6.300 Signal Processing

Week 4, Lecture B:
Discrete Time Fourier Transform

- Definition
- Examples
- DT vs CT; FS vs FT
- DT Impulse

Quiz 1: Thursday October 3, 2-4pm 50-340

- Closed book except for one page of notes (8.5'' x 11'' both sides)
- No electronic devices (No headphones, cell phones, calculators, …)
- Coverage up to Week #3 (DTFS)
- practice quiz as a study aid, no HW#4

Geometric Series

Closed form sums of geometric sequences.

$$A = \sum_{n=0}^{N-1} \alpha^n$$

If the series has finite length (here N terms), it will converge for finite α.

$$\begin{aligned}
A &= 1 + \alpha + \alpha^2 + \cdots + \alpha^{N-1} \\
\alpha A &= \phantom{1 +{}} \alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N \\
A - \alpha A &= 1 \phantom{+ \alpha + \alpha^2 + \cdots + \alpha^{N-1}} - \alpha^N \\
A &= \begin{cases} \dfrac{1-\alpha^N}{1-\alpha} & \text{if } \alpha \neq 1 \\ N & \text{if } \alpha = 1 \end{cases}
\end{aligned}$$

If the series has infinite length, it will converge if $|\alpha| < 1$.

$$\sum_{n=0}^{\infty} \alpha^n = \lim_{N \to \infty} \sum_{n=0}^{N-1} \alpha^n = \lim_{N \to \infty} \frac{1-\alpha^N}{1-\alpha} = \frac{1}{1-\alpha} \quad \text{if } |\alpha| < 1$$

From Fourier Series to Fourier Transform (DT)

- Last time: use continuous-time Fourier transform to represent arbitrary (aperiodic) CT signals as sums of sinusoidal components

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$$

Synthesis equation

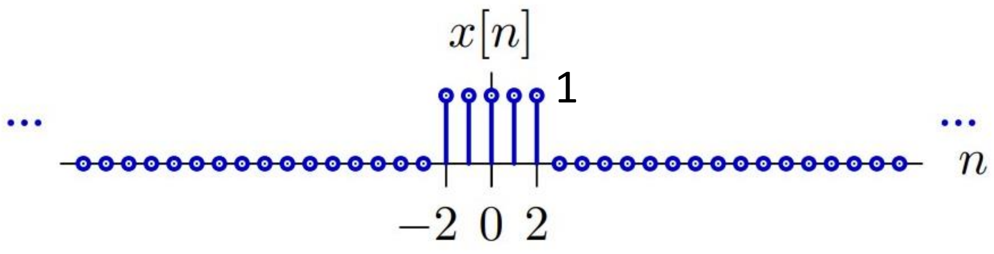
$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

Analysis equation

Today: generalize the **Fourier Transform** idea to **discrete-time** signals.

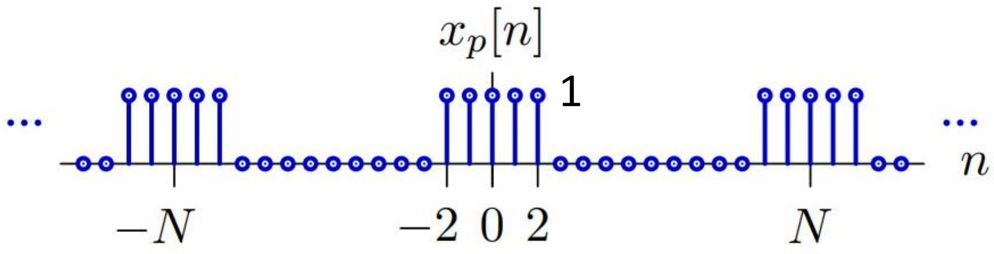
Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Strategy: make a periodic version of $x[n]$ by summing shifted copies:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$



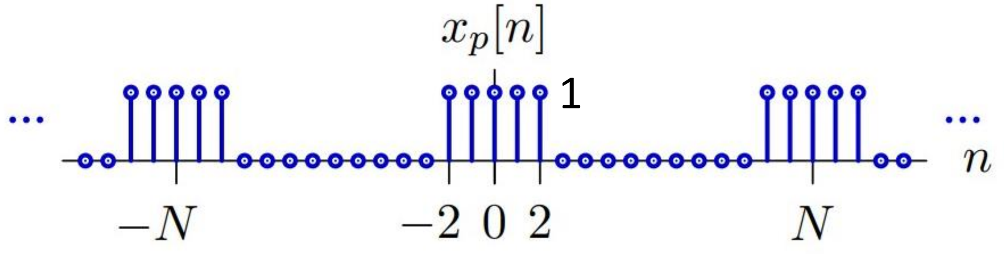
Since $x_p[n]$ is periodic, it has a Fourier series (which depends on N)

