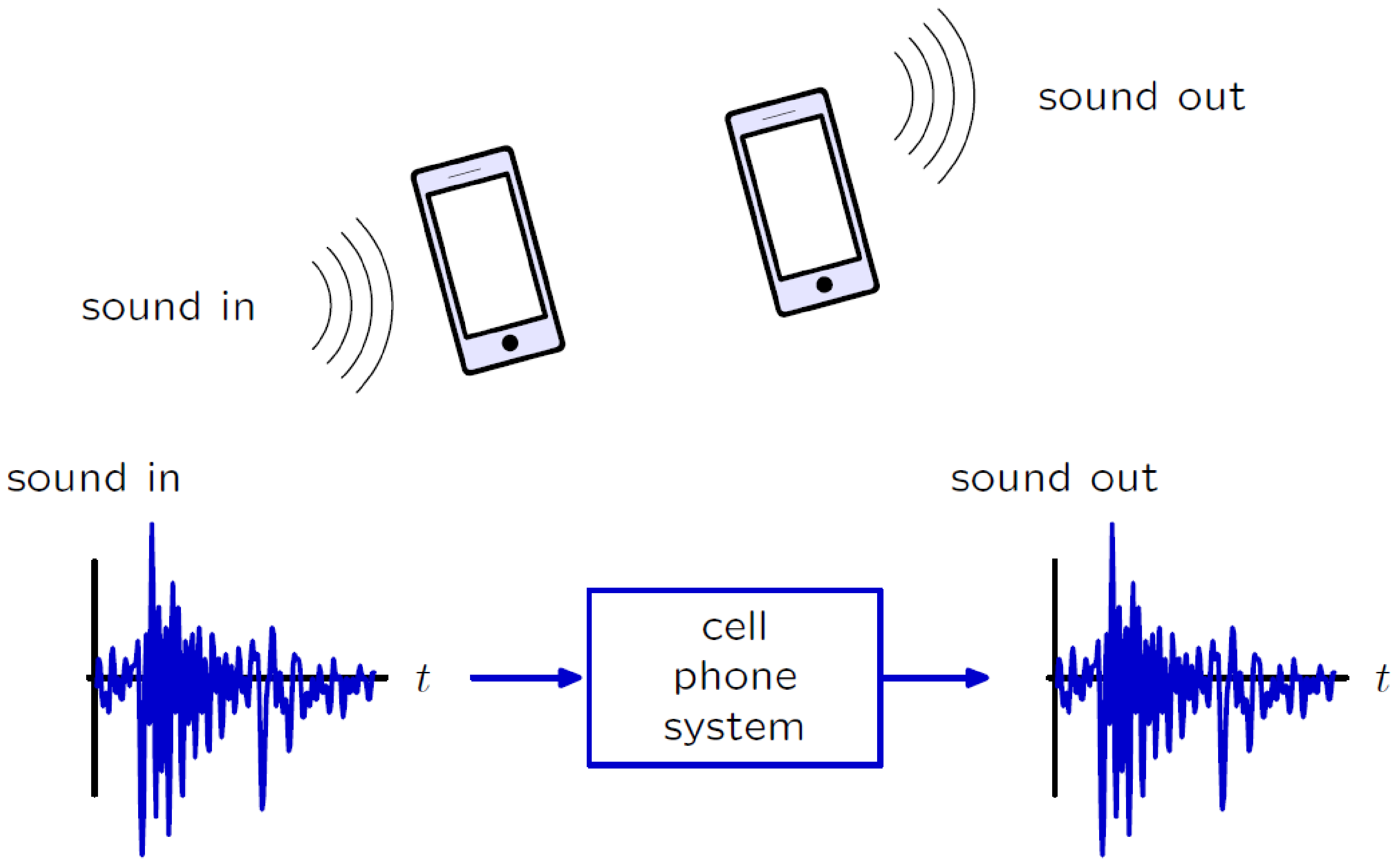


Example: Cell Phone System



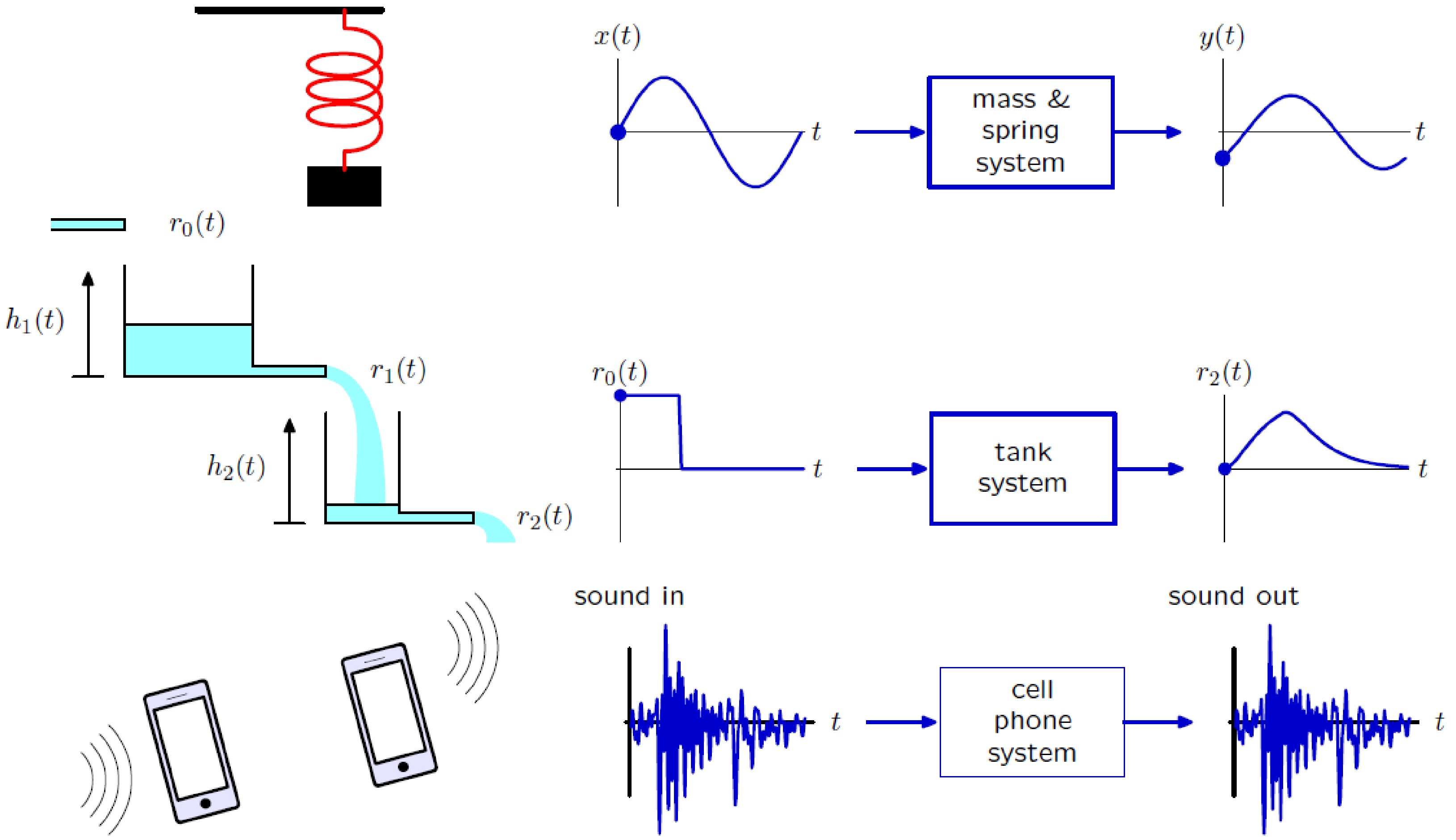


Example: Cell Phone System



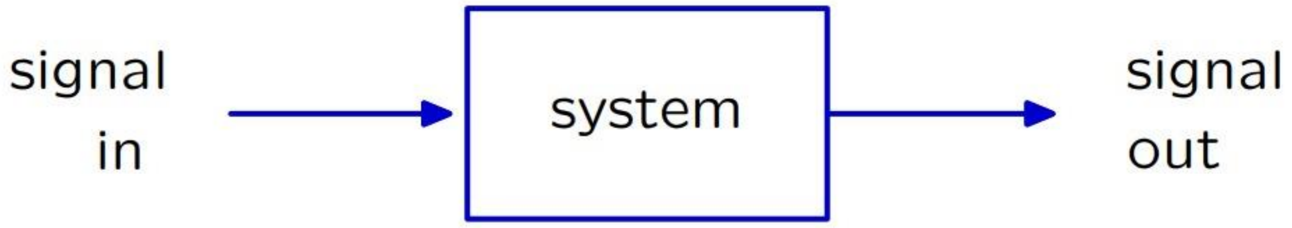
Signals and Systems: Widely Applicable

The Signals and Systems approach has broad applications: electrical, mechanical, optical, acoustic, biological, financial, …



