6.300 Signal Processing

Week 5, Lecture A:
Quiz Review-Properties of Fourier Series

Lecture slides are available on CATSOOP:

https://sigproc.mit.edu/fall24

Quiz 1: Thursday October 3, 2-4pm 50-340

- Closed book except for one page of notes (8.5’’ x 11’’ both sides)
- No electronic devices (No headphones, cell phones, calculators, …)
- Coverage up to Week #3 (DTFS) today’s lecture and recitation also useful
- practice quiz as a study aid, no HW#4

Fourier Representations

Signals: periodic vs aperiodic
continuous vs discrete

Synthesis Equation: reconstruct signal from Fourier components
Analysis Equation: Finding the Fourier components

	Representing continuous time signal requires frequency contents from $-\infty$ *to* ∞	Representing discrete time signal $X[k]$ & $X(\Omega)$ periodic
Time domain Periodic, Frequency domain Discrete	**CTFS** $x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi kt}{T}}$ $X[k] = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$ $\omega_0 = \frac{2\pi}{T}$	**DTFS** $x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\Omega_0 kn}$ $\Omega_0 = \frac{2\pi}{N}$ $X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$
Time domain Aperiodic, Frequency domain Continuous	**CTFT** $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$ $X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$	**DTFT** $x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega) \cdot e^{j\Omega n}\, d\Omega$ $X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$

CT signals, DT signals, sampling

A CT signal $x(t) = \cos(\omega t)$ sampled at $t = n\Delta T$, the resulting DT signal $x[n] = \cos(\Omega n)$ with $\Omega = \omega \Delta T$

$$x(t) = \cos(\omega t) \xrightarrow[f_s = \frac{1}{\Delta T}]{\Omega = \omega / f_s} x[n] = \cos(\Omega n)$$

Aliasing and Nyquist frequency:

$$x[n] = \cos(\Omega n) = \cos((\Omega + 2\pi)n) = \cos((\Omega + 2k\pi)n)$$

Nyquist frequency: $\frac{1}{2} f_s$

- when the highest frequency of a signal is less than the Nyquist frequency, the resulting DT signal is free of aliasing.
- Or, the sampling rate need to be larger than twice the highest frequency in the signal to prevent aliasing

Properties (I): Linearity

- Consider $y(t) = Ax_1(t) + Bx_2(t),$ where $x_1(t)$ and $x_2(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X_1[k]$ and $X_2[k]$?

Properties (II): Time flip(reversal)

- Consider $y(t) = x(-t)$, where $x(t)$ is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Properties (III): Real-valued periodic signal

If $f(t)$ is real valued periodic signal:

How to go from trig form to CE form for CTFS

Substitute complex exponentials for trigonometric functions.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \right)$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k \underbrace{\frac{1}{2}(e^{jk\omega_o t} + e^{-jk\omega_o t})}_{\cos(k\omega_o t)} + d_k \underbrace{\frac{1}{2j}(e^{jk\omega_o t} - e^{-jk\omega_o t})}_{\sin(k\omega_o t)} \right)$$

$$= c_o + \sum_{k=1}^{\infty} \frac{c_k - jd_k}{2} e^{jk\omega_o t} + \sum_{k=1}^{\infty} \frac{c_k + jd_k}{2} e^{-jk\omega_o t}$$

$$= c_o + \sum_{k=1}^{\infty} \frac{c_k - jd_k}{2} e^{jk\omega_o t} + \sum_{k=-1}^{-\infty} \frac{c_{-k} + jd_{-k}}{2} e^{+jk\omega_o t}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \quad \text{where} \quad a_k = \begin{cases} \frac{1}{2}(c_k - jd_k) & \text{if } k > 0 \\ c_0 & \text{if } k = 0 \\ \frac{1}{2}(c_{-k} + jd_{-k}) & \text{if } k < 0 \end{cases}$$

The trig form of the Fourier series (top of page) has an equivalent form with complex exponentials (red).

Let's try it!