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as N → ∞

As N → ∞, $x_p[n] \rightarrow x[n]$ and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

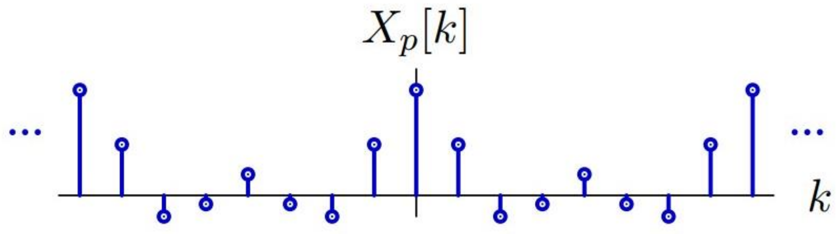
$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n-mN]$$



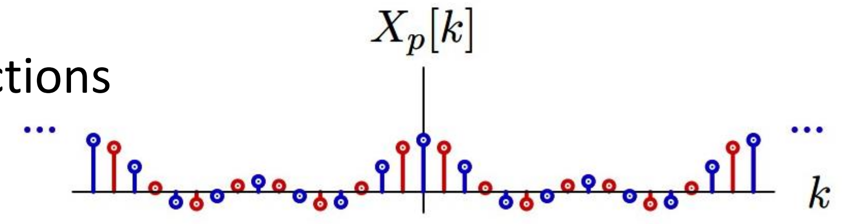
Calculate the Fourier series coefficients $X_p[k]$: $X_p[k] = \frac{1}{N} \sum_{n=<N>} x_p[n] \cdot e^{-j\frac{2\pi}{N}kn}$

$$X_p[k] = \frac{1}{N} \sum_{n=<N>} x[n] \cdot e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\left(\frac{2\pi k}{N}\right) + \frac{2}{N}\cos\left(\frac{4\pi k}{N}\right)$$

Plot the resulting Fourier Series coefficients for N=8.



What happens if you double the period N?



There will be twice as many samples per period of the cosine functions

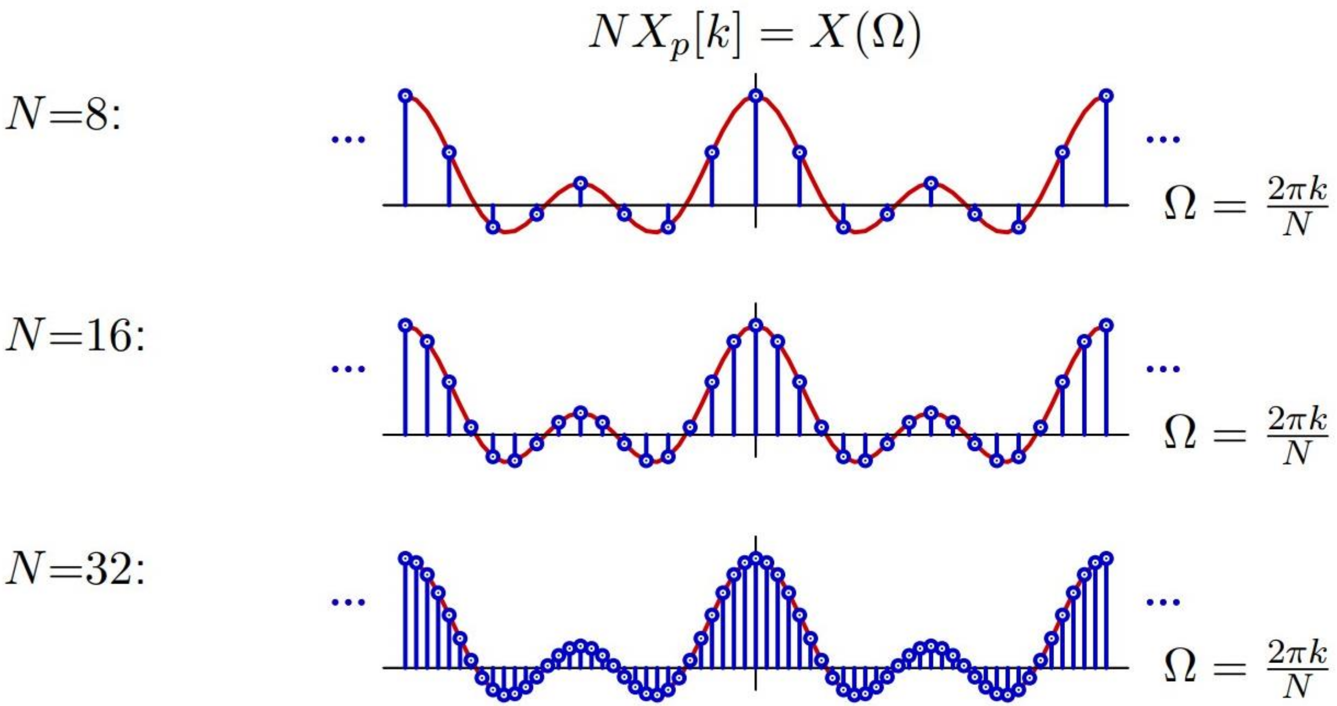
The red samples are at new intermediate frequencies