The System Abstraction

Many applications of signal processing can be formulated as systems that convert an input signal into an output signal.



Examples:

- **audio**: equalization, noise reduction, reverberation reduction, echo cancellation, pitch shift (auto-tune)
- **image:** smoothing, edge enhancement, unsharp masking, feature detection
- **video:** image stabilization, motion magnification

Example

Audio: Vocal removal

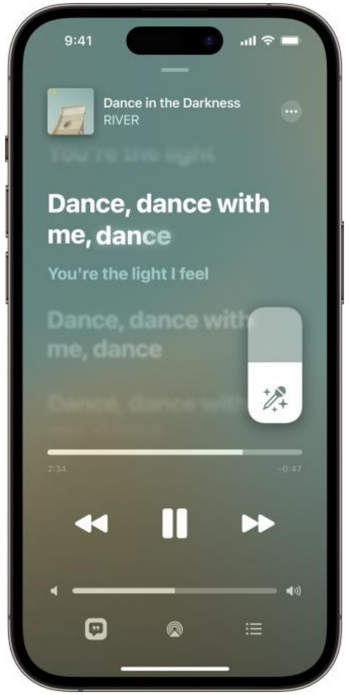
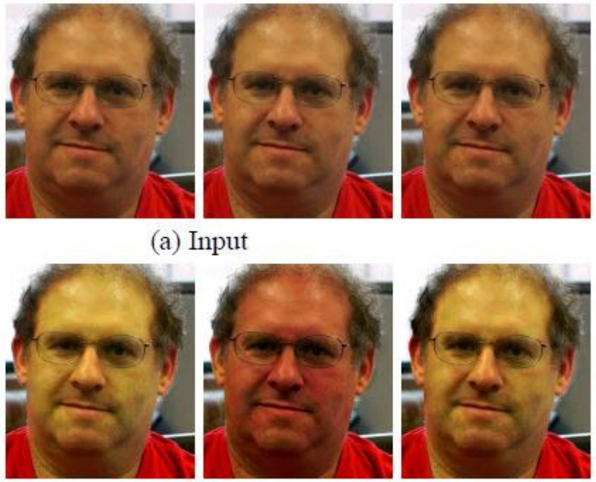


