6.300 Signal Processing

Week 2, Lecture B:
Fourier Series – Complex Exponential Form

- Complex numbers
- Fourier series: Sinusoids to complex exponentials
- Delay property of Fourier series

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

Fourier Series

Previously: Representing periodic signals as weighted sums of sinusoids.

Synthesis Equation

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t) \quad \text{where } \omega_o = \frac{2\pi}{T}$$

Analysis Equations

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt$$

Q1: How to go from sinusoids to complex numbers?
Q2: Is it really simpler?

Today: Simplifying the math with complex numbers.

Simplifying Math By Using Complex Numbers – How?

Our biggest simplification comes from **Euler's formula**, which relates complex exponentials to trigonometric functions (Leonhard Euler, 1748).

$$e^{j\theta} = \cos\theta + j\sin\theta$$

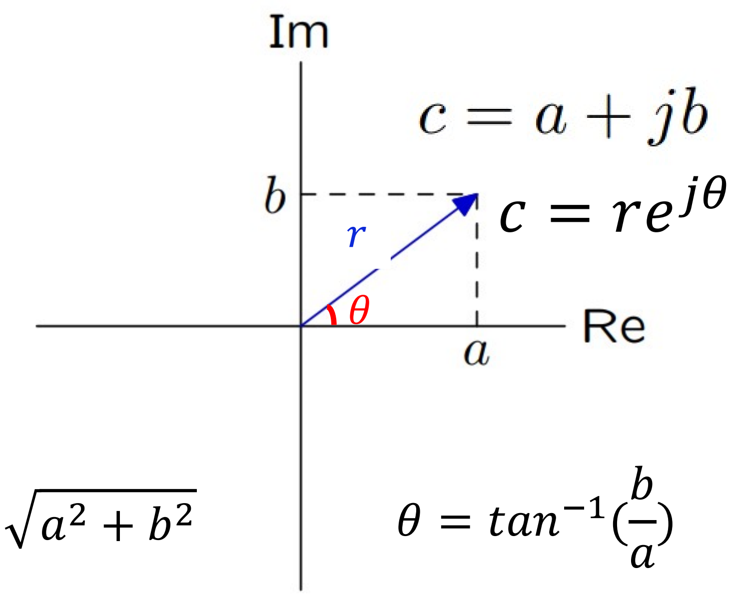
where $j = \sqrt{-1}$.

Richard Feynman called this ”the most remarkable formula in mathematics.”

Geometric Interpretation of Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Rectangular form



$r = \sqrt{a^2 + b^2}$

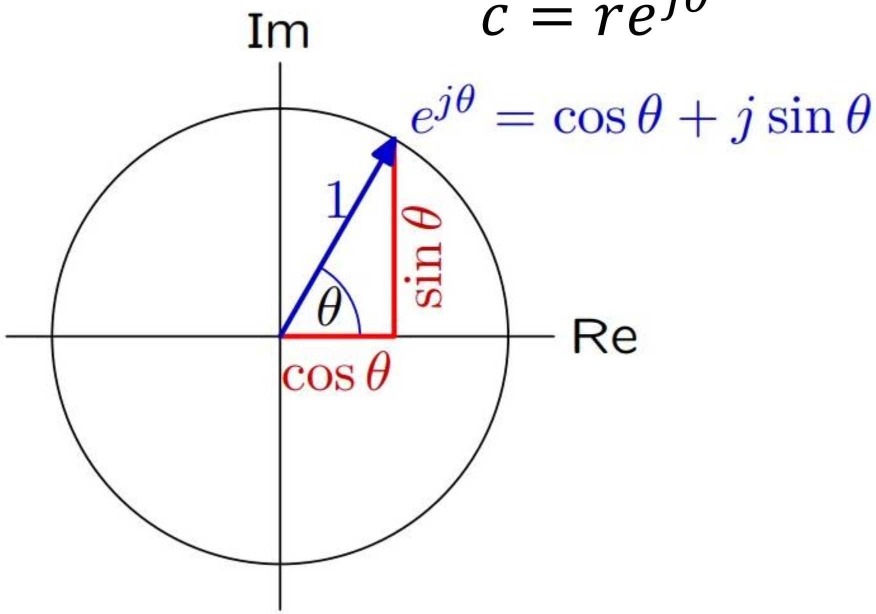
$\theta = tan^{-1}(\frac{b}{a})$

$a = rcos\theta$

$b = rsin\theta$

Polar form

$c = re^{j\theta}$



- Complex numbers are two-dimensional, and can be described as points in the complex plane.
- Two ways of describing a unit vector at angle θ in the complex plane: rectangular and polar form.