Fourier Representations of Aperiodic Signals

$$X_p[k] = \frac{1}{N} + \frac{2}{N}\cos\left(\frac{2\pi k}{N}\right) + \frac{2}{N}\cos\left(\frac{4\pi k}{N}\right)$$

$let\ \Omega = \frac{2\pi k}{N}$, Define a new function $X(\Omega) = N \cdot X_p[k] = 1 + 2\cos(\Omega) + 2\cos(2\Omega)$

If we consider Ω and $X(\Omega) = 1 + 2\cos(\Omega) + 2\cos(2\Omega)$ to be continuous, the discrete function $NX_p[k]$ is a sampled version of $X(\Omega)$.



As N increases, the resolution in Ω increases

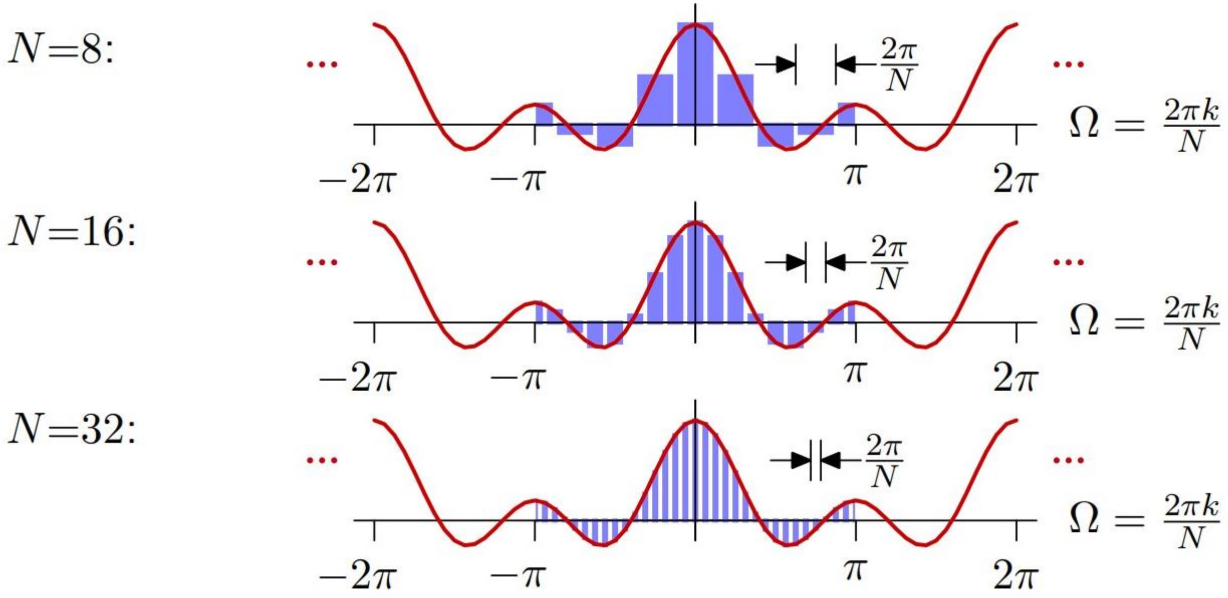
Fourier Representations of Aperiodic Signals

We can reconstruct $x[n]$ from $\mathrm{X}(\Omega)$ using Riemann sums (approximating an integral by a finite sum).

$$x_p[n] = \sum_{k=\langle N \rangle} X_p[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} N X_p[k] e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right)$$

$$x[n] = \lim_{N\to\infty} x_p[n] = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{k=\langle N \rangle} N X_p[k] e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$N X_p[k] = X(\Omega)$$



As N → ∞,

- $k\Omega_0 = \frac{2\pi k}{N}$ becomes a continuum, $\frac{2\pi k}{N} \to \Omega$.
- The sum takes the from of an integral, $\Omega_0 = \frac{2\pi}{N} \to d\Omega$
- We obtain a spectrum of coefficients: $X(\Omega)$.

