# 6.300 Signal Processing

## Week 3, Lecture A:
## Sampling and Aliasing

- Continuous signals → discrete signals
- Sampling effect
- Quantization effect

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

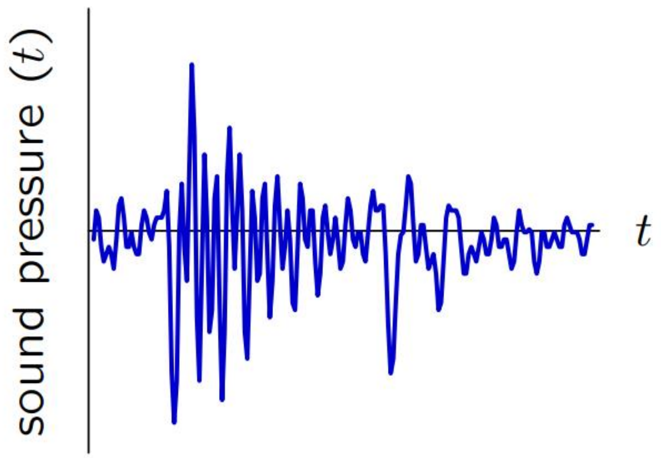
# Today: from continuous to discrete signals

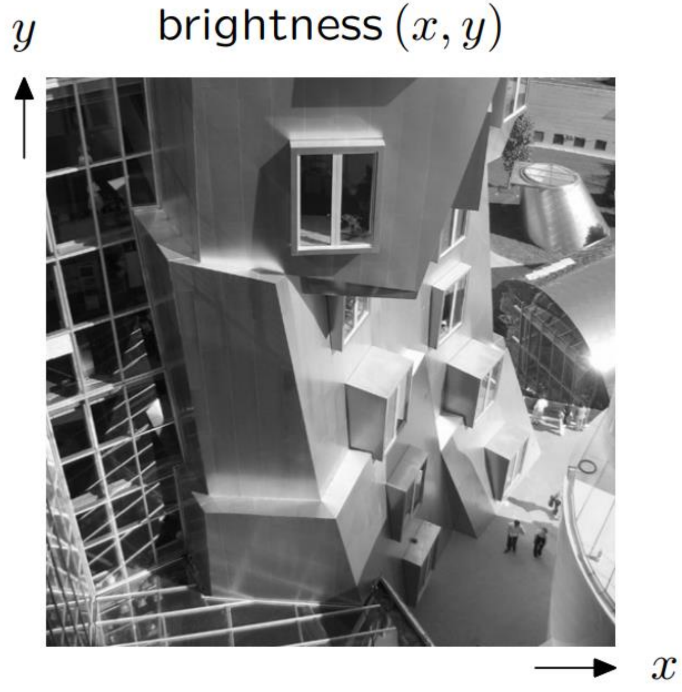
Physical signals are often of continuous domain:

- continuous spatial coordinates (in meters)
- continuous values

Computations manipulate functions of discrete domain:

- discrete spatial coordinates (in pixels)
- discrete values

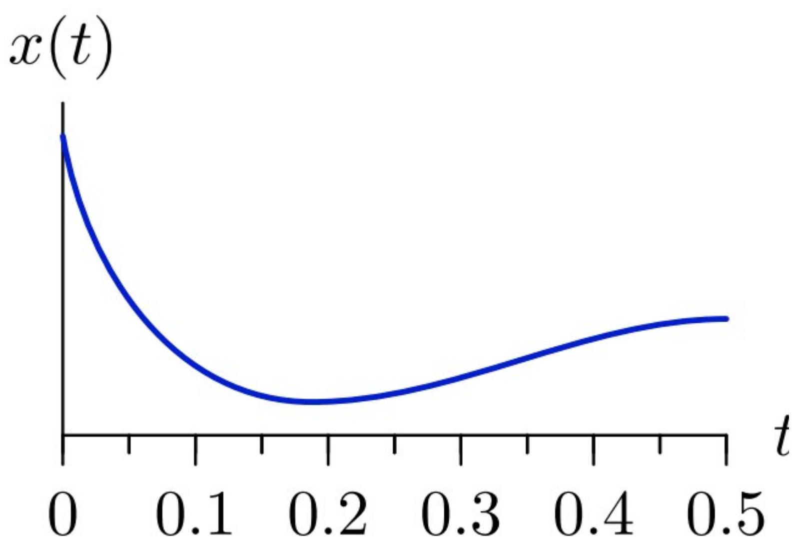


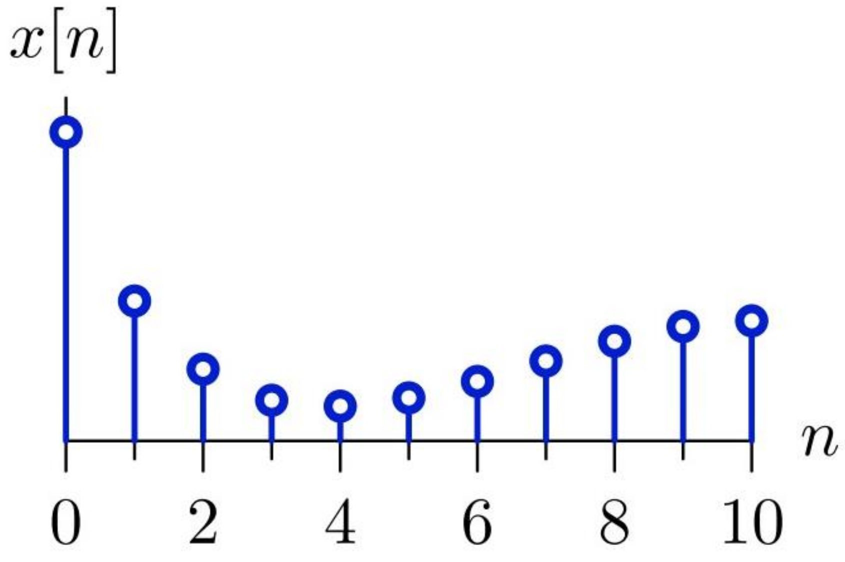



# Today: from continuous to discrete signals

- Signal processing requires the operation: continuous → discrete
- Today: Understand the relationship between continuous signals to discrete signals.
- Question: How to convert continuous signals to discrete signals?