Addition

Q: which way is easier? Rectangular or Polar?

Addition: the real part of a sum is the sum of the real parts, and the imaginary part of a sum is the sum of the imaginary parts.

Let c_1 and c_2 represent complex numbers:

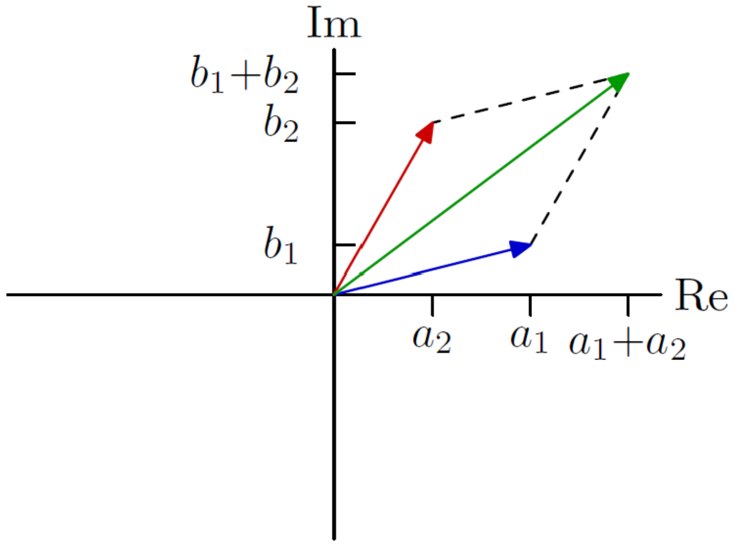
$$c_1 = a_1 + jb_1$$

$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1{+}a_2) + j(b_1{+}b_2)$$

Rules for adding complex numbers are same as those for adding vectors.



Multiplication

Q: which way is easier? Rectangular or Polar?

Multiplication is more complicated.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$

$$c_2 = a_2 + jb_2$$

Then

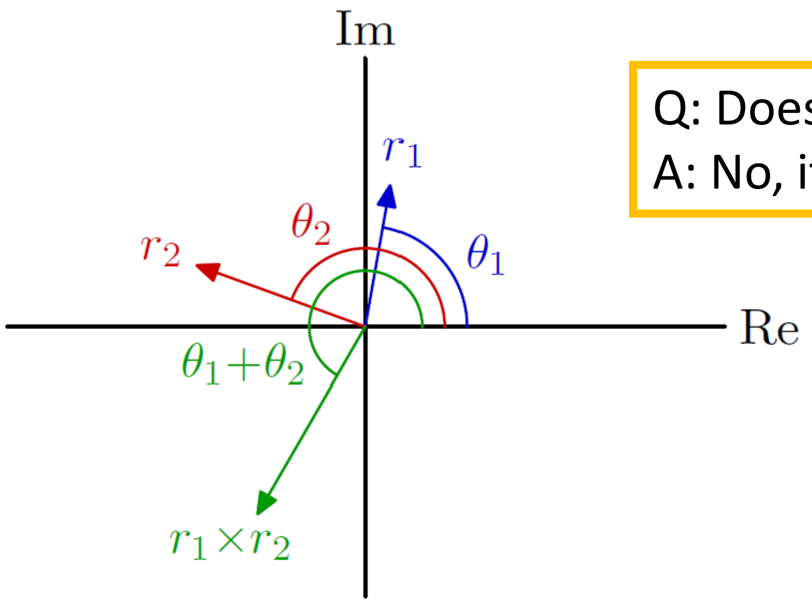
$$\begin{aligned} c_1 \times c_2 &= (a_1 + jb_1) \times (a_2 + jb_2) \\ &= a_1 \times a_2 + a_1 \times jb_2 + jb_1 \times a_2 + jb_1 \times jb_2 \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \end{aligned}$$

Although the rules of algebra still apply, the result is complicated:

- the real part of a product is NOT the product of the real parts, and
- the imaginary part is NOT the product of the imaginary parts.