Discrete-Time Fourier Transform

$$x[n] = \lim_{N\to\infty} x_p[n] = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{k=\langle N\rangle} N X_p[k] e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Since $X(\Omega) = N \cdot X_p[k]$ $\qquad X_p[k] = \frac{1}{N} \sum_{n=<N>} x[n] \cdot e^{-j\frac{2\pi}{N}kn}$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

Fourier Series and Fourier Transform

Fourier series and transforms are similar:
both represent signals by their frequency content.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of harmonics.
Aperiodic signals generally require all possible frequencies.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

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Discrete-Time Fourier Series

$$x[n] = x[n+N] = \sum_{k=<N>} X[k]e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\Omega_0 kn}$$

Analysis equation

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period.
The information in an aperiodic signal is spread across all time.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

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Discrete-Time Fourier Series

$$x[n] = x[n+N] = \sum_{k=<N>} X[k] e^{j\frac{2\pi}{N}kn}$$

Synthesis equation

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n] e^{-j\Omega_0 kn}$$

Analysis equation

Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_0$ are samples of continuous frequency Ω

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\,d\Omega$$

Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

Analysis equation

Discrete-Time Fourier Series

$$x[n] = x[n+N] = \sum_{k=<N>} X[k]e^{j\Omega_0 kn}$$

Synthesis equation

$$\Omega_0 = \frac{2\pi}{N}$$

$$X[k] = X[k+N] = \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\Omega_0 kn}$$

Analysis equation

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{j\Omega n}$.
Because of aliasing, we need only integrate dΩ over a 2π interval.

Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$$ Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$ Analysis equation

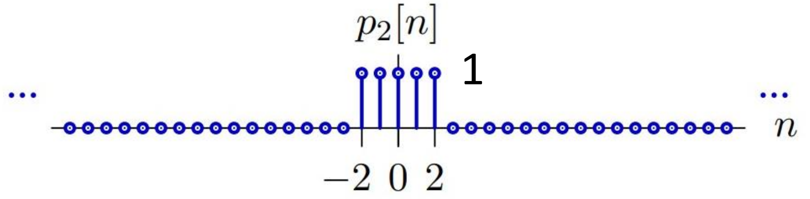
Continuous-Time Fourier Transform

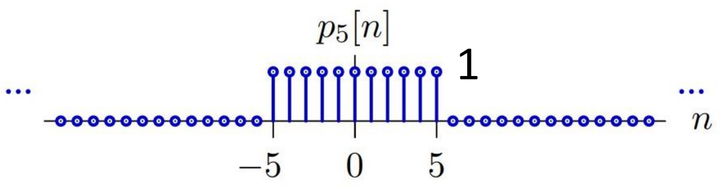
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}\, d\omega$$ Synthesis equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$$ Analysis equation

Fourier Transform of a Rectangular Pulse (width 2S+1)

$$p_S[n] = \begin{cases} 1 & -S \le N \le S \\ 0 & \text{otherwise} \end{cases}$$





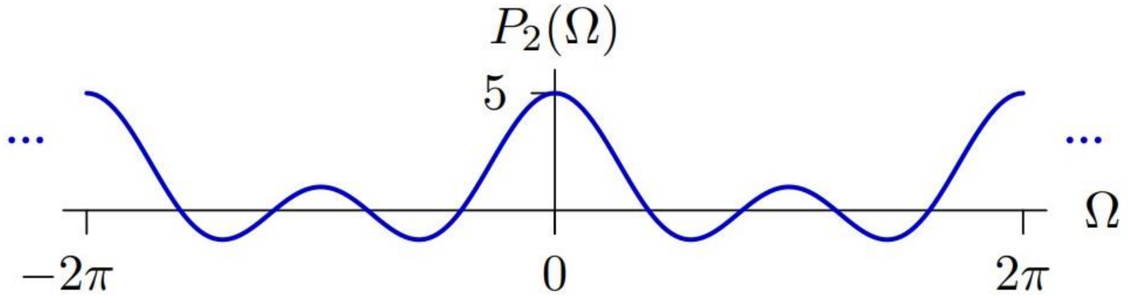
$$let\ m = n + S, n = m - S$$

$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n] \cdot e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n} = e^{j\Omega S} \sum_{m=0}^{2S} e^{-j\Omega m}$$