Sampling & Quantization

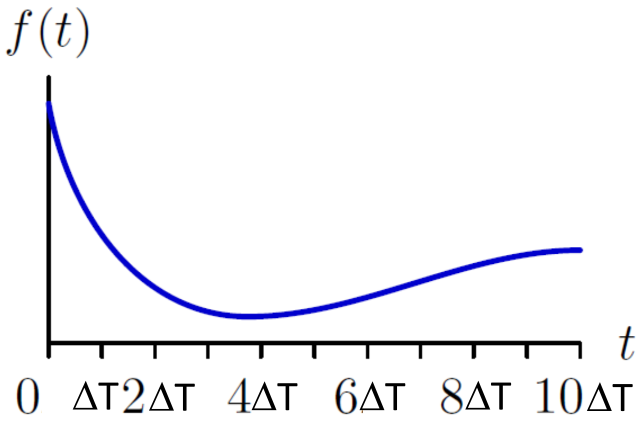


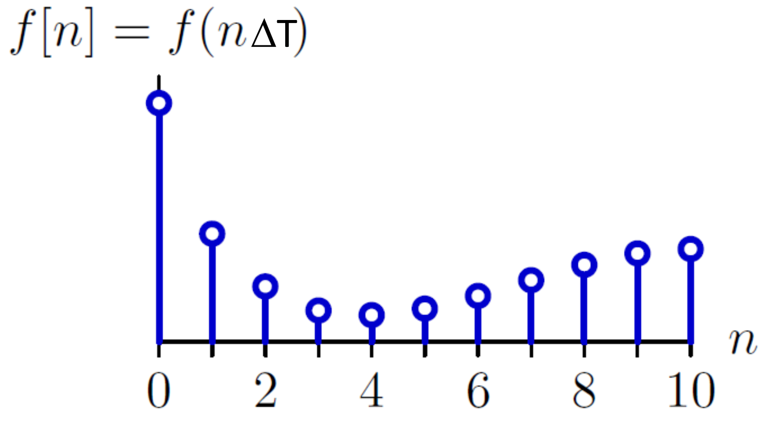



# Sampling

Sampling refers to the process by which a continuous-time signal $f(t)$ is converted to a discrete-time signal $f[n]$.

We use parentheses to denote functions of continuous domain (e.g., $f(t)$) and square brackets to denote functions of discrete domain (e.g., $f[n]$).




$$\Delta T = \text{sampling interval}$$

$$f_s = \frac{1}{\Delta T} = \text{sampling frequency}$$

How does sampling affect the information contained in a signal?

# Effect of sampling are easily heard

- How does sampling affect the information contained in a signal?

Sampling Music

$$f_s = \frac{1}{\Delta T}$$

- $f_s = 44.1$ kHz
- $f_s = 22$ kHz
- $f_s = 11$ kHz
- $f_s = 5.5$ kHz
- $f_s = 2.8$ kHz

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

# Effect of sampling are easily seen

Sampling images: original 4112 x 3088

# Effect of sampling are easily seen

Sampling images: undersample 2x

# Effect of sampling are easily seen

Sampling images: undersample 8x

# Effect of sampling are easily seen

Sampling images: undersample 16x

# Effect of sampling are easily seen

Sampling images: undersample 32x

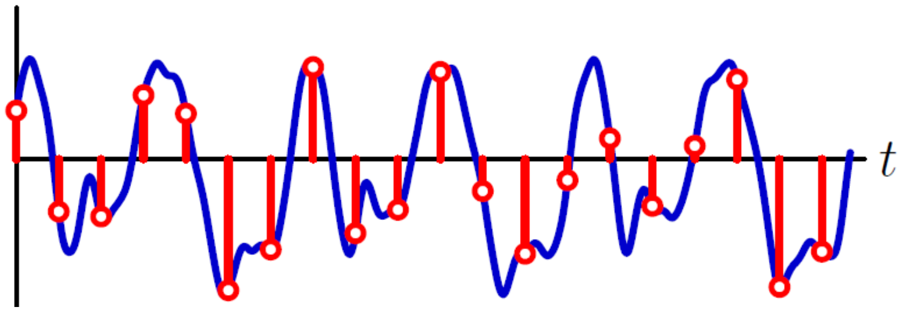
- Information loss
- Distortion

# Effect of sampling

We would like to sample in a way that preserves **information**.

However, information is generally **lost** in the sampling process.

Example: samples (red) provide no information about intervening values.



# Effect of sampling

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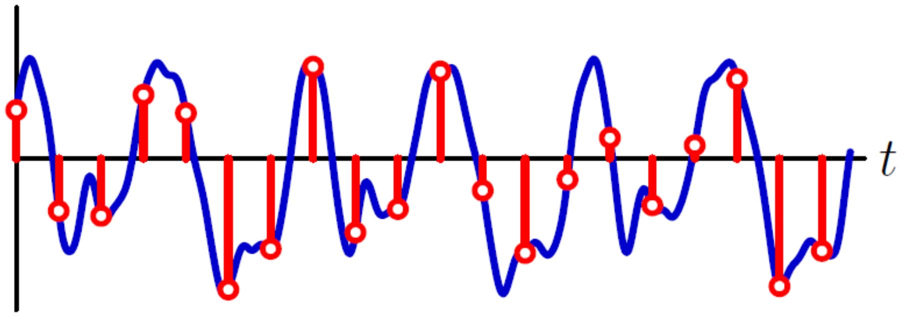
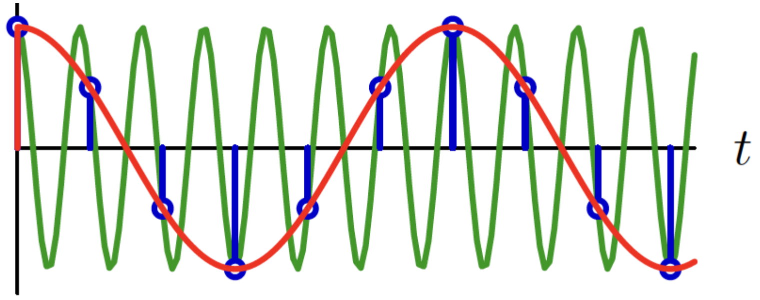



Furthermore, information that is retained by sampling can be misleading.