Multiplication: Polar form

The magnitude of the product of complex numbers is the **product** of their magnitudes. The angle of a product is the **sum** of the angles.



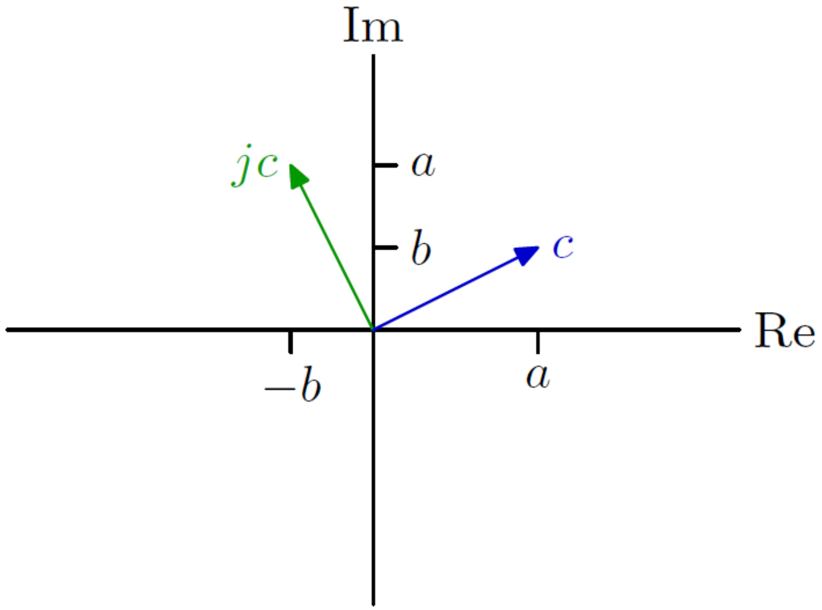
Q: Does phase change lead to magnitude change?
A: No, it's just a rotation of the vector!

$$
\begin{aligned}
r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} &= r_1(\cos\theta_1 + j\sin\theta_1) \times r_2(\cos\theta_2 + j\sin\theta_2) \\
&= r_1 r_2 (\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2}_{j\sin(\theta_1+\theta_2)}) \\
&= r_1 r_2 e^{j(\theta_1+\theta_2)}
\end{aligned}
$$

Multiplication of Complex Numbers

E.g. Multiply a complex number by j. let's first try rectangular form:

$$c = a + jb$$
$$jc = ja - b$$



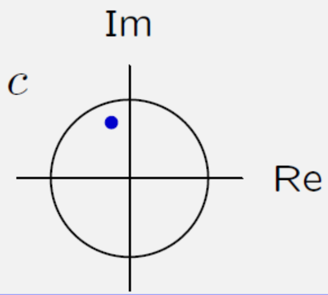
$e^{j2\pi} = 1; e^{j\pi} = -1; e^{j\pi/2} = j;$

Q: Is there an easier way to do it?

