6.300 Signal Processing

Week 5, Lecture A:
Quiz Review-Properties of Fourier Series

Lecture slides are available on CATSOOP:

https://sigproc.mit.edu/fall24

Quiz 1: Thursday October 3, 2-4pm 50-340

- Closed book except for one page of notes (8.5'' x 11'' both sides)
- No electronic devices (No headphones, cell phones, calculators, …)
- Coverage up to Week #3 (DTFS) today's lecture and recitation also useful
- practice quiz as a study aid, no HW#4

Fourier Representations

Signals: periodic vs aperiodic
continuous vs discrete

Synthesis Equation: reconstruct signal from Fourier components
Analysis Equation: Finding the Fourier components

	Representing continuous time signal requires frequency contents from $-\infty\ to\ \infty$	Representing discrete time signal $X[k]$ & $X(\Omega)$ periodic
Time domain Periodic, Frequency domain Discrete	**CTFS** $x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$ $X[k] = \frac{1}{T}\int_{T} x(t) e^{-j\frac{2\pi}{T}kt} dt$ $\omega_0 = \frac{2\pi}{T}$	**DTFS** $x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\Omega_0 kn}$ $\Omega_0 = \frac{2\pi}{N}$ $X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$
Time domain Aperiodic, Frequency domain Continuous	**CTFT** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega$ $X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$	**DTFT** $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$ $X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$

CT signals, DT signals, sampling

A CT signal $x(t) = \cos(\omega t)$ sampled at $t = n\Delta T$, the resulting DT signal $x[n] = \cos(\Omega n)$ with $\Omega = \omega \Delta T$

$$x(t) = \cos(\omega t) \xrightarrow[f_s = \frac{1}{\Delta T}]{\Omega = \omega / f_s} x[n] = \cos(\Omega n)$$

Aliasing and Nyquist frequency:

$$x[n] = \cos(\Omega n) = \cos((\Omega + 2\pi)n) = \cos((\Omega + 2k\pi)n)$$

Nyquist frequency: $\frac{1}{2} f_s$

- when the highest frequency of a signal is less than the Nyquist frequency, the resulting DT signal is free of aliasing.
- Or, the sampling rate need to be larger than twice the highest frequency in the signal to prevent aliasing

Properties (I): Linearity

- Consider $y(t) = Ax_1(t) + Bx_2(t)$, where $x_1(t)$ and $x_2(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?

First, $y(t)$ must also be periodic in T

$$Y[k] = \frac{1}{T}\int_T y(t)e^{-j\frac{2\pi kt}{T}}\,dt \;=\; \frac{1}{T}\int_T (Ax_1(t) + Bx_2(t))e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= A\frac{1}{T}\int_T x_1(t)e^{-j\frac{2\pi kt}{T}}\,dt + B\frac{1}{T}\int_T x_2(t)e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= AX_1[k] + BX_2[k]$$

If $y(t) = Ax_1(t) + Bx_2(t)$, then $Y[k] = AX_1[k] + BX_2[k]$

Properties (II): Time flip(reversal)

- Consider $y(t) = x(-t)$, where $x(t)$ is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y(t)$ must also be periodic in T

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi k(-t)}{T}} = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi(-k)t}{T}}$$

Let $m = -k$

$$y(t) = x(-t) = \sum_{m=\infty}^{-\infty} X[-m]e^{j\frac{2\pi mt}{T}} = \sum_{m=-\infty}^{\infty} X[-m]e^{j\frac{2\pi mt}{T}}$$

Since we know

$$y(t) = \sum_{m=-\infty}^{\infty} Y[m]e^{j\frac{2\pi mt}{T}} \Longrightarrow Y[k] = X[-k]$$

If $y(t) = x(-t), Y[k] = X[-k]$

Properties (III): Real-valued periodic signal

If $f(t)$ is real valued periodic signal:

$$F[k] = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi kt}{T}}dt$$

$$F[-k] = \frac{1}{T}\int_T f(t)e^{j\frac{2\pi kt}{T}}dt$$

$$F^*[-k] = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi kt}{T}}dt$$

$$= F[k]$$

If $f(t)$ is real valued periodic signal, $F[k] = F^*[-k]$

How to go from trig form to CE form for CTFS

Substitute complex exponentials for trigonometric functions.