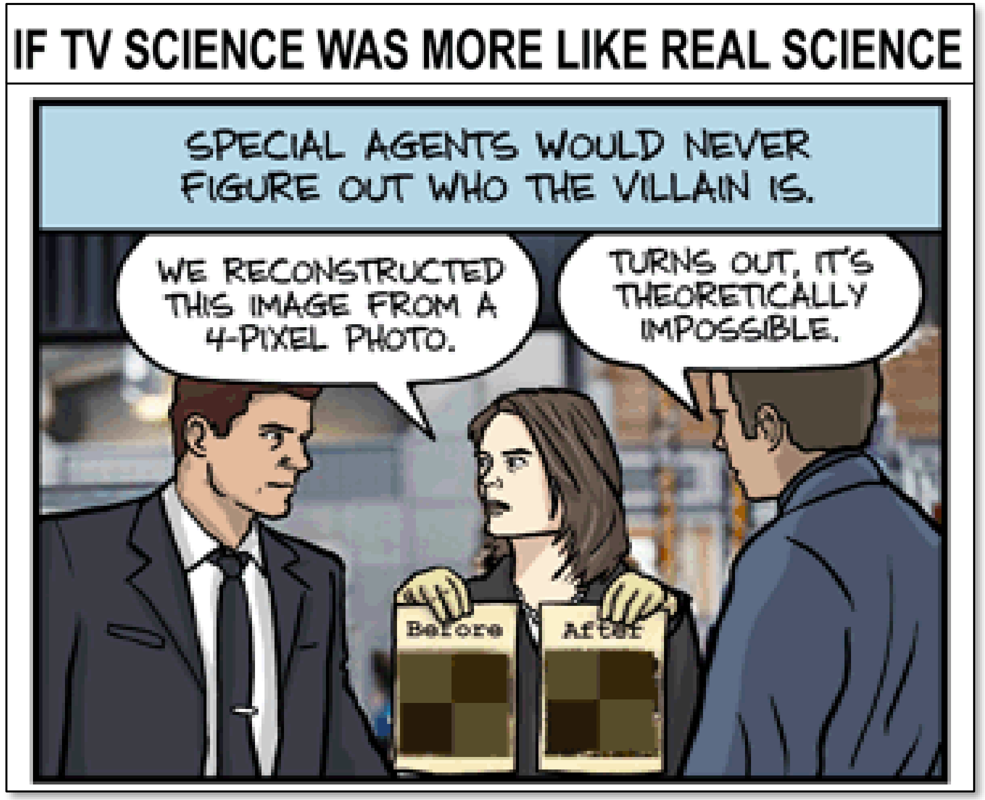
Example: samples can suggest patterns not contained in the original.




Samples (blue) of the original high-frequency signal (green)

# The artifacts of sampling in real life

Loss of information




Aliasing

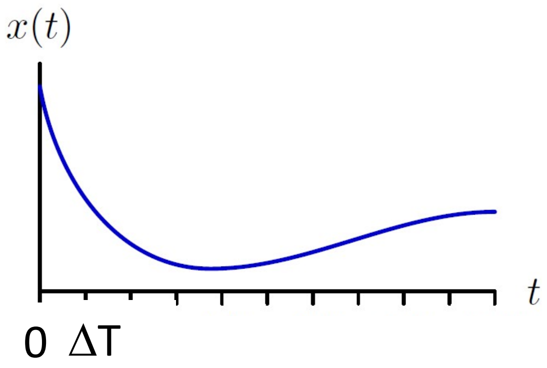
When fps = rpm

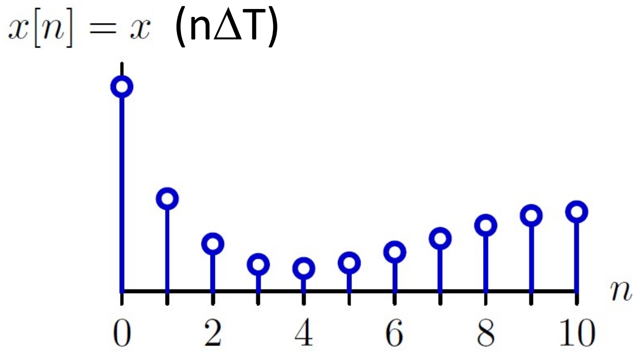
https://www.youtube.com/watch?v=_xTjyV8F6XU

# To mitigate artifacts, we need to understand sampling

What is sampling:

- Continuous coordinate → discrete coordinate
- Measuring the value every ΔT seconds



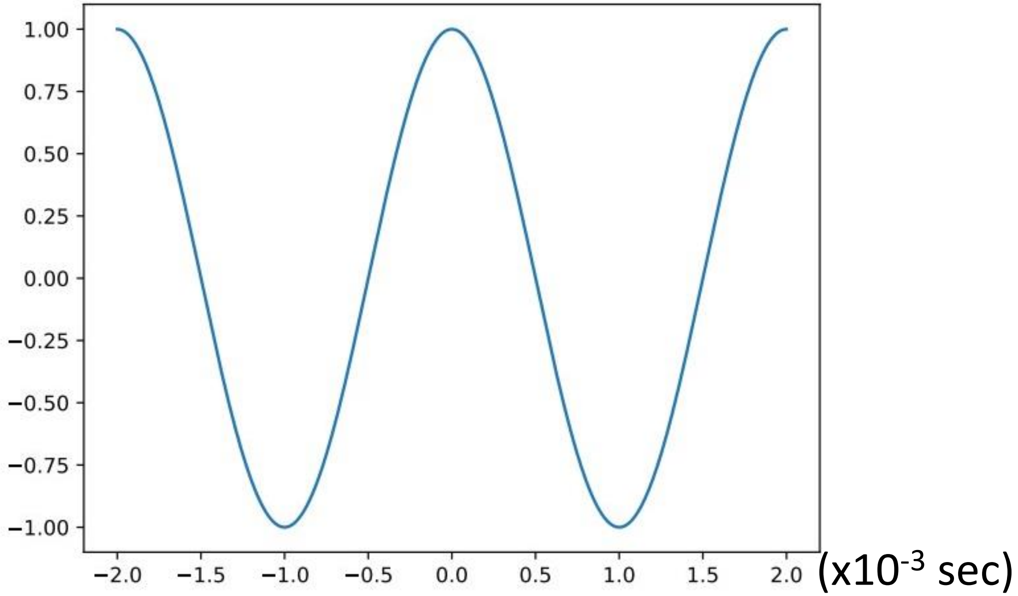




Parameters that matter here:

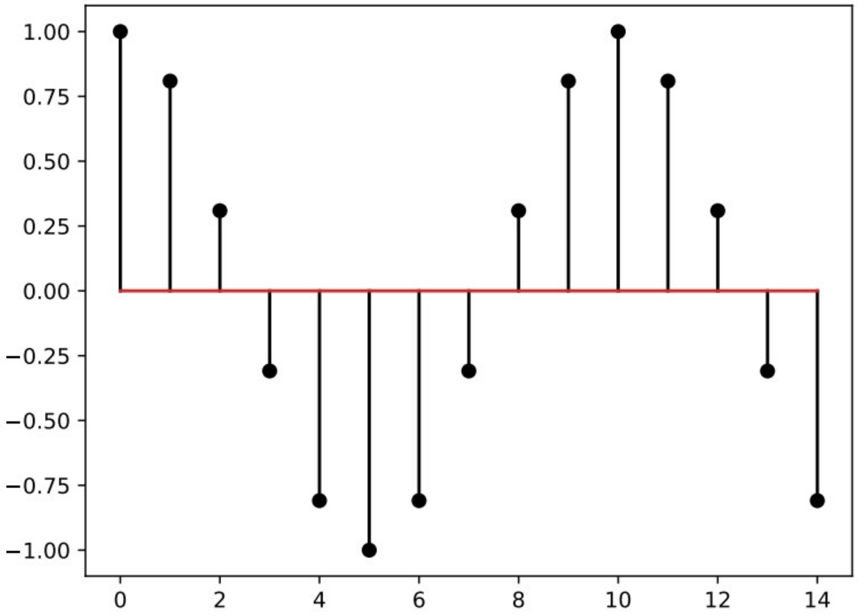
$$\Delta T \text{ (seconds / sample)} = \text{sampling interval}$$
$$f_s \text{ (samples / second)} = \text{sampling rate}$$

# Let's use sinusoids an example from CT to DT

$x(t) = \cos(\omega t)$




$x[n] = \cos(\Omega n)$



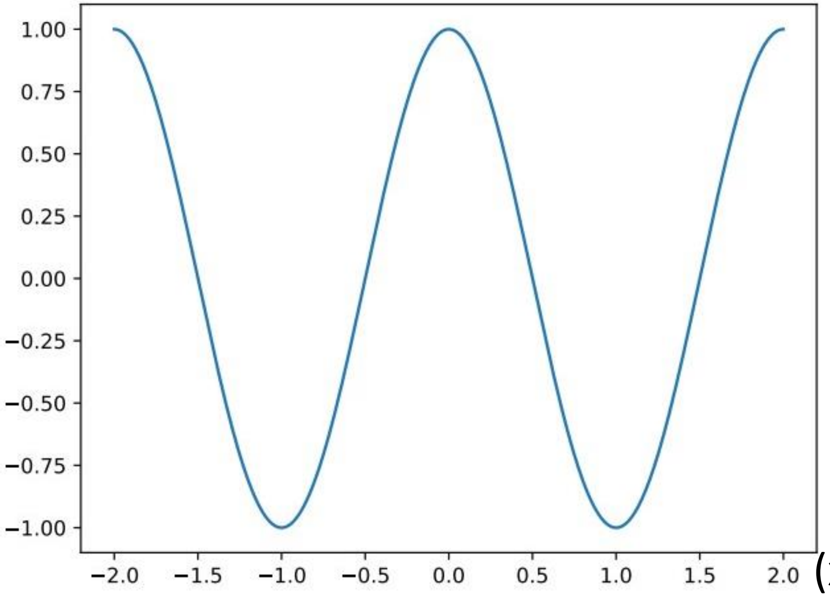

Question: what is the relationship between $\omega$ and $\Omega$?

$$\boxed{\begin{array}{l}\text{Sample } x(t) = \cos(\omega t) \text{ every } \Delta T \text{ seconds to obtain } x[n]: \\ \quad x[n] = x(n\Delta T) = \cos(\omega n \Delta T) = \cos\Big((\omega \Delta T) n\Big)\end{array}}$$

$$\boxed{\Omega = \omega \Delta T = \omega / f_s}$$

# Check yourself

$x(t) = \cos(\omega t)$



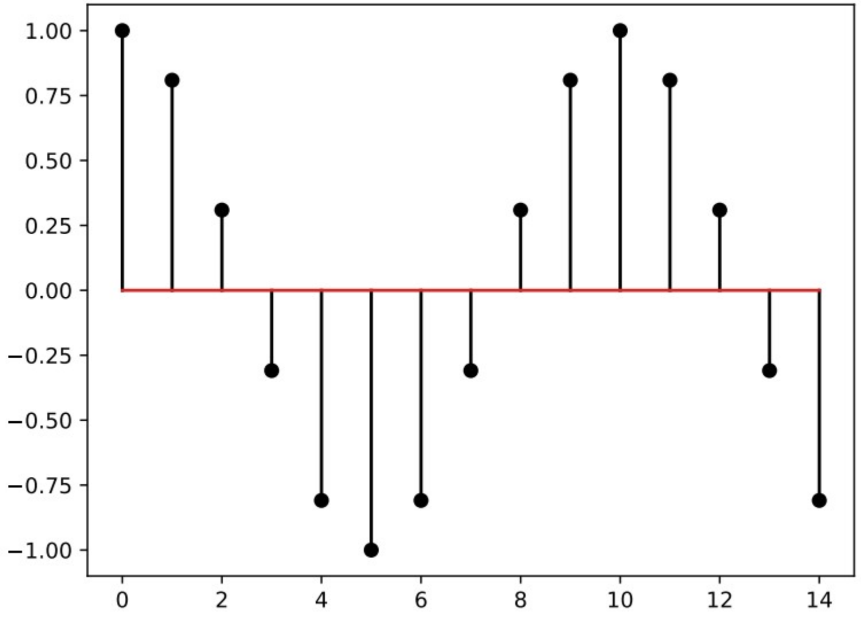

(x$10^{-3}$ sec)

$\Omega = \omega/f_s$

$\Delta T = 2 \times 10^{-3}$ sec/10

$f_s = \frac{1}{\Delta T} = 5000$

$x[n] = \cos(\Omega n)$




Question: We have the CT signal shown on the left and its corresponding sampled DT signal shown on the right. What is the sampling rate $f_s$ ?