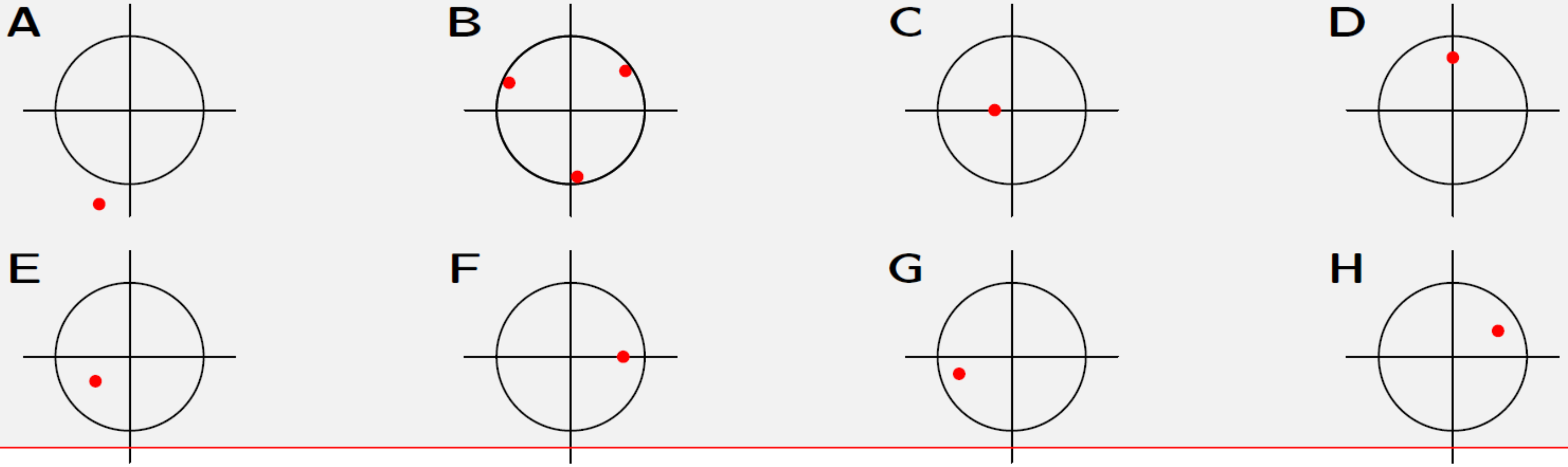
Multiplying by j rotates an arbitrary complex number by $\pi/2$.

Check yourself

Let c represent the complex number shown by a filled dot in the complex plane below, where the circle has radius 1.



Which if any of the following figures shows the value of jc?
Which if any of the following figures shows the value of $\text{Im}(c)$?
Which if any of the following figures shows the value of $1/c$?



How to go from trig form to CE form for CTFS

Substitute complex exponentials for trigonometric functions.

$$f(t) = c_0 + \sum_{k=1}^{\infty} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$

$$= c_0 + \sum_{k=1}^{\infty} \Big(c_k \underbrace{\frac{1}{2}(e^{jk\omega_o t} + e^{-jk\omega_o t})}_{\cos(k\omega_o t)} + d_k \underbrace{\frac{1}{2j}(e^{jk\omega_o t} - e^{-jk\omega_o t})}_{\sin(k\omega_o t)} \Big)$$

$$= c_o + \sum_{k=1}^{\infty} \frac{c_k - jd_k}{2} e^{jk\omega_o t} + \sum_{k=1}^{\infty} \frac{c_k + jd_k}{2} e^{-jk\omega_o t}$$

$$= c_o + \sum_{k=1}^{\infty} \frac{c_k - jd_k}{2} e^{jk\omega_o t} + \sum_{k=-1}^{-\infty} \frac{c_{-k} + jd_{-k}}{2} e^{+jk\omega_o t}$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \quad \text{where} \quad a_k = \begin{cases} \frac{1}{2}(c_k - jd_k) & \text{if } k > 0 \\ c_0 & \text{if } k = 0 \\ \frac{1}{2}(c_{-k} + jd_{-k}) & \text{if } k < 0 \end{cases}$$

The trig form of the Fourier series (top of page) has an equivalent form with complex exponentials (red).

Let's try it!

$$e^{j\theta} = cos\theta + jsin\theta$$

$$e^{-j\theta} = cos\theta - jsin\theta$$

$$cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} = -j\frac{e^{j\theta} - e^{-j\theta}}{2}$$

Meaning for Negative k

The complex exponential form of the series has positive and negative k's.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Only positive values of k are used in the trig form.

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

Q: Why? What does negative k mean?

The negative k's are required by Euler's formula.