Let's try it!

$$f(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\right)$$

$$= c_0 + \sum_{k=1}^{\infty}\left(c_k\underbrace{\frac{1}{2}(e^{jk\omega_o t}+e^{-jk\omega_o t})}_{\cos(k\omega_o t)} + d_k\underbrace{\frac{1}{2j}(e^{jk\omega_o t}-e^{-jk\omega_o t})}_{\sin(k\omega_o t)}\right)$$

$$= c_o + \sum_{k=1}^{\infty}\frac{c_k - jd_k}{2}e^{jk\omega_o t} + \sum_{k=1}^{\infty}\frac{c_k + jd_k}{2}e^{-jk\omega_o t}$$

$$= c_o + \sum_{k=1}^{\infty}\frac{c_k - jd_k}{2}e^{jk\omega_o t} + \sum_{k=-1}^{-\infty}\frac{c_{-k} + jd_{-k}}{2}e^{+jk\omega_o t}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \quad \text{where} \quad a_k = \begin{cases} \frac{1}{2}(c_k - jd_k) & \text{if } k > 0 \\ c_0 & \text{if } k = 0 \\ \frac{1}{2}(c_{-k} + jd_{-k}) & \text{if } k < 0 \end{cases}$$

$$e^{j\theta} = cos\theta + jsin\theta$$

$$e^{-j\theta} = cos\theta - jsin\theta$$

$$cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = -j\frac{e^{j\theta} - e^{-j\theta}}{2}$$

The trig form of the Fourier series (top of page) has an equivalent form with complex exponentials (red).

Properties (IV): Symmetric and Antisymmetric Parts

- If $f(t) = f_S(t) + f_A(t)$ is a real valued signal and periodic in time with fundamental period T, what are the Fourier coefficients of $f_S(\cdot)$ and $f_A(\cdot)$, in terms of F[k]?

If $f(t)$ is real valued periodic signal, $F[k] = F^*[-k]$

$$f_S(t) = \frac{f(t) + f(-t)}{2} \xrightarrow[\text{time flip}]{\text{Linearity}} F_S[k] = \frac{F[k] + F[-k]}{2} = \frac{F[k] + F^*[k]}{2} = \frac{2Re(F[k])}{2} = Re(F[k])$$

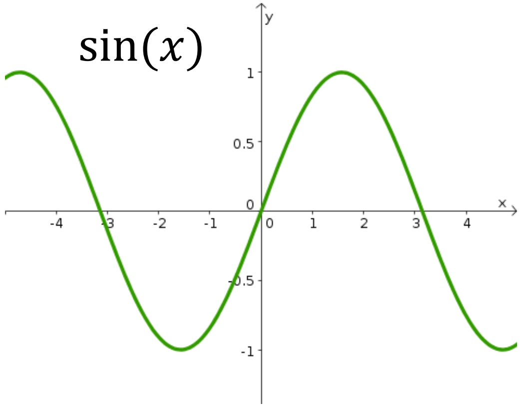
$$f_A(t) = \frac{f(t) - f(-t)}{2} \xrightarrow[\text{time flip}]{\text{Linearity}} F_A[k] = \frac{F[k] - F[-k]}{2} = \frac{F[k] - F^*[k]}{2} = \frac{2j \cdot Im(F[k])}{2} = j \cdot Im(F[k])$$