$\omega = 2\pi/(2 \times 10^{-3}\text{ sec}) = 1000\pi$

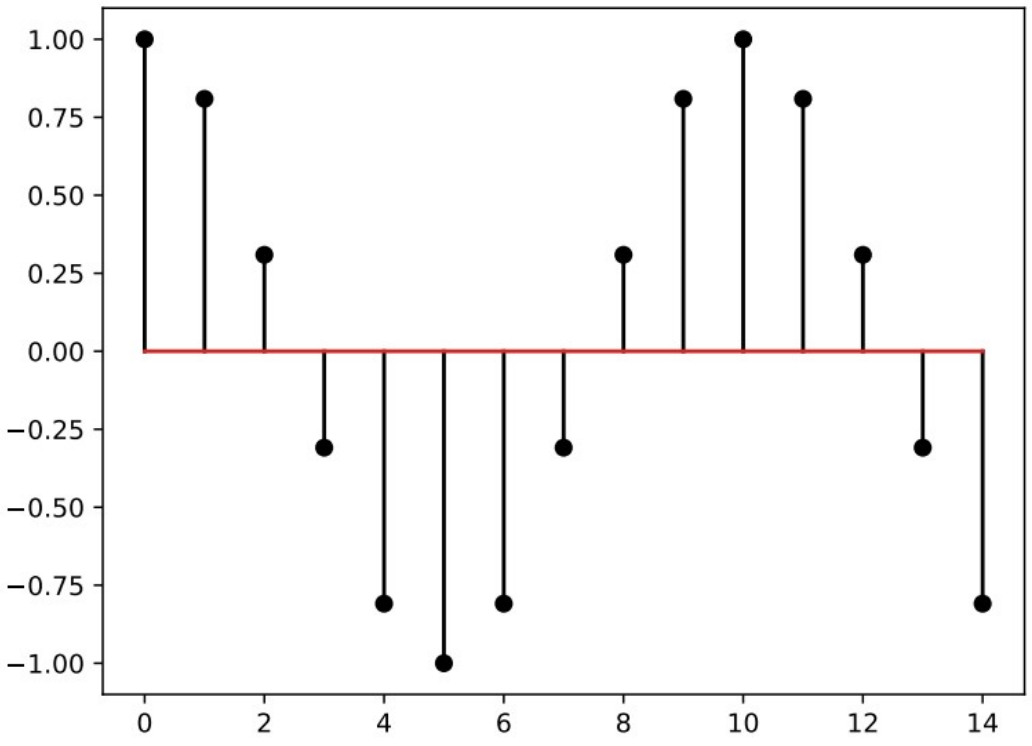
$\Omega = 2\pi/10 = 0.2\pi$

$\omega = 2\pi f$, $f = 500\text{Hz}$

$\boxed{\Omega = 2\pi f/f_s}$

# Intro to aliasing

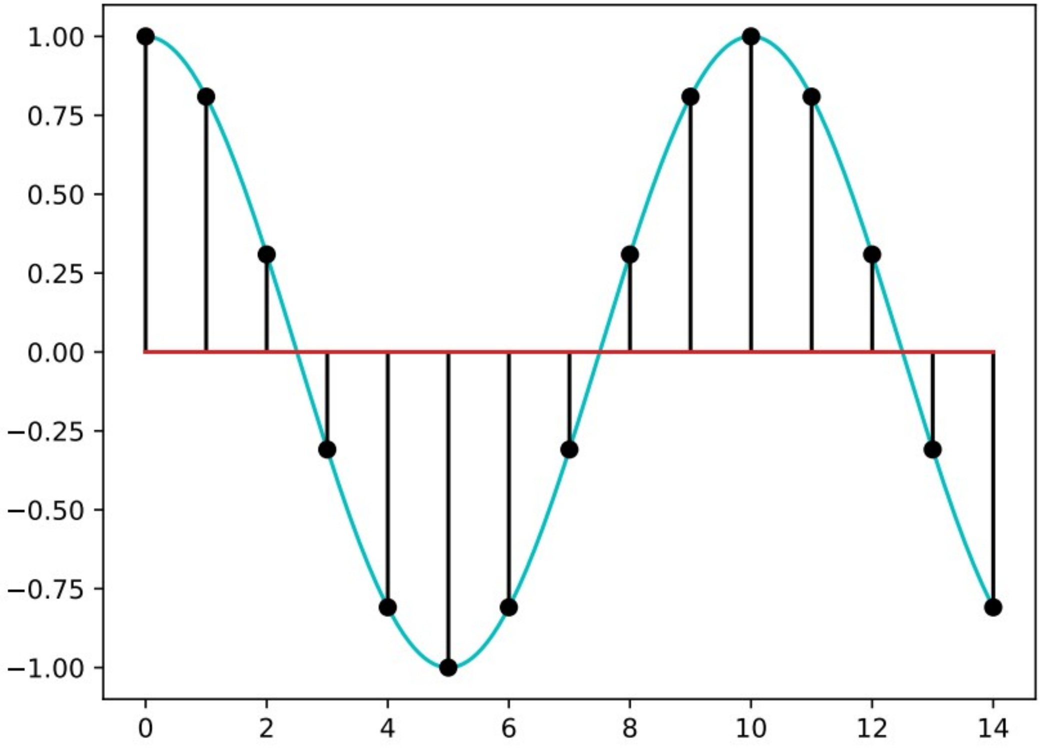
$$x[n] \ = \cos(\Omega n)$$




- The same DT signal can be used to obtain CT signals of different frequency!