When $\Omega = 0, (or\ 2k\pi),\ P_S(\Omega) = 2S + 1$

When $\Omega \neq 0\ or\ 2k\pi$

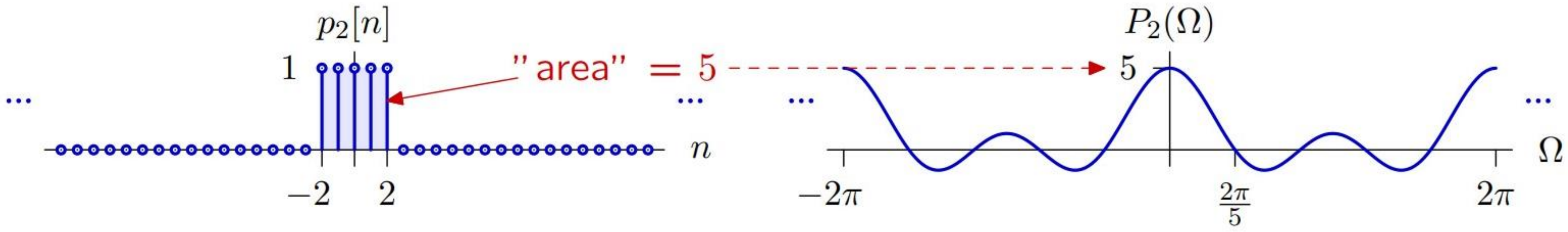
$$P_S(\Omega) = \frac{e^{j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2}} \cdot \frac{1 - e^{-j\Omega(2S+1)}}{1 - e^{-j\Omega}} = \frac{e^{j\Omega(S+\frac{1}{2})} - e^{-j\Omega(S+\frac{1}{2})}}{e^{j\Omega/2} - e^{-j\Omega/2}} = \frac{\sin(\Omega\left(S + \frac{1}{2}\right))}{\sin(\frac{\Omega}{2})}$$



Fourier Transform of a rectangular pulse

Similar to CT, the value of X(Ω) at Ω = 0 is the sum of x[n] over all time.

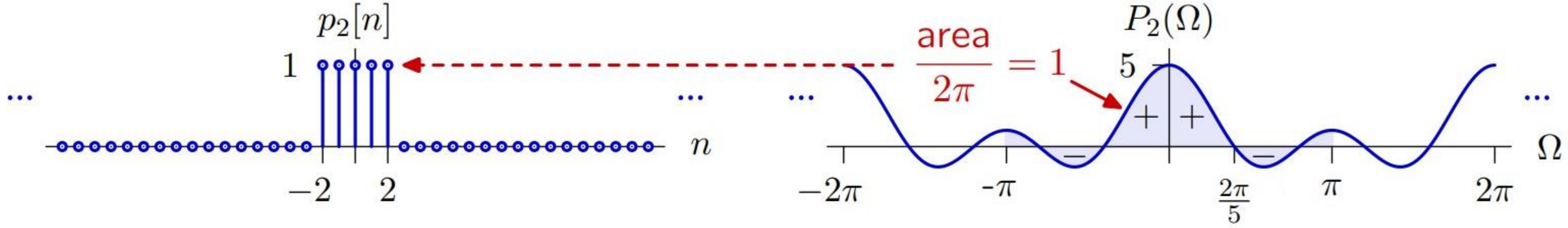
$$X(0) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]$$



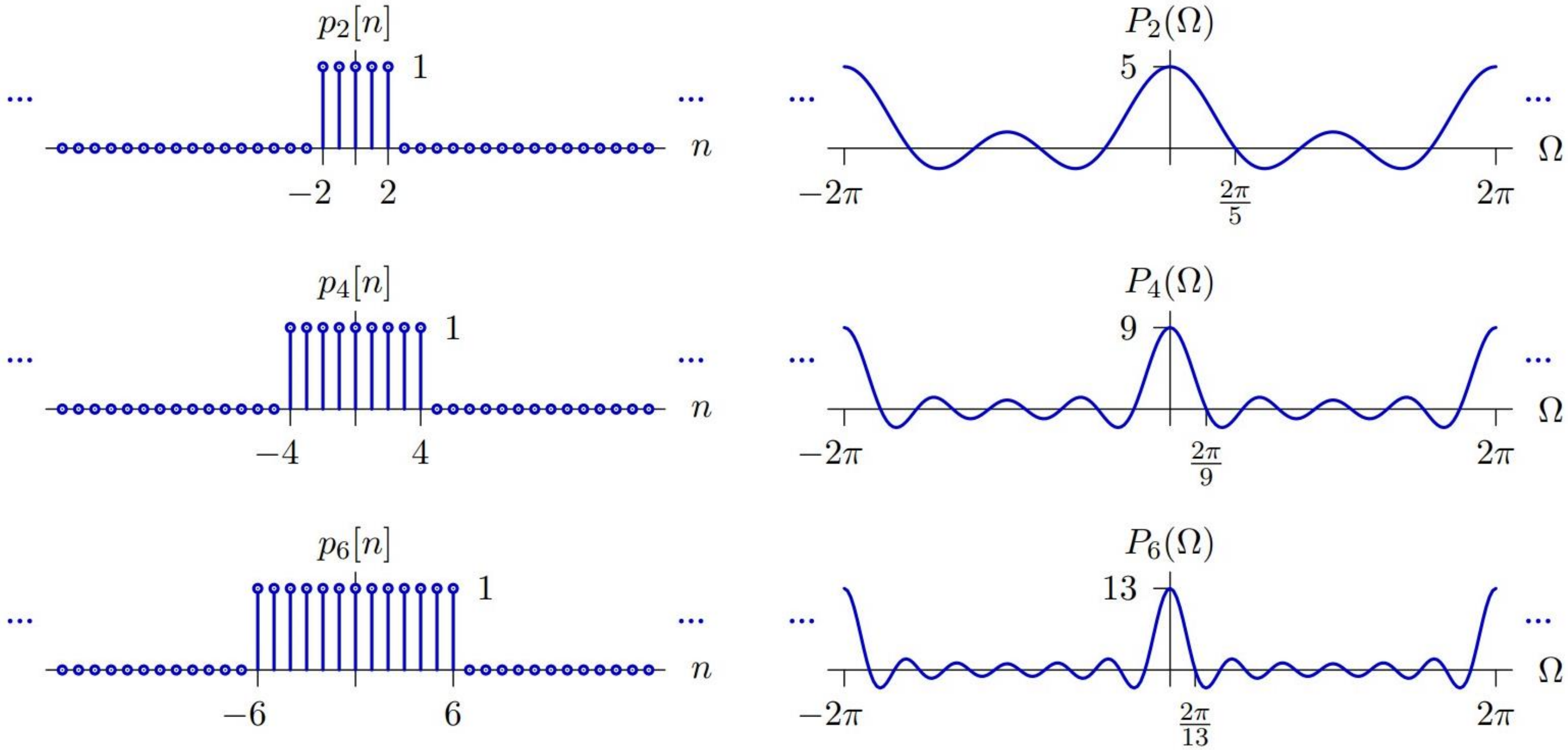
Fourier Transform of a rectangular pulse