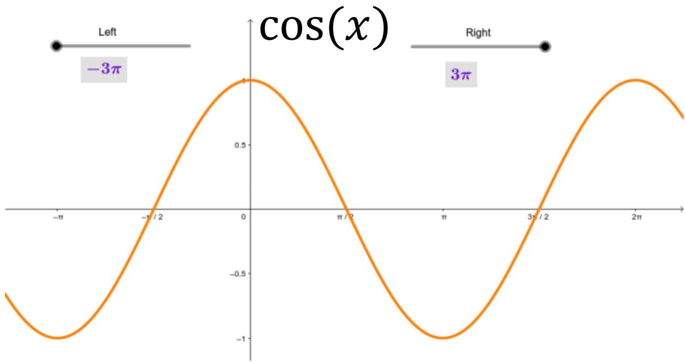
The real part of F[k] comes from the symmetric part of the signal,
the imaginary part of F[k] comes from the antisymmetric part of the signal

Symmetric and Antisymmetric Parts in CTFS

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

$$f(-t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) - d_k \sin(k\omega_0 t))$$





- c_k's (cosines) alone only represent the symmetric part of the signal.
- d_k 's (sines) alone only represent the antisymmetric part of the signal.

$$f_S(t) = \frac{f(t) + f(-t)}{2} \qquad f_A(t) = \frac{f(t) - f(-t)}{2}$$

The symmetric part shows up in the c_k coefficients, and the antisymmetric part shows up in the d_k coefficients.

Properties (V): Time Shift

- Consider $y(t) = x(t - t_0)$, where x is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

$$Y[k] = \frac{1}{T}\int_T y(t)e^{-j\frac{2\pi kt}{T}}\,dt = \frac{1}{T}\int_T x(t-t_0)e^{-j\frac{2\pi kt}{T}}\,dt$$

$let\ \ u = t - t_0,$
then $t = u + t_0,$
$\mathrm{d}t = du$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi k(u+t_0)}{T}}\,du$$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}e^{-j\frac{2\pi kt_0}{T}}\,du$$

$$= e^{-j\frac{2\pi kt_0}{T}}\frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}\,du = e^{-j\frac{2\pi kt_0}{T}}X[k]$$

Each coefficient $Y[k]$ in the series for $y(t)$ is a constant $e^{-jk\omega_0\tau}$ times the corresponding coefficient $X[k]$ in the series for $x(t)$.

Properties (VI): Time Derivative

- Consider $y(t) = \frac{d}{dt}x(t)$, where $x(t)$ and $y(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Start with the synthesis equation: $$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}$$

Then, from the definition of y(·), we have:

$$y(t) = \dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi kt}{T}}\right) = \sum_{k=-\infty}^{\infty}\left(j\frac{2\pi k}{T}X[k]\right)e^{j\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} Y[k]e^{j\frac{2\pi kt}{T}}$$

From this form, we can see that $Y[k] = j\frac{2\pi k}{T}X[k]$.

Properties of Fourier Transforms

Continuous-Time Fourier Transform

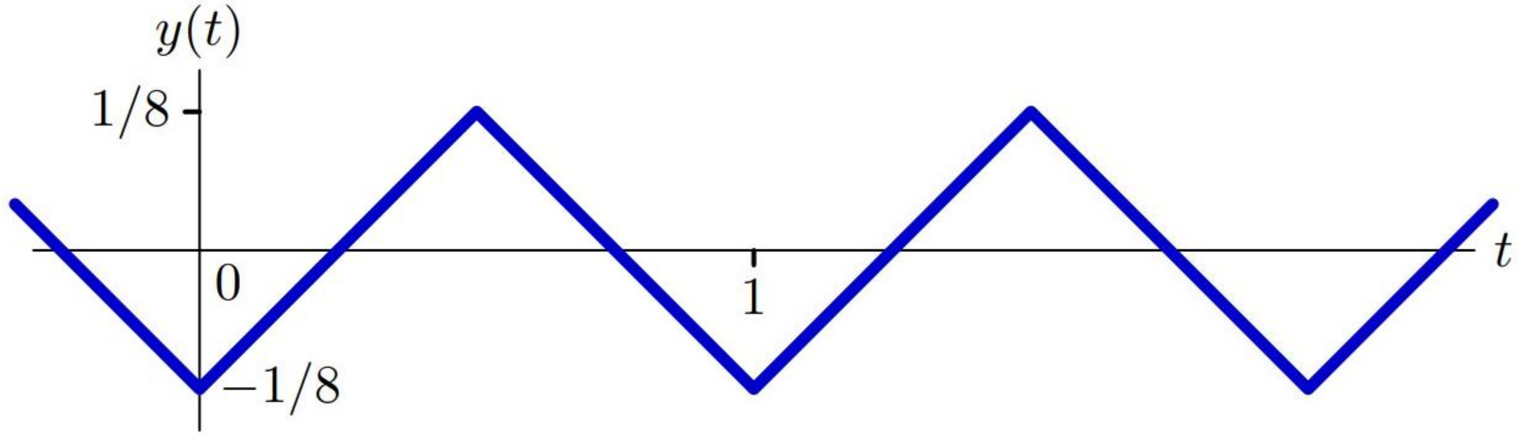
Property	$y(t)$	$Y(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time delay	$x(t - t_0)$	$e^{-j\omega t_0} X(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Scaling time	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Time derivative	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency derivative	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$