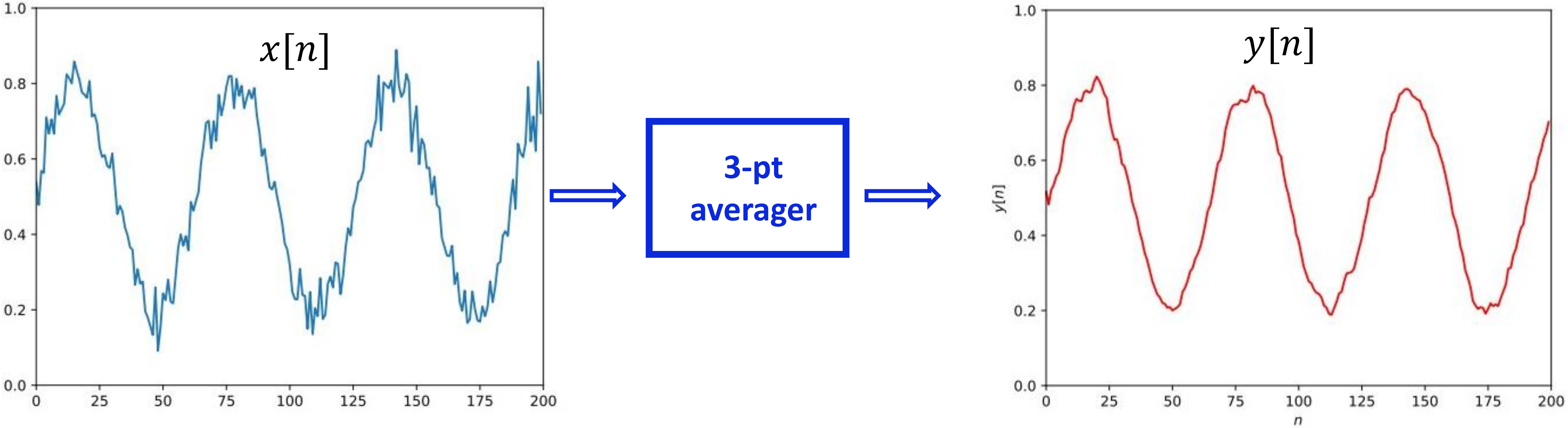
Image: Denoising

Video: Motion magnification


(a) Input
(b) Magnified

Example: Running Average

Noisy sensor data can be "smoothed" to reduce the impact of noise on the signal. For example, consider the following data on the left, consisting of a sinusoid corrupted with noise:



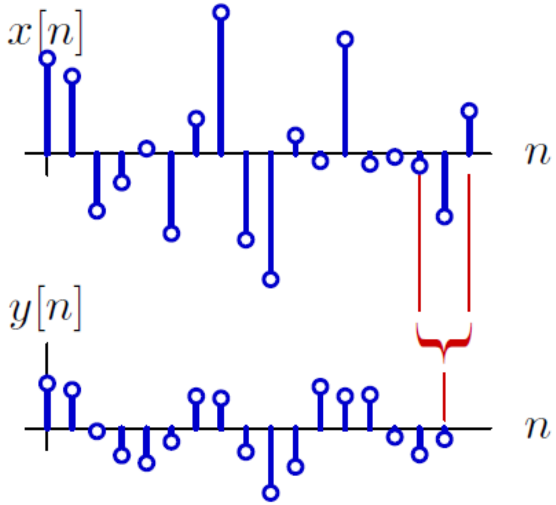
Consider the case where this signal is the input to a system described as "three point averager", whose output at time n is the average of three consecutive input samples:

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$

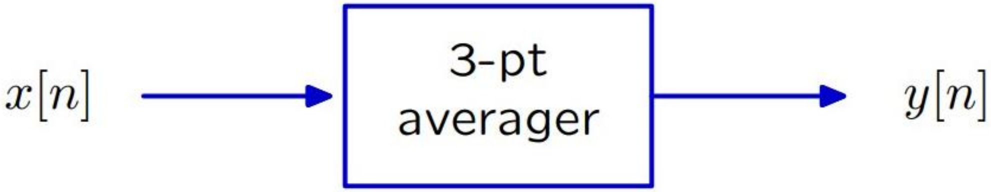
Example System: Three-point Averaging

The output at time *n* is average of inputs at times *n*-1, *n*, and *n*+1.

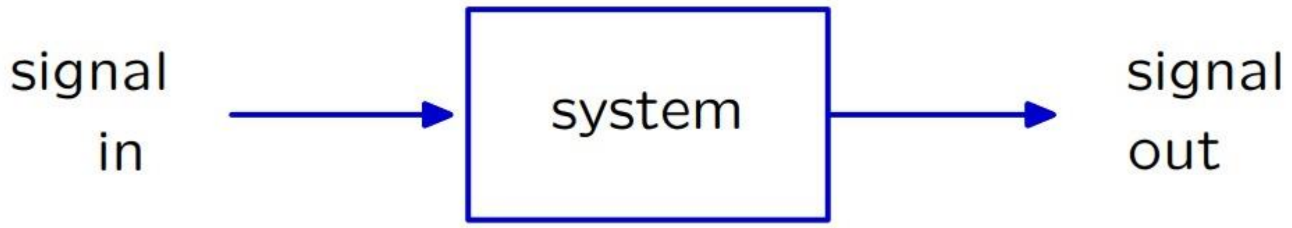
$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Think of this process as a system with input $x[n]$ and output $y[n]$.



Multiple Representation of Systems



We can represent a system in the following three ways:

- Difference Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system by its amplification or attenuation of frequency components

Linear, Time-Invariant(LTI) System

Arbitrary systems are arbitrarily difficult to describe.