# Intro to aliasing

$$x[n] = \cos(\Omega n)$$

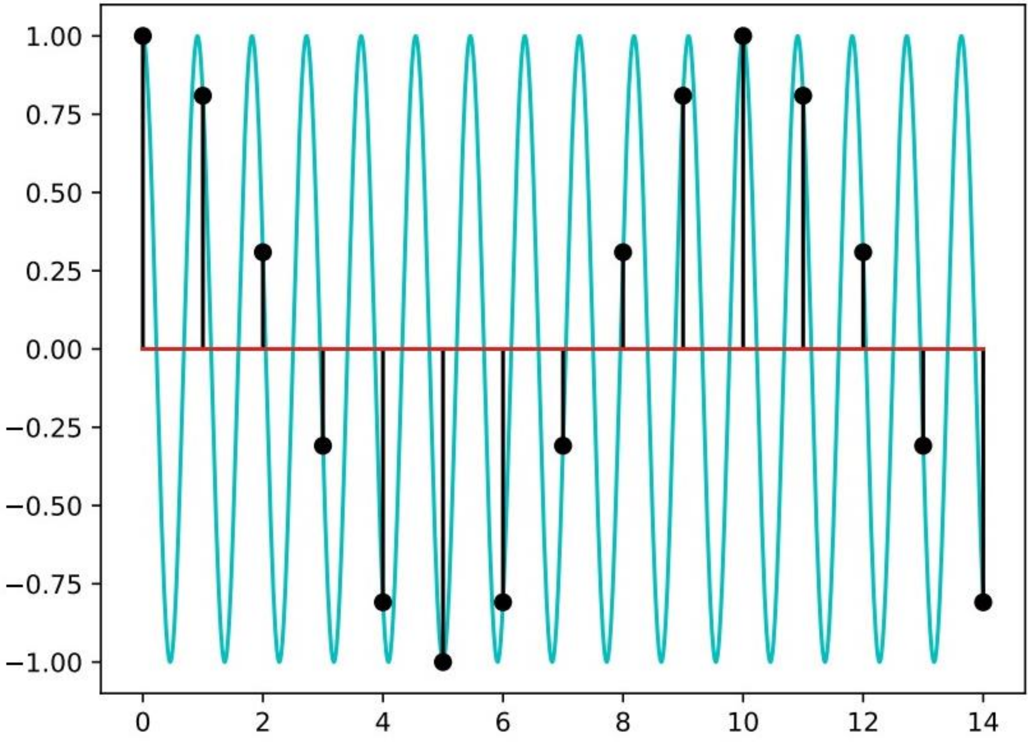



1 cycle ($2\pi$ radians) in 10 samples: $\Omega = 0.2\pi$

- The same DT signal can be used to obtain CT signals of different frequency!

# Intro to aliasing

$$x[n] = \cos(\Omega n)$$

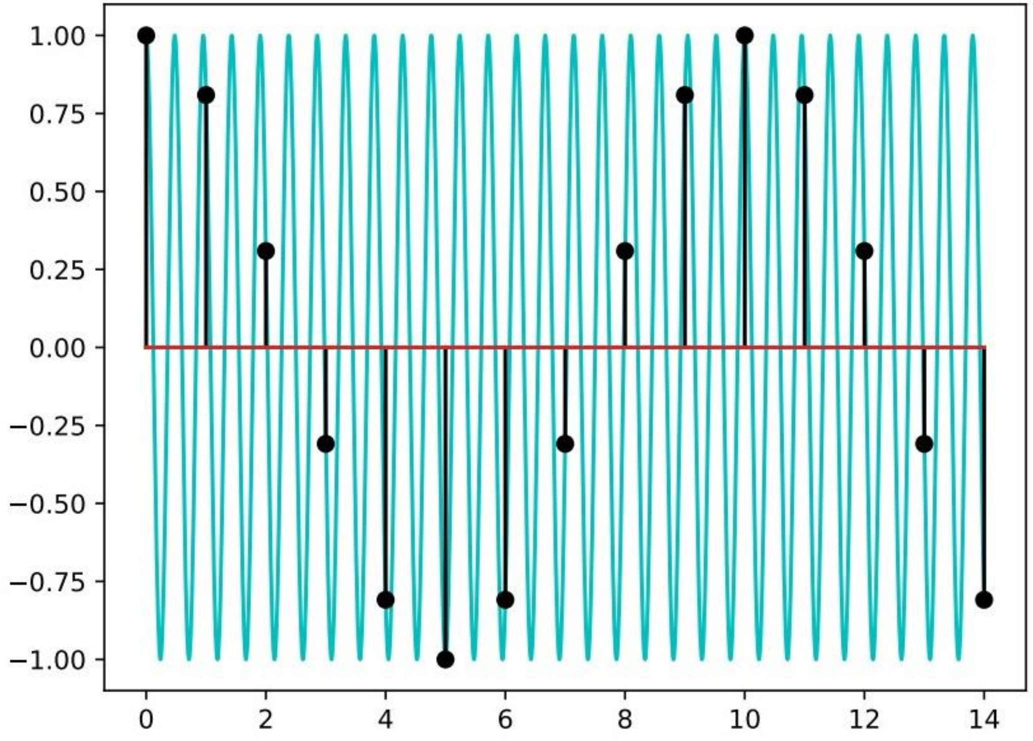



- ❑ The same DT signal can be used to obtain CT signals of different frequency!

11 cycles ($22\pi$ radians) in 10 samples: $\Omega = 2.2\pi$

# Intro to aliasing

$$x[n] = \cos(\Omega n)$$




- The same DT signal can be used to obtain CT signals of different frequency!

21 cycles ($42\pi$ radians) in 10 samples: $\Omega = 4.2\pi$

Q: Why this can happen?

# Why aliasing can happen

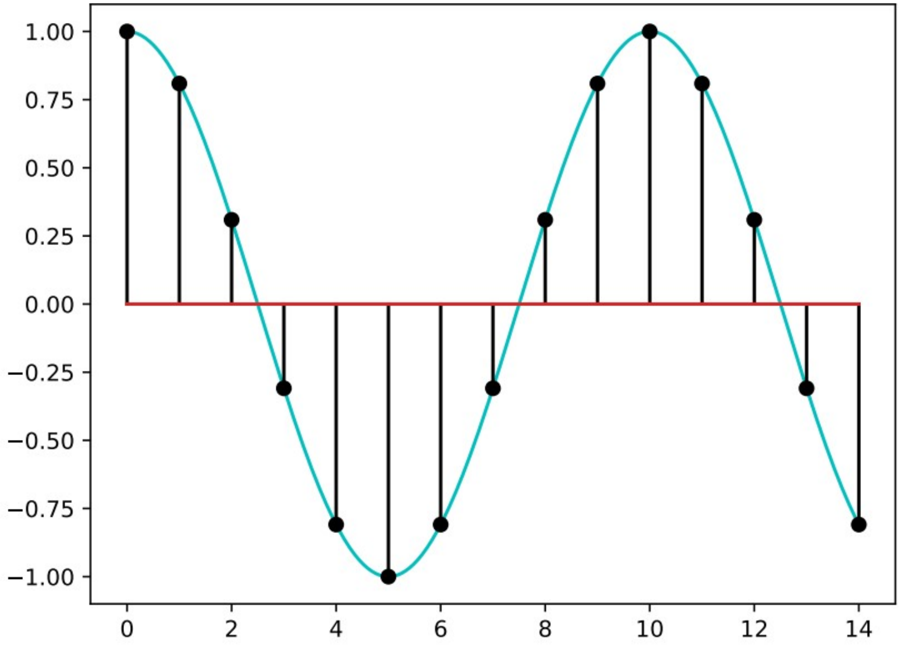
$x(t) = \cos(\omega t)$ v.s. $x[n] = \cos(\Omega n)$

Q: What happens when we increase $\omega$?

The frequency of the sound increases as $\omega$ increases

Q: What happens when we increase $\Omega$:

$\cos((\Omega + 2\pi)n) = \cos(\Omega n + 2\pi n) = \cos(\Omega n)$




- Compared to CT signals: n is always an integer
  - We only have values at integer multiples of $\Omega$
  - There are multiple $\Omega$ values that lead to the exact same set of discrete points
- Consequences
  - This graph could be described by an infinite number of different $\Omega$ values
  - Hence aliasing: the same signal can be described by different “names”.