The value of x[0] is 1/2π times the integral of X(Ω) over Ω = [−π, π].

$$x[0] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, d\Omega$$



Fourier Transforms of Pulses with Different Widths



$$P_S(\Omega) = \frac{\sin(\Omega\left(S + \frac{1}{2}\right))}{\sin(\frac{\Omega}{2})}$$

As the function widens in n(time) the Fourier transform narrows in Ω (freq).

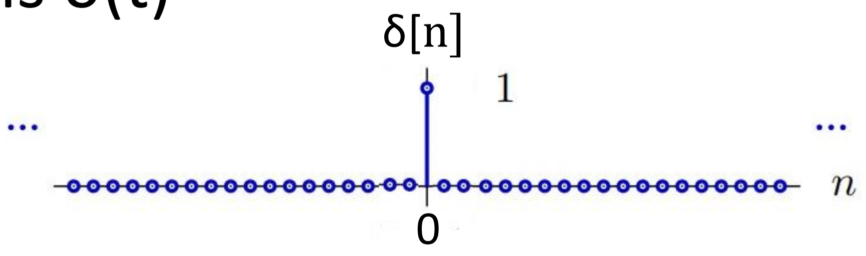
How about going the other way?

In the extreme of S=0, the signal becomes a unit impulse δ[n]

DT Impulse

The DT impulse is δ[n], its CT equivalent is δ(t)

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$



The DTFT of δ[n]:

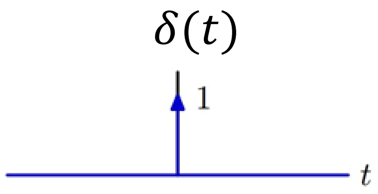
$$X(\Omega) = \sum_{n=-\infty}^{\infty} \delta[n] \cdot e^{-j\Omega n} = 1$$

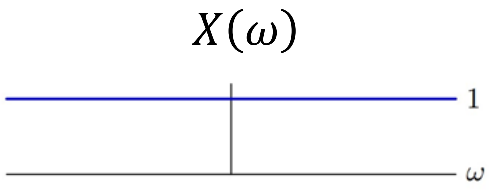
δ[n] still has the "sifting property:"

$$\sum_{n=-\infty}^{\infty} \delta[n-a] f[n] = f[a]$$

In comparison to its CT counterpart δ(t):

$$\int_{-\infty}^{\infty} \delta(t)dt = \int_{0_-}^{0_+} \delta(t)dt = 1$$





$$\int_{-\infty}^{\infty} \delta(t-a) f(t)dt = f(a)$$

Special Cases

The Fourier transform of the shortest possible CT signal $f(t) = \delta(t)$ is the widest possible CT transform $F(\omega) = 1$.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0} = 1$$

A similar result holds in DT.

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega 0} = 1$$

Special Cases

The Fourier transform of the widest possible CT signal $f(t) = 1$ is the narrowest possible CT transform $F(\omega) = 2\pi\delta(\omega)$.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{-j\omega t}d\omega = \int_{-\infty}^{\infty}\delta(\omega)e^{-j0t}d\omega = 1$$

A similar result holds in DT.