Fortunately, many useful systems have two important properties:

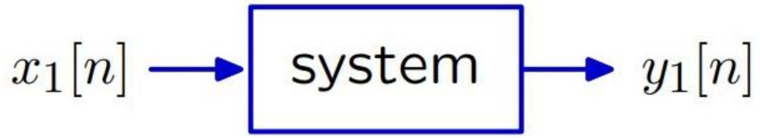
- **Linearity** (additivity and homogeneity)
- **Time invariance**

In 6.300, we will focus on systems that have both of these properties, which are called **LTI systems**.

Additivity

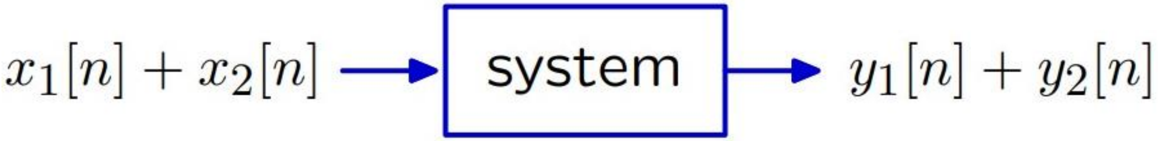
A system is additive if its response to a sum of inputs is equal to the sum of its responses to each input taken one at a time.

Given



And

the system is additive if

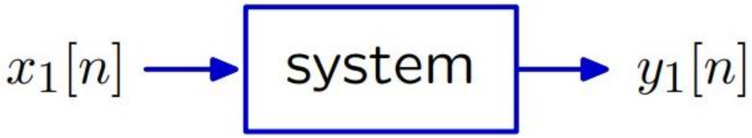


is true for all possible inputs.

Homogeneity

A system is homogeneous if multiplying its input by a constant multiplies its output by the same constant.

Given



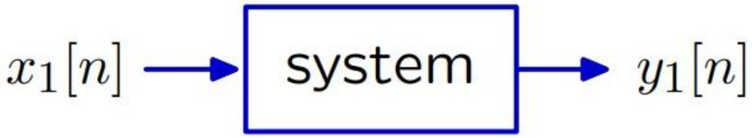
the system is homogeneous if

is true for all α and all possible inputs.

Linearity

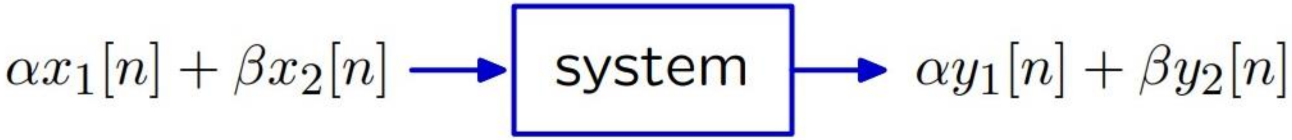
A system is linear if its response to a weighted sum of inputs is equal to the weighted sum (i.e. superposition) of its responses to each of the inputs.

Given



And

the system is linear if

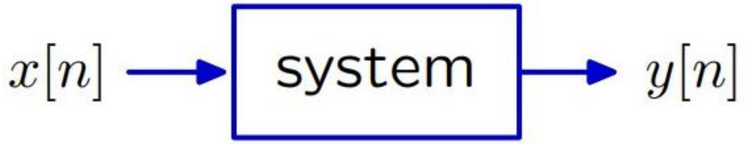


is true for all α and β and all possible inputs.

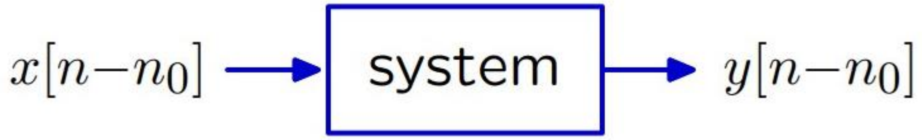
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given



the system is time-invariant if



is true for all n_0 and for all possible inputs.

Check yourself (I)

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1] \qquad \text{(for all n)}$$

is this system linear?

Check yourself (II)

Consider a system represented by the following difference equation:

$$y[n] = x[n] + 1 \qquad \text{(for all n)}$$

is this system linear?

Check yourself (III)

Consider a system represented by the following difference equation:

$$y[n] = x[n] \times x[n-1] \qquad \text{(for all n)}$$

is this system linear?