# Aliasing

In our example, we had:

$\omega_1$ =1000$\pi$, f=500Hz

$\omega_2$ =11000$\pi$, f=5500Hz

$\omega_3$ =21000$\pi$, f=10500Hz

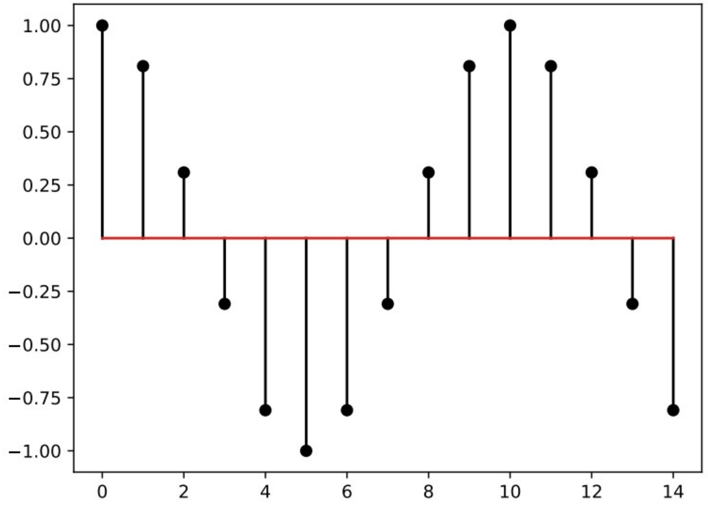
$f_s = 5000$

$\omega = \Omega \cdot fs$

$$x_1[n] = \cos(0.2\pi n)$$

$$x_2[n] = \cos(2.2\pi n)$$

$$x_3[n] = \cos(4.2\pi n)$$

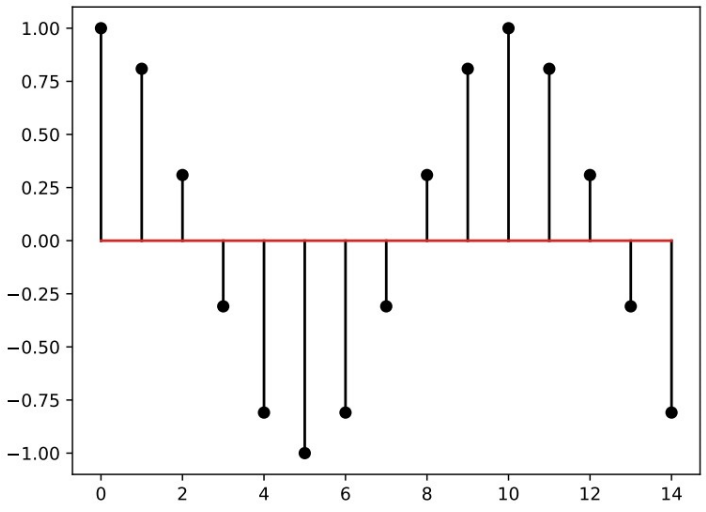



These all represent the exact same signal! They are all *aliases* for that signal.

**Consider we obtained $x_1[n]$, $x_2[n]$, $x_3[n]$ by sampling from continuous time signals, if a sampling rate fs of 5000 Hz was used, what frequency ($\omega$ or f) should the original continuous time signal has?**

**Participation question for Lecture**

# Base band




Multiple frequencies $\Omega$ that we could use to refer to this signal

We can remove the ambiguity of which frequency is represented by a set of samples by choosing the one in the range $0 \leq \Omega \leq \pi$.

We call that range of frequencies the **base band** of frequencies, and the value of $\Omega$ that falls in that range is often referred to as the *principle alias*.

# Maximum Frequency

If we limit our attention to frequencies in the base band, then there is a maximum possible discrete frequency $\Omega_{\text{max}} = \pi$.

Note this difference from CT, where there is no maximum frequency.

# Let's compute maximum frequency: Nyquist frequency

If a CT signal $x(t) = \cos \omega t$ is sampled at times $t = n\Delta T$, the resulting DT signal is $x[n] = \cos \Omega n$ where

$$\Omega = \omega \, \Delta T$$

If we restrict DT frequencies to the range $0 \leq \Omega \leq \pi$, then the corresponding CT frequencies are in the range $0 \leq \omega \leq \omega_N$ where

$$\omega_N = \frac{\pi}{\Delta T}$$

and

$$f_N = \frac{\omega_N}{2\pi} = \frac{1}{2\Delta T} = \frac{1}{2} f_s$$

The Nyquist frequency is basically half the sampling rate.

# Anti-Aliasing

- If there are frequencies in the CT signal that are greater than the Nyquist frequency, they will alias to frequencies in the base band (that really don't have anything to do with the original frequency!)
- Anti-aliasing to prevent distortions: remove frequencies above Nyquist frequency before sampling so that they won't alias into the base band