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{-j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\Omega)e^{-j\Omega n}d\Omega = \int_{2\pi}\delta(\Omega)e^{-j0n}d\Omega = 1$$

Unit Impulse in Frequency Domain

Because DT Fourier Transforms are periodic in 2π, it becomes an impulse train repeated every 2π.

$$X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$$

The DT signal whose Fourier transform is the above unit impulse is:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{0_-}^{0_+} \delta(\Omega)e^{j0n}d\Omega = \frac{1}{2\pi}\int_{0_-}^{0_+} \delta(\Omega)d\Omega = \frac{1}{2\pi}$$

Therefore if x[n] = 1 for all n, the transform is a delta function in frequency.

$$1 \overset{\text{DTFT}}{\Longleftrightarrow} \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi m)$$

This is in contrast to the CT case:

$$1 \overset{\text{CTFT}}{\Longleftrightarrow} 2\pi\delta(\omega)$$

Math With Impulses

This is what we learned previously:

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_o)e^{j\omega t}d\omega \\
&= \int_{-\infty}^{\infty} \delta(\omega-\omega_o)e^{j\omega_o t}d\omega \\
&= e^{j\omega_o t}\int_{-\infty}^{\infty} \delta(\omega-\omega_o)d\omega \\
&= e^{j\omega_o t}
\end{aligned}
$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\omega_o t} \stackrel{\text{CTFT}}{\Longrightarrow} 2\pi\delta(\omega-\omega_o)$$

The impulse in frequency has infinite value at $\omega = \omega_o$ and is zero at all other frequencies.

Math With Impulses

A similar construction applies in DT.

$$\begin{aligned}
f[n] &= \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{2\pi} 2\pi\delta(\Omega-\Omega_o)e^{j\Omega n}d\Omega \\
&= \int_{2\pi} \delta(\Omega-\Omega_o)e^{j\Omega_o n}d\Omega \\
&= e^{j\Omega_o n}\int_{2\pi} \delta(\Omega-\Omega_o)d\Omega \\
&= e^{j\Omega_o n}
\end{aligned}$$

Thus, the Fourier transform of a complex exponential is a delta function at the frequency of the complex exponential:

$$e^{j\Omega_o n} \stackrel{\text{DTFT}}{\Longrightarrow} 2\pi\delta(\Omega-\Omega_o)$$

The impulse in frequency shows that the transform is infinite at $\Omega = \Omega_o$ and is zero at all other frequencies.

Relations Between Fourier Series and Fourier Transforms

If a periodic signal $f(t) = f(t+T)$ has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_o t} \overset{\text{FT}}{\Longrightarrow} 2\pi\delta(\omega - \omega_o)$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad F[k]$$

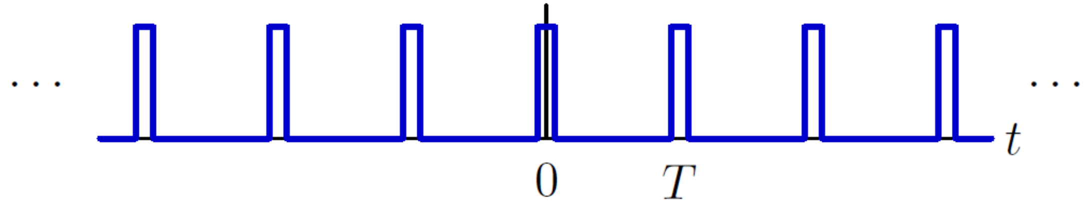
$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} F[k] e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta\left(\omega - \frac{2\pi}{T}k\right)$$

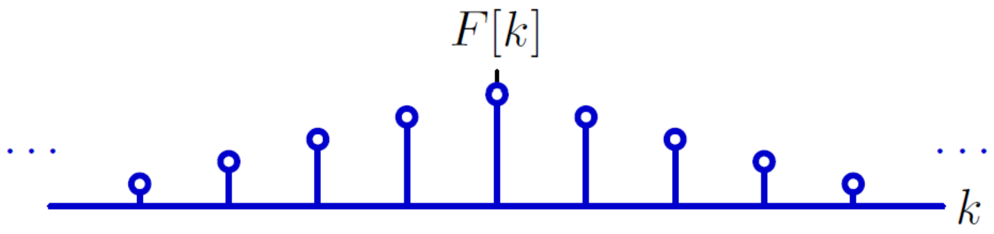
Each term in the Fourier series is replaced by an impulse in the Fourier transform.