Discrete-Time Fourier Transform

Property	$y[n]$	$Y(\Omega)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\Omega) + bX_2(\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time delay	$x[n - n_0]$	$e^{-j\Omega n_0} X(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Frequency derivative	$nx[n]$	$j\frac{d}{d\Omega}X(\Omega)$

Exercise I

Let $Y[k]$ represent the Fourier series coefficients of the following signal:



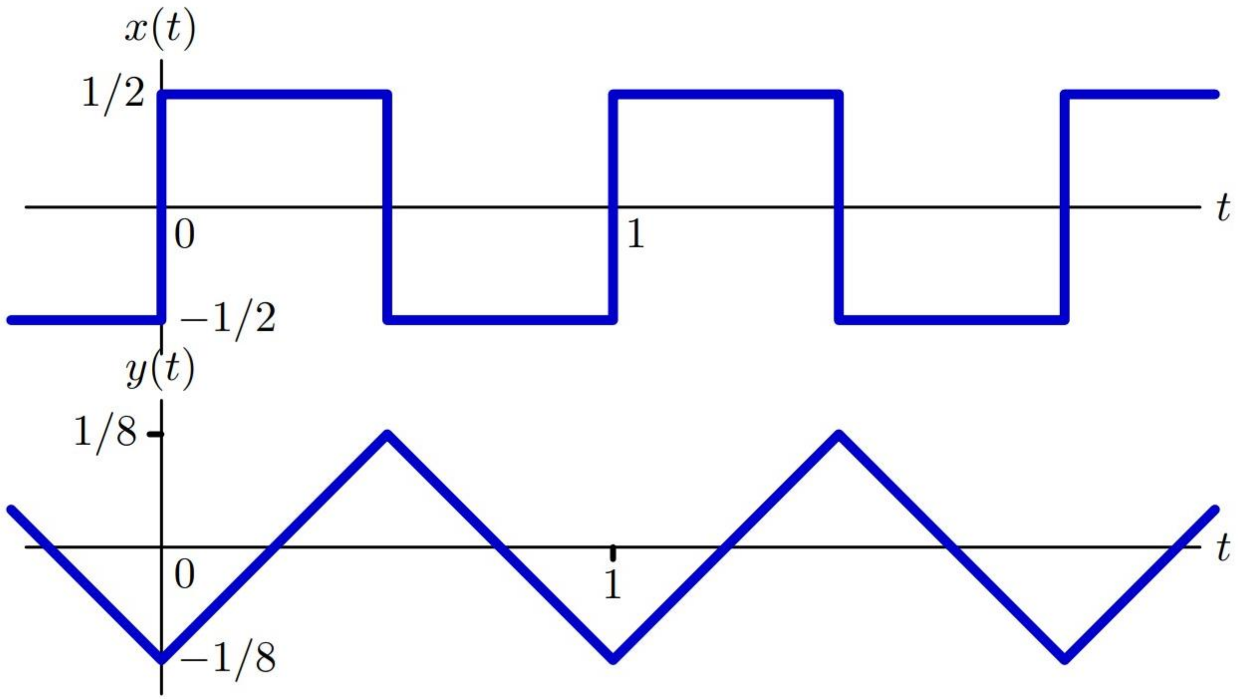
Which of the following is/are true?