Sampling Music

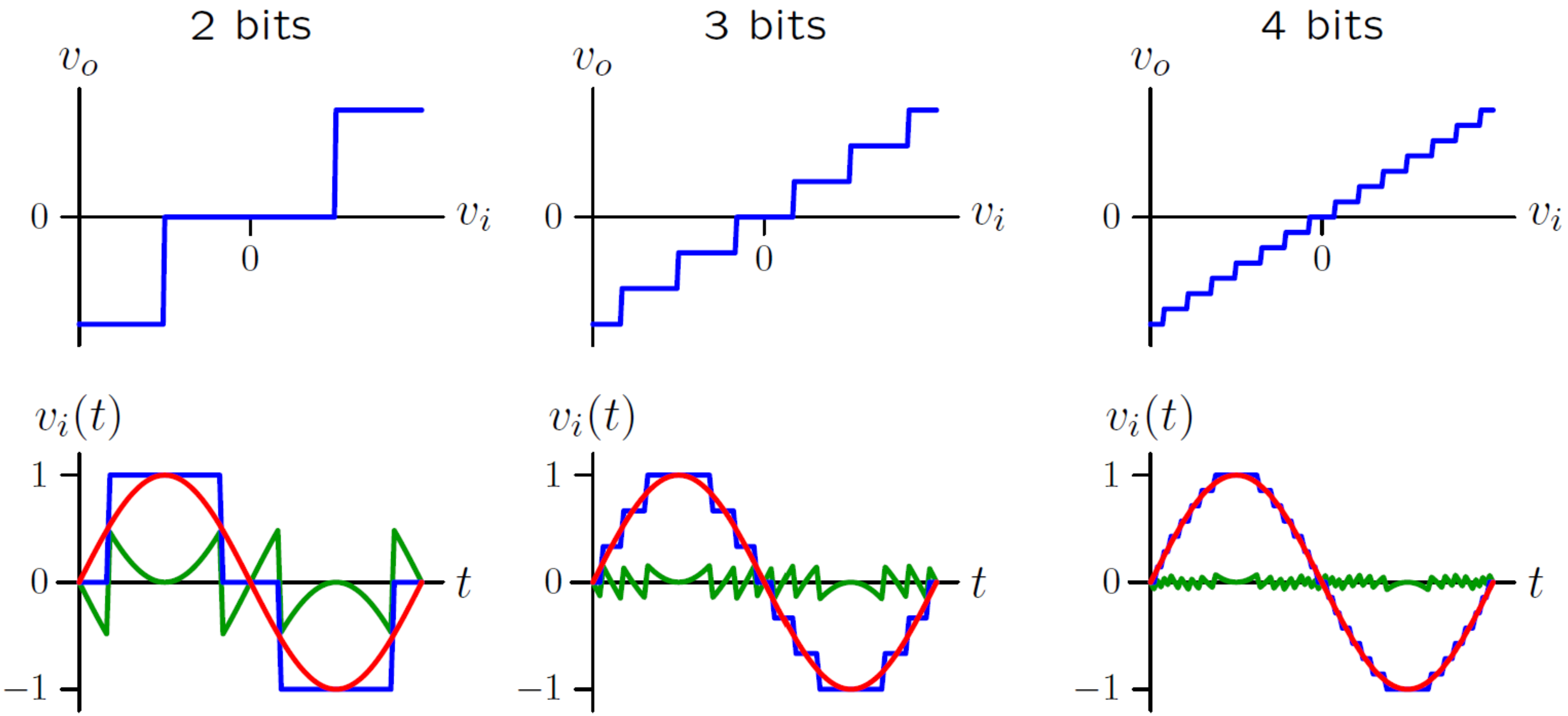
$$f_N = \frac{1}{2} f_s$$

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

| | No anti-aliasing | with anti-aliasing |
|---|---|---|
| • $f_s$ = 5.5kHz | | |
| • $f_s$ = 2.8kHz | | |

# Quantization

The information content of a signal depends not only with sample rate but also with the number of bits used to represent each sample.




Bit rate = (# bits/sample)×(# samples/sec)

# Check yourself

We hear sounds that range in amplitude from 1,000,000 to 1.

How many bits are needed to represent this range?

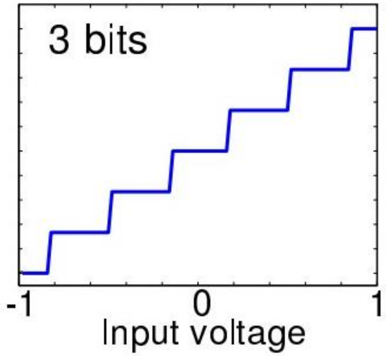
1. 5 bits
2. 10 bits
3. 20 bits
4. 30 bits
5. 40 bits

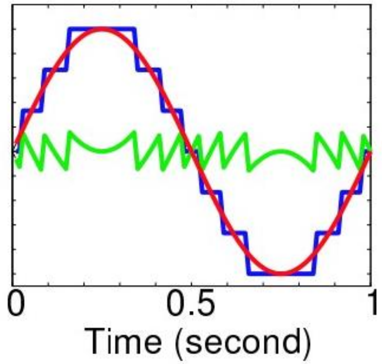
| bits | range |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1,024 |
| 11 | 2,048 |
| 12 | 4,096 |
| 13 | 8,192 |
| 14 | 16,384 |
| 15 | 32,768 |
| 16 | 65,536 |
| 17 | 131,072 |
| 18 | 262,144 |
| 19 | 524,288 |
| 20 | 1,048,576 |

# Quantization demonstration in music

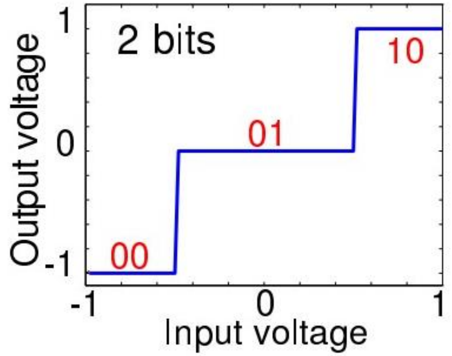
16 bits/sample

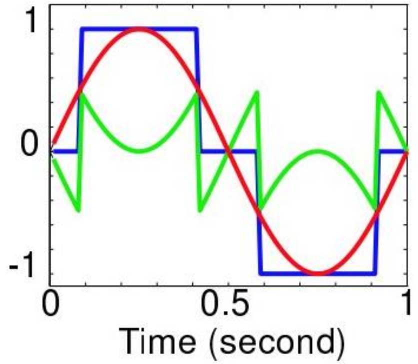
3 bits/sample




2 bits/sample




J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

# Quantization of Images

**8 bit 100%**

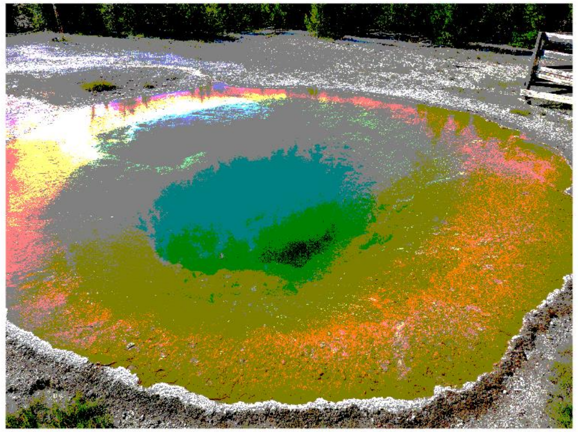
**5 bit 63%**

**4 bit 50%**

**3 bit 38%**

**2 bit 25%**

**1 bit 13%**



# Summary

- Converting continuous time signals into discrete signals
  - ➢ Sampling
  - ➢ Quantization
- Important new concepts:
  - ➢ Sampling rate (sampling frequency) $f_s$
  - ➢ Base band, Nyquist frequency
  - ➢ Aliasing and Anti-aliasing

We will now go to 4-370 for recitation & common hour