Relations Between Fourier Series and Fourier Transforms

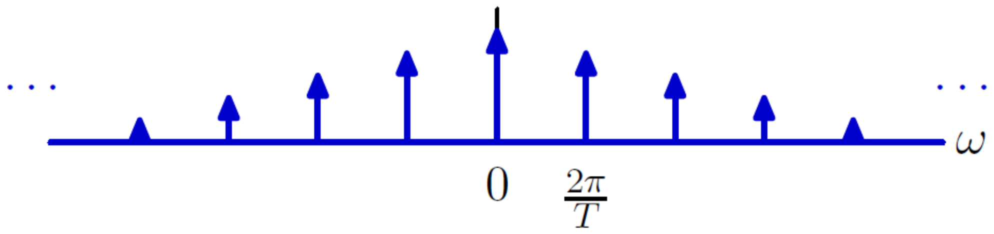
Each Fourier series term is replaced by an impulse in the Fourier transform.

$$f(t) = \sum_{m=-\infty}^{\infty} f(t - mT)$$





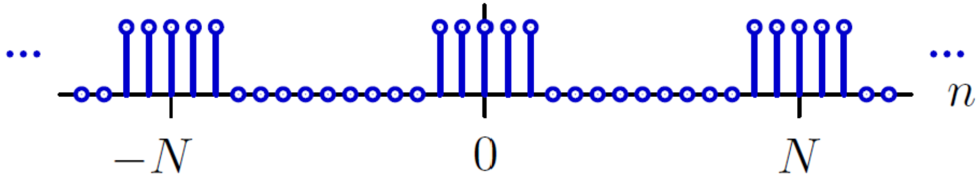
$$F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi \, F[k] \, \delta(\omega - k\frac{2\pi}{T})$$



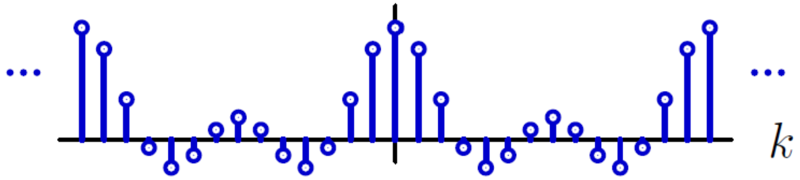
Relations Between Fourier Series and Fourier Transforms

Each Fourier series term is replaced by an impulse in the Fourier transform.

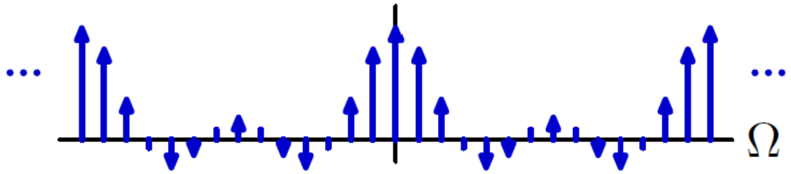
$f[n] = f[n-N]$ for all integers n



$F[k] = F[k-N]$ for all integers k



$$F(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi F[k]\delta\left(\Omega - k\frac{2\pi}{N}\right)$$



Periodic DT signals that have Fourier series representations also have Fourier transform representations.

Summary

We will now go to 4-370 for recitation & common hour

- Discrete-Time Fourier Transform: Fourier representation to all DT signals!

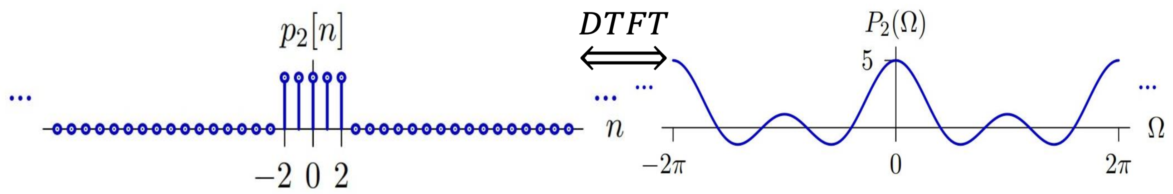
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\cdot e^{j\Omega n}\, d\Omega$$

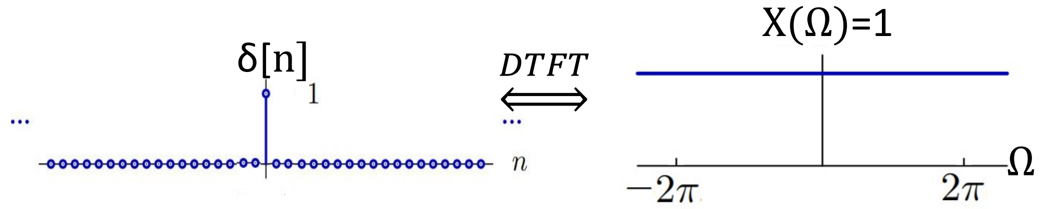
Synthesis equation

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\cdot e^{-j\Omega n}$$

Analysis equation

- Very useful signals:
  - Rectangular pulse and its FT(sinc)
  - Delta function (Unit impulse) and its FT





- If a periodic signal $f[n] = f[n+N]$ has a Fourier Series representation, then it can also be represented by an equivalent Fourier Transform.