Participation question for Lecture

1. $Y[k] = 0$ if k is even
2. $Y[k]$ is real-valued
3. $|Y[k]|$ decreases with k^2
4. there are an infinite number of non-zero $Y[k]$

Exercise I

What is the relationship between the two following signals?



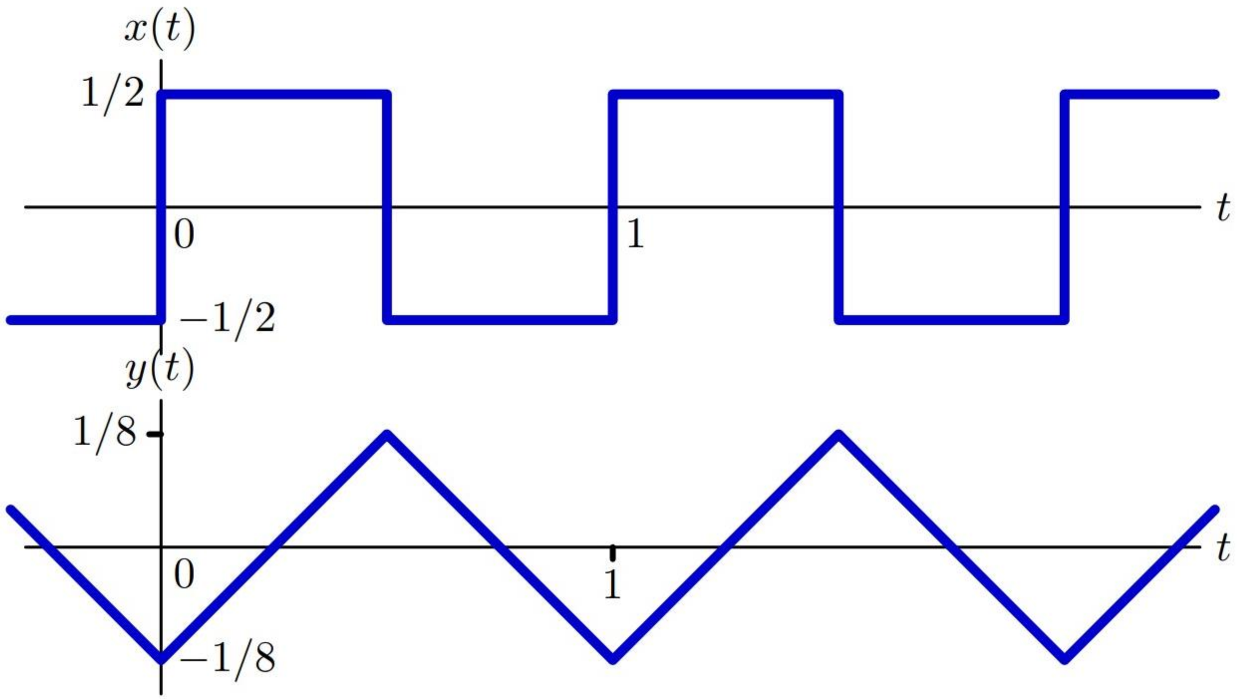
$$X[k] = \begin{cases} 0, & k \text{ is even} \\ \dfrac{1}{j\pi k}, & k \text{ is odd} \end{cases}$$

$T = 1$

$$X[k] = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}dt = \int_0^{1/2} \frac{1}{2} e^{-j\frac{2\pi kt}{T}}dt + \int_{1/2}^{1} \left(-\frac{1}{2}\right) e^{-j\frac{2\pi kt}{T}}dt = \begin{cases} 0 & k \text{ is even} \\ \dfrac{1}{jk\pi} & k \text{ is odd} \end{cases}$$