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = -j\frac{e^{j\theta} - e^{-j\theta}}{2}$$

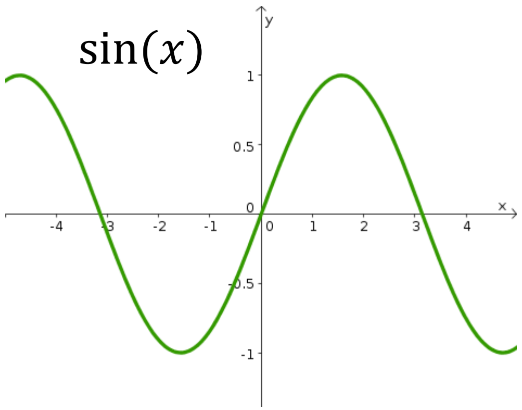
Properties (IV): Symmetric and Antisymmetric Parts

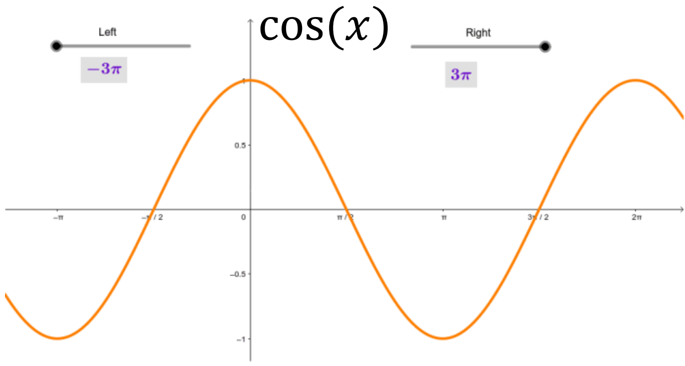
- If $f(t) = f_S(t) + f_A(t)$ is a real valued signal and periodic in time with fundamental period T, what are the Fourier coefficients of $f_S(\cdot)$ and $f_A(\cdot)$, in terms of $\mathrm{F[k]}$?

Symmetric and Antisymmetric Parts in CTFS

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

$$f(-t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) - d_k \sin(k\omega_0 t))$$





- c_k's (cosines) alone only represent the symmetric part of the signal.
- d_k 's (sines) alone only represent the antisymmetric part of the signal.

$$f_S(t) = \frac{f(t) + f(-t)}{2} \qquad f_A(t) = \frac{f(t) - f(-t)}{2}$$

The symmetric part shows up in the c_k coefficients, and the antisymmetric part shows up in the d_k coefficients.

Properties (V): Time Shift

- Consider $y(t) = x(t - t_0)$, where x is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

$$Y[k] = \frac{1}{T}\int_T y(t)e^{-j\frac{2\pi kt}{T}}\,dt \; = \frac{1}{T}\int_T x(t-t_0)e^{-j\frac{2\pi kt}{T}}\,dt$$

let $u = t - t_0$,
then $t = u + t_0$,
$dt = du$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi k(u+t_0)}{T}}\,du$$

$$= \frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}e^{-j\frac{2\pi kt_0}{T}}\,du$$

$$= e^{-j\frac{2\pi kt_0}{T}}\frac{1}{T}\int_T x(u)e^{-j\frac{2\pi ku}{T}}\,du = e^{-j\frac{2\pi kt_0}{T}}X[k]$$

Each coefficient $Y[k]$ in the series for $y(t)$ is a constant $e^{-jk\omega_0\tau}$ times the corresponding coefficient $X[k]$ in the series for $x(t)$.

Properties (VI): Time Derivative

- Consider $y(t) = \frac{d}{dt}x(t)$, where $x(t)$ and $y(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Properties of Fourier Transforms