Participation question for Lecture

Check yourself (IV)

Consider a system represented by the following difference equation:

$$y[n] = nx[n] \qquad \text{(for all n)}$$

is this system linear?

Check yourself (V)

Consider a system represented by the following difference equation:

$$y[n] = nx[n] \qquad \text{(for all n)}$$

is this system time-invariant?

Represent a LTI system with Difference Equations

A system is linear and time-invariant if it can be expressed in terms of a linear difference equation with constant coefficients of the following form:

$$\sum_m C_m\, y[n-m] = \sum_k d_k\, x[n-k]$$

e.g. 3-pt averager:

$$y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$$

Linearity: weighted sum of outputs equal to weighted sum of inputs

$$\sum_m C_m\,(\alpha y_1[n-m] + \beta y_2[n-m]) = \sum_k d_k \cdot \alpha x_1\,[n-k] + \sum_k d_k \cdot \beta x_2\,[n-k]$$

Time invariance: delaying an input delays its output

$$\sum_m C_m\, y[n-n_0-m] = \sum_k d_k\, x[n-n_0-k]$$

The three-point averager is a linear, time-invariant system, but the system $y[n] = x[n] + 1$ is not a LTI system.

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients as follows, then the system is LTI.

$$\sum_l c_l \frac{d^l}{dt^l} y(t) = \sum_m d_m \frac{d^m}{dt^m} x(t)$$

Additivity: output of sum is sum of outputs.

$$\sum_l c_l \frac{d^l}{dt^l}(y_1(t) + y_2(t)) = \sum_m d_m \frac{d^m}{dt^m}(x_1(t) + x_2(t))$$

Homogeneity: scaling an input scales its output.

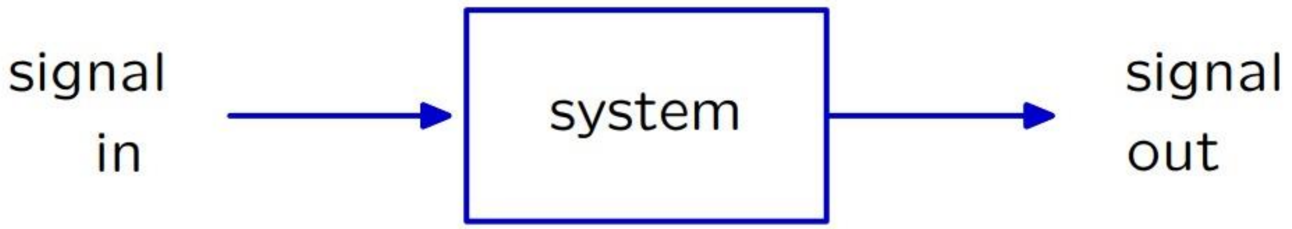
$$\sum_l c_l \frac{d^l}{dt^l}(\alpha y(t)) = \sum_m d_m \frac{d^m}{dt^m}(\alpha x(t))$$

Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l}{dt^l} y(t - t_0) = \sum_m d_m \frac{d^m}{dt^m} x(t - t_0)$$

Multiple Representation of LTI Systems

Next: Representing a system by its unit-sample response

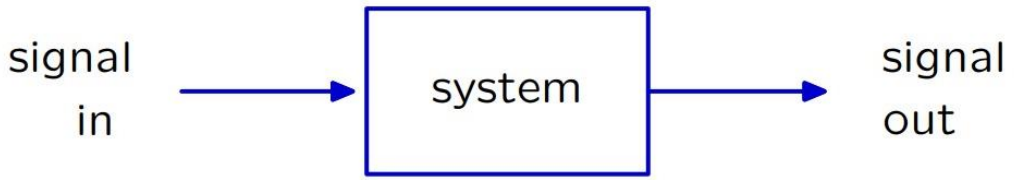


✓ Difference (differential) Equation: represent system by algebraic constraints on samples

- **Convolution: represent a system by its unit-sample response**
- Filter: represent a system as by its amplification or attenuation of frequency components

Summary

The concept of “system” to represent the process/method to manipulate signals:



Linear, Time-Invariant Systems

Three ways of representing a LTI system:

- Difference (differential) Equation: represent system by algebraic constraints on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system as by its frequency response

We will now go to 4-370 for recitation & common hour