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$

$$\cos(k\omega_o t) = \mathrm{Re}\{e^{jk\omega_o t}\} = \frac{1}{2}\left(e^{jk\omega_o t} + e^{-jk\omega_o t}\right)$$

$$\sin(k\omega_o t) = \mathrm{Im}\{e^{jk\omega_o t}\} = \frac{1}{2j}\left(e^{jk\omega_o t} - e^{-jk\omega_o t}\right)$$

The negative k do not indicate negative frequencies. They are the mathematical result of representing sinusoids with complex exponentials.

Simplifying Math By Using Complex Numbers

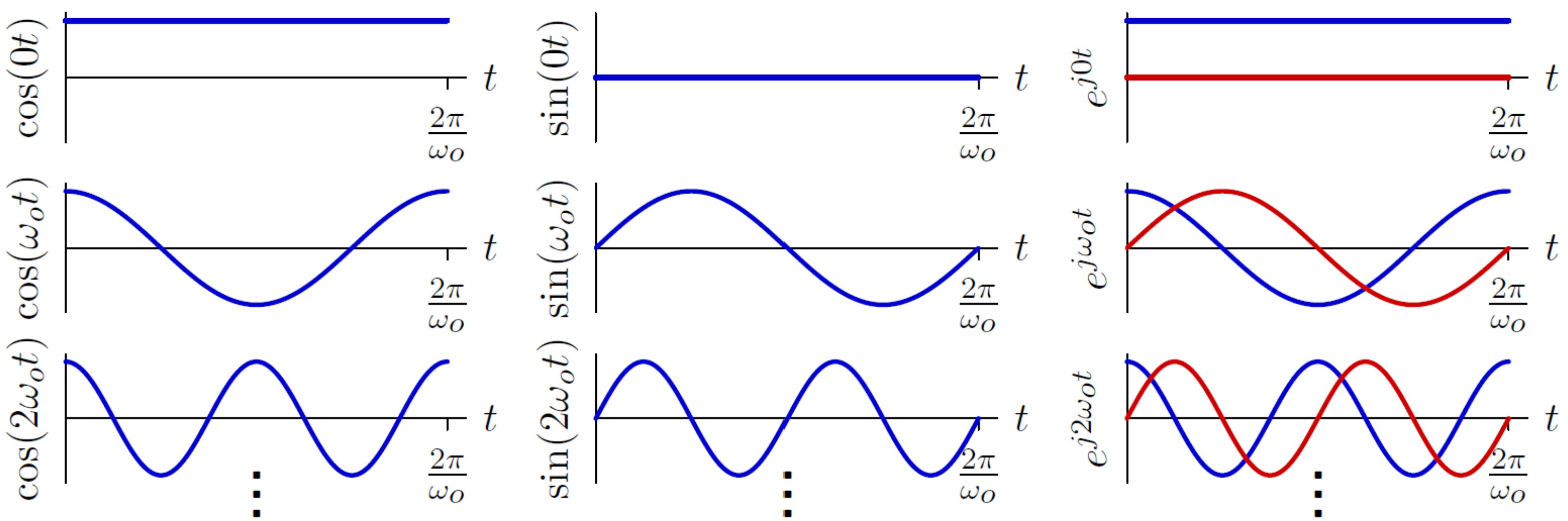
Euler's formula allows us to represent both sine and cosine basis functions with a single complex exponential:

$$f(t) = \sum \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big) = \sum a_k e^{jk\omega_o t}$$

Q: What's the difference?

Real-valued basis functions

Complex basis functions



- Potentially simpler math
  - Cosine and Sine all-in-one
  - No need to memorize trig identities
- Negative k

Fourier Series Directly from Complex Exponential Form

Assume that $f(t)$ is periodic in T and is composed of a weighted sum of harmonically related complex exponentials.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}$$

We can "sift" out the component at $l\omega_o$ by multiplying both sides by $e^{-jl\omega_o t}$ and integrating over a period.

$$\int_T f(t)e^{-j\omega_o lt}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt} e^{-j\omega_o lt} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_o (k-l)t} dt$$

$$= \begin{cases} Ta_l & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

Solving for a_l provides an explicit formula for the coefficients:

$$a_k = \frac{1}{T}\int_T f(t)e^{-j\omega_o kt}dt\,; \quad \text{where } \omega_o = \frac{2\pi}{T}\,.$$