Continuous-Time Fourier Transform

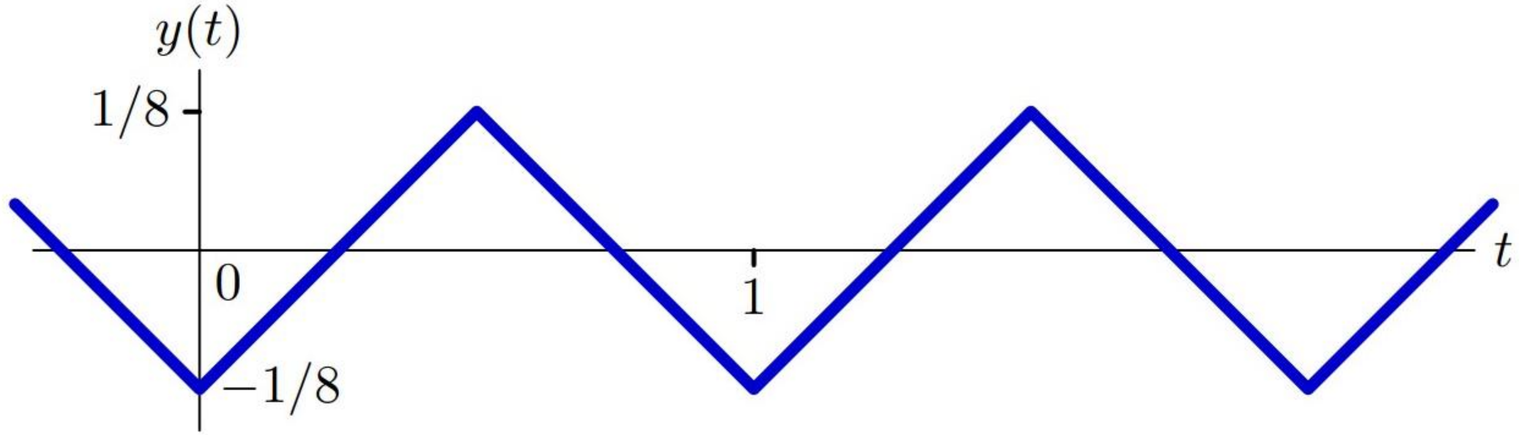
Property	$y(t)$	$Y(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time reversal	$x(-t)$	$X(-\omega)$
Time delay	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Scaling time	$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{\omega}{a}\right)$
Time derivative	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Frequency derivative	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$

Discrete-Time Fourier Transform

Property	$y[n]$	$Y(\Omega)$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\Omega) + bX_2(\Omega)$
Time reversal	$x[-n]$	$X(-\Omega)$
Time delay	$x[n-n_0]$	$e^{-j\Omega n_0}X(\Omega)$
Conjugation	$x^*[n]$	$X^*(-\Omega)$
Frequency derivative	$nx[n]$	$j\frac{d}{d\Omega}X(\Omega)$

Exercise I

Let $Y[k]$ represent the Fourier series coefficients of the following signal:



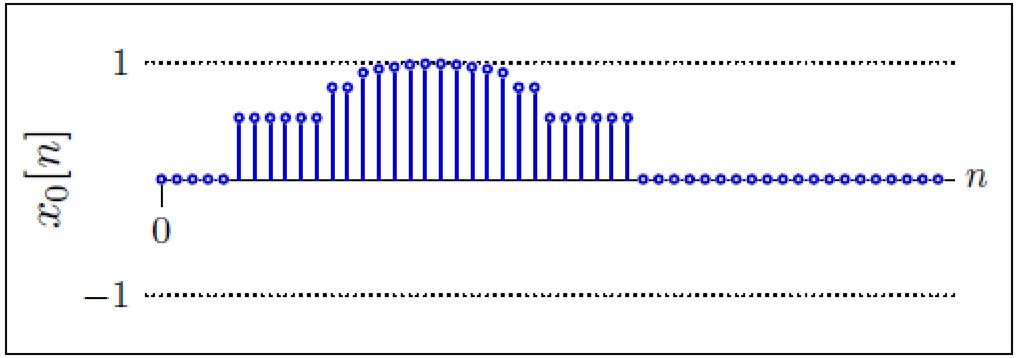
Participation question for Lecture

Which of the following is/are true?

1. $Y[k] = 0$ if k is even
2. $Y[k]$ is real-valued
3. $|Y[k]|$ decreases with k^2
4. there are an infinite number of non-zero $Y[k]$

Exercise II

Ben Bitdiddle created a signal $x_0[n]$ representing the MIT dome, but he only saved the DTFS coefficients $X_0[k]$ (and not the original signal). However, he knew that one period of the original signal (which is periodic in $N = 51$) looked like this:



a) $X_A[k] = \text{Re}\left(X_0[k]\right)$

b) $X_B[k] = \text{Im}\left(X_0[k]\right)$

c) $X_C[k] = j\text{Im}\left(X_0[k]\right)$

d) $X_D[k] = \begin{cases} 0 & \text{if } k = 0 \\ X_0[k] & \text{otherwise} \end{cases}$

e) $X_E[k] = \begin{cases} 0 & \text{if } k = 25 \\ X_0[k] & \text{otherwise} \end{cases}$

f) $X_F[k] = X_0[k] + 1/51$

g) $X_G[k] = e^{j\pi} X_0[k]$

h) $X_H[k] = \begin{cases} X_0[0] & \text{if } k = 0 \\ e^{j\pi} X_0[k] & \text{otherwise} \end{cases}$

i) $X_I[k] = |X_0[k]| e^{j(-\angle X_0[k])}$