Exercise I

The triangle waveform is the integral of the square wave.



$$X[k] = \begin{cases} 0, & k \text{ is even} \\ \frac{1}{j\pi k}, & k \text{ is odd} \end{cases}$$

$$x(t) = \frac{d}{dt} y(t)$$

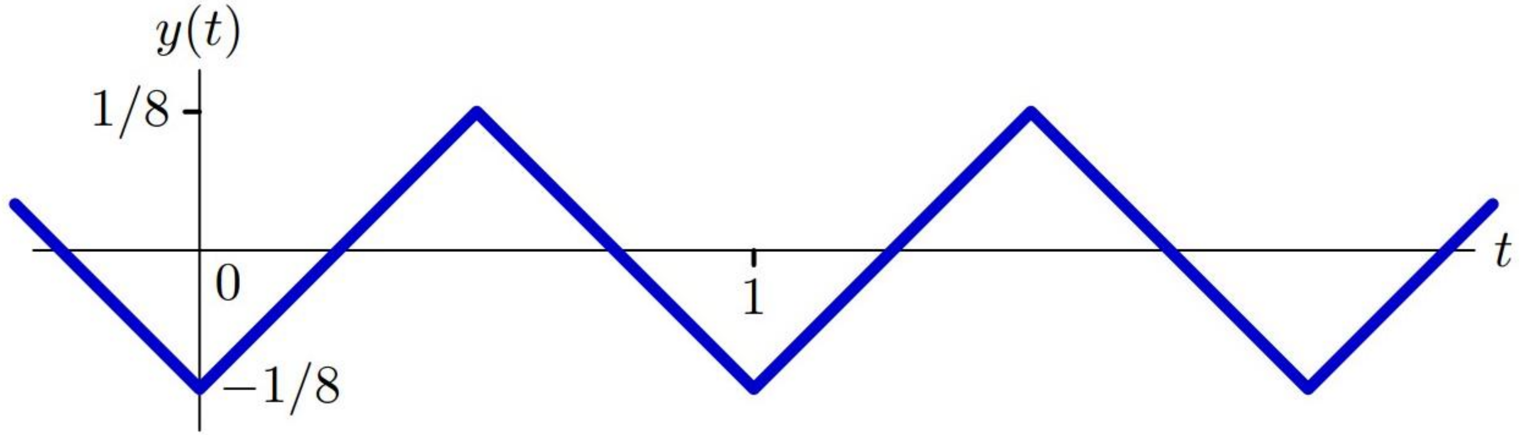
$$X[k] = j\frac{2\pi k}{T} Y[k], \qquad T = 1$$

$$Y[k] = \frac{1}{j2\pi k} X[k] \;= \begin{cases} 0, & k \text{ is even} \\ -\frac{1}{2\pi^2 k^2}, & k \text{ is odd} \end{cases}$$

$y(t)$ is symmetric around t=0, thus its Fourier Series coefficients are purely real

Exercise I

Let $Y[k]$ represent the Fourier series coefficients of the following signal:



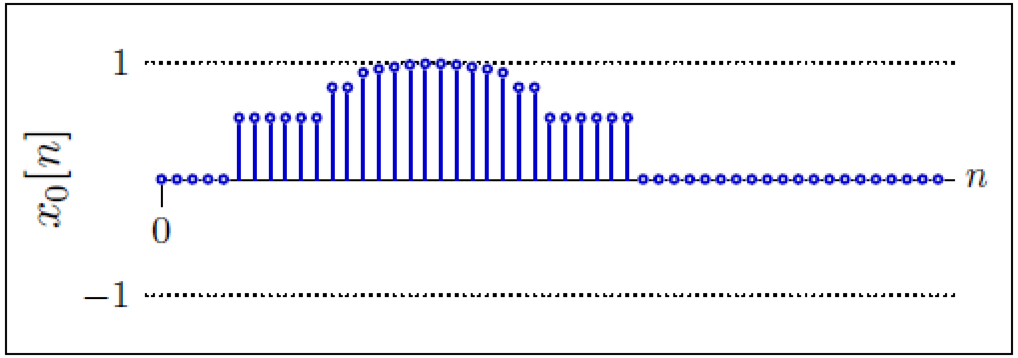
Which of the following is/are true?