Orthogonal decompositions

Vector representation: let $\bar{r}$ represent a vector with components a and b in the $\hat{x}$ and $\hat{y}$ directions, respectively.

$$a = \bar{r} \cdot \hat{x}$$
$$b = \bar{r} \cdot \hat{y}$$

("analysis" equations)

$$\bar{r} = a\hat{x} + b\hat{y}$$

("synthesis" equation)

Fourier series: let $f(t)$ represent a signal with harmonic components $a_0, a_1, \ldots, a_k$ for harmonics $e^{j0t}, e^{j\frac{2\pi}{T}t}, \ldots, e^{j\frac{2\pi}{T}kt}$ respectively.

$$a_k = \frac{1}{T} \int_T f(t) e^{-j\frac{2\pi}{T}kt} dt$$

("analysis" equation)

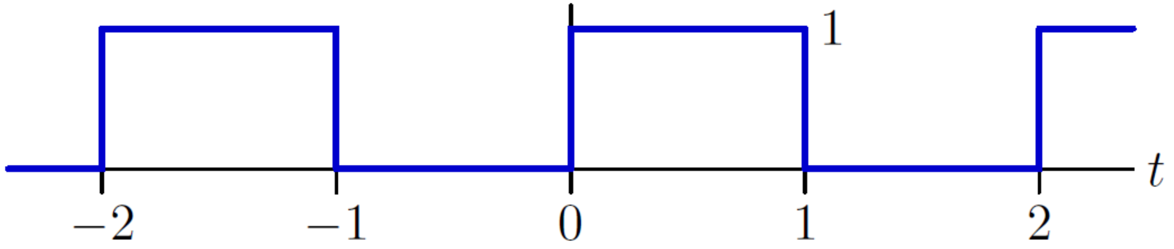
$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

("synthesis" equation)

Fourier analysis of a square wave

We previously used trig functions to find the Fourier series for $f(t)$ below:

$$f(t) = f(t+2)$$



$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

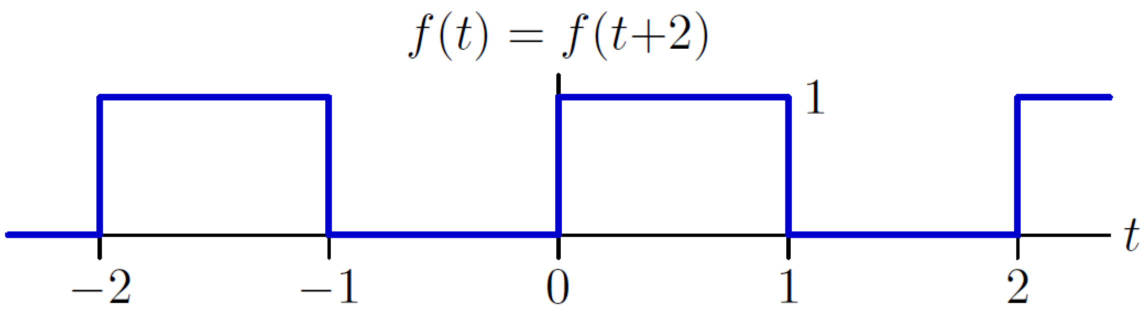
$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

Fourier analysis of a square wave

Now try complex exponentials.



Continuous Time Fourier series (CTFS)

Comparison of trigonometric and complex exponential forms.

Complex Exponential Form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T f(t) e^{-jk\omega_o t} dt$$

Or more often:

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi k t}{T}}$$

$$X[k] = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi k t}{T}} dt$$

Trigonometric Form

$$f(t) = f(t+T) = c_o + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

$$c_0 = \frac{1}{T}\int_T f(t)\, dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\, dt; \quad k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\, dt; \quad k = 1, 2, 3, \ldots$$

Is the complex exponential form actually easier?