1. $Y[k] = 0$ if k is even
2. $Y[k]$ is real-valued
3. $|Y[k]|$ decreases with k^2
4. there are an infinite number of non-zero $Y[k]$

$$Y[k] = \begin{cases} 0, & k \text{ is even} \\ -\dfrac{1}{2\pi^2 k^2}, & k \text{ is odd} \end{cases}$$

Exercise II

Ben Bitdiddle created a signal $x_0[n]$ representing the MIT dome, but he only saved the DTFS coefficients $X_0[k]$ (and not the original signal). However, he knew that one period of the original signal (which is periodic in $N = 51$) looked like this:



a) $X_A[k] = \text{Re}\left(X_0[k]\right)$

b) $X_B[k] = \text{Im}\left(X_0[k]\right)$

c) $X_C[k] = j\text{Im}\left(X_0[k]\right)$

d) $X_D[k] = \begin{cases} 0 & \text{if } k = 0 \\ X_0[k] & \text{otherwise} \end{cases}$

e) $X_E[k] = \begin{cases} 0 & \text{if } k = 25 \\ X_0[k] & \text{otherwise} \end{cases}$

f) $X_F[k] = X_0[k] + 1/51$

g) $X_G[k] = e^{j\pi} X_0[k]$

h) $X_H[k] = \begin{cases} X_0[0] & \text{if } k = 0 \\ e^{j\pi} X_0[k] & \text{otherwise} \end{cases}$

i) $X_I[k] = |X_0[k]| e^{j(-\angle X_0[k])}$

a) $x_A[n]$: symmetric part of $x_0[n]$, $x_A[n] = \frac{x_0[n]+x_0[-n]}{2}$

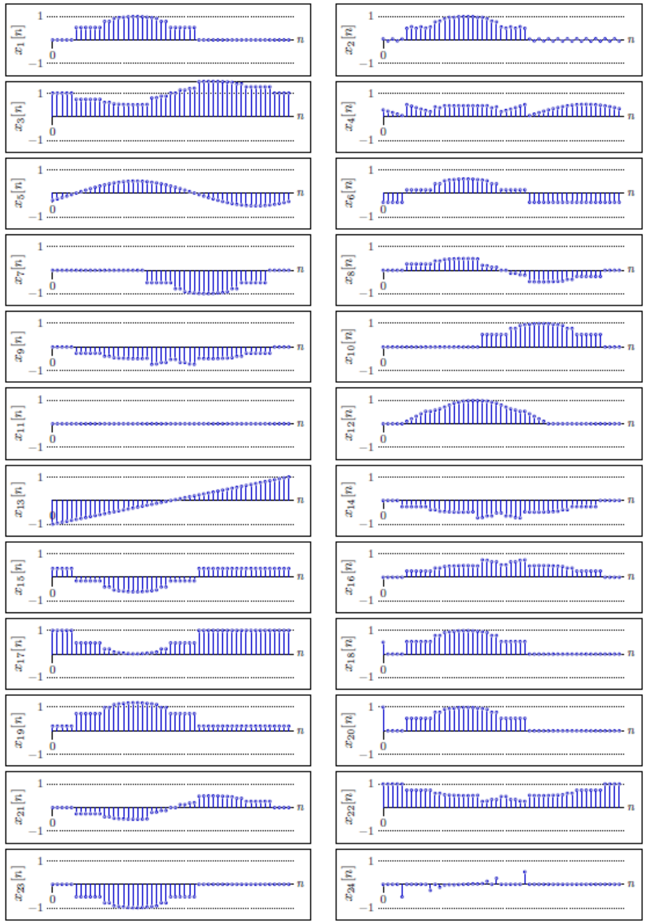
c) $x_C[n]$: antisymmetric part of $x_0[n]$, $x_C[n] = \frac{x_0[n]-x_0[-n]}{2}$

d) $x_D[n]$: DC part becomes zero

f) $x_F[n]$: constant in freq domain => δ[n] in time domain

g) $x_F[n]$: $e^{j\pi} = -1$

i) $X_I[k] = X_0^*[k] = X_0\ [-k]$



Additional slides to show the properties with DTFS

Properties of DTFS: Linearity

- Consider $y[n] = Ax_1[n] + Bx_2[n]$, where $x_1[n]$ and $x_2[n]$ are periodic in N. What are the DTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} (Ax_1[n] + Bx_2[n])e^{-j\frac{2\pi}{N}kn}$$

$$= A\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x_1[n]e^{-j\frac{2\pi}{N}kn} + B\frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x_2[n]e^{-j\frac{2\pi}{N}kn}$$

$$= AX_1[k] + BX_2[k]$$

If $y[n] = Ax_1[n] + Bx_2[n], then\ Y[k] = AX_1[k] + BX_2[k]$

Properties of DTFS: Time flip

- Consider $y[n] = x[-n]$, where $x[n]$ is periodic in N. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[-n]e^{-j\frac{2\pi k}{N}n}$$

Let $m = -n$

$$Y[k] = \frac{1}{N}\sum_{m=-n_0}^{-(n_0+N-1)} x[m]e^{-j\frac{2\pi k}{N}(-m)}$$

$$= \frac{1}{N}\sum_{m=-n_0}^{-n_0-N+1} x[m]e^{-j\frac{2\pi(-k)}{N}m} \quad = X[-k]$$

If $y[n] = x[-n]$, $then\ Y[k] = X[-k]$

Flipping in time flips in frequency.

Properties of DTFS: Time Shift

- Consider $y[n] = x[n-m]$, where $x[n]$ is periodic in N, m is an integer. What are the DTFS coefficients $Y[k]$, in terms of $X[k]$?

First, $y[n]$ must also be periodic in N

$$Y[k] = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} y[n]e^{-j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n-m]e^{-j\frac{2\pi k}{N}n}$$

Let $l = n - m, then\ \ n = l + m$

$$Y[k] = \frac{1}{N}\sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi k}{N}(l+m)} = e^{-j\frac{2\pi k}{N}m}\cdot\frac{1}{N}\cdot\sum_{l=n_0-m}^{n_0-m+N-1} x[l]e^{-j\frac{2\pi k}{N}l}$$

$$= e^{-j\frac{2\pi k}{N}m}\cdot X[k]$$

If $y[n] = x[n-m],\ \ then\ Y[k] = e^{-j\frac{2\pi km}{N}}X[k]$

Shifting in time changes phase of Fourier Series Coefficient.

Properties of DTFS: Complex-conjugate Coefficients

If x[n] is real-valued periodic signal, $X[k] = X^*[-k]$.

$$X[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi k}{N} n}$$

$$X[-k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi (-k)}{N} n}$$

$$X[-k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{j\frac{2\pi k}{N} n}$$

$$X^*[-k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j\frac{2\pi k}{N} n} = X[k]$$

Properties of DTFS: Symmetric and Antisymmetric Parts

- A real-valued signal $x[n]$ written in terms of the symmetric and antisymmetric parts: $x[n] = x_S[n] + x_A[n]$

$$x_S[n] = \frac{1}{2}(x[n] + x[-n]) \overset{\text{DTFS}}{\Longleftrightarrow} \frac{1}{2}(X[k] + X[-k]) = \frac{1}{2}(X[k] + X^*[k])$$

$$= Re(X[k])$$

$$x_A[n] = \frac{1}{2}(x[n] - x[-n]) \overset{\text{DTFS}}{\Longleftrightarrow} \frac{1}{2}(X[k] - X[-k]) = \frac{1}{2}(X[k] - X^*[k])$$

$$= j \cdot Im(X[k])$$

The real part of X[k] comes from the symmetric part of the signal,
the imaginary part of X[k] comes from the antisymmetric part of the signal