Let’s consider the effect of a half-period shift on the Fourier coefficients of the trig form vs CE form:

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T)\,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2)\,.$$

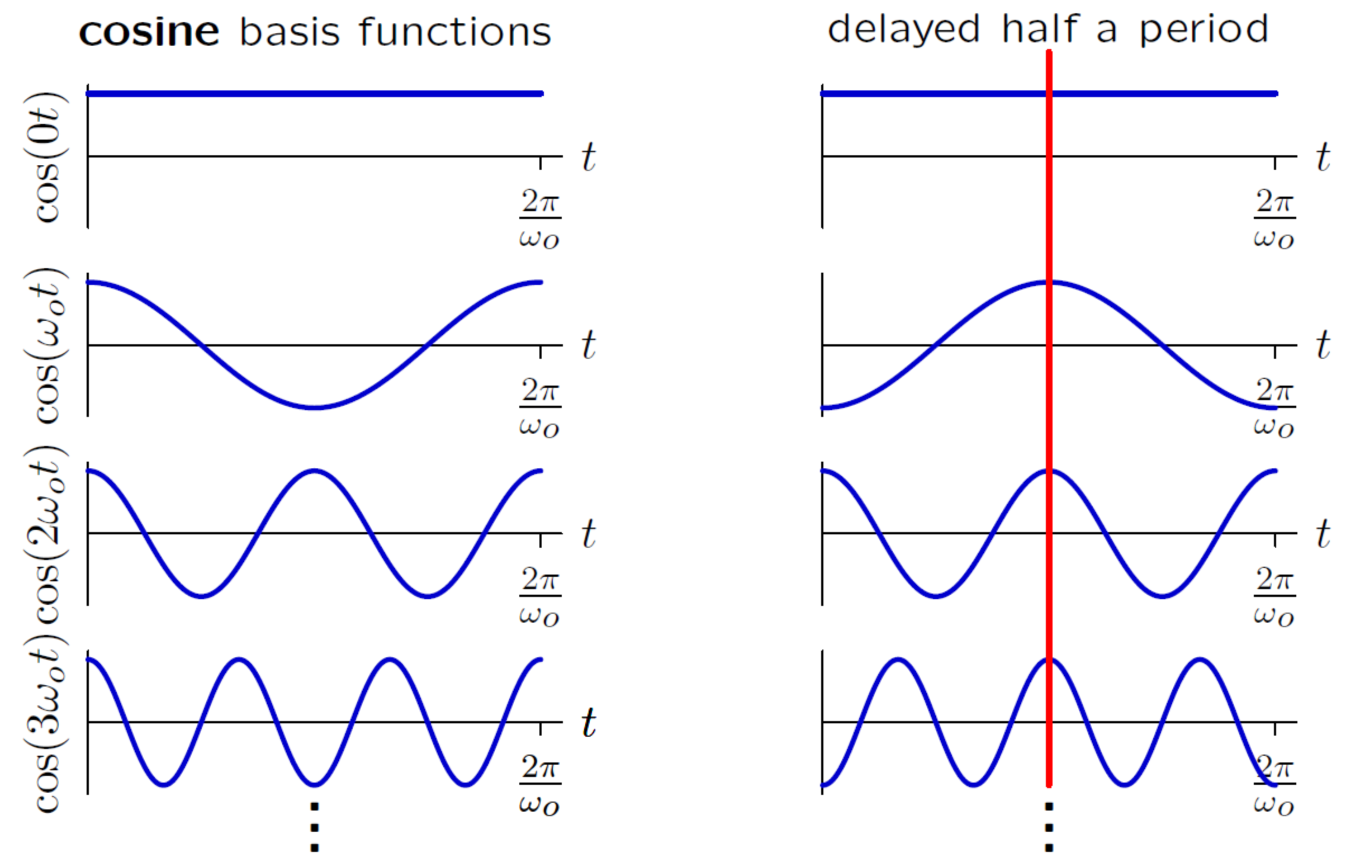
How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated
- sine coefficients d_k are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$

Alternative (more intuitive) approach

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2) = \sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t-T/2)\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t-T/2)\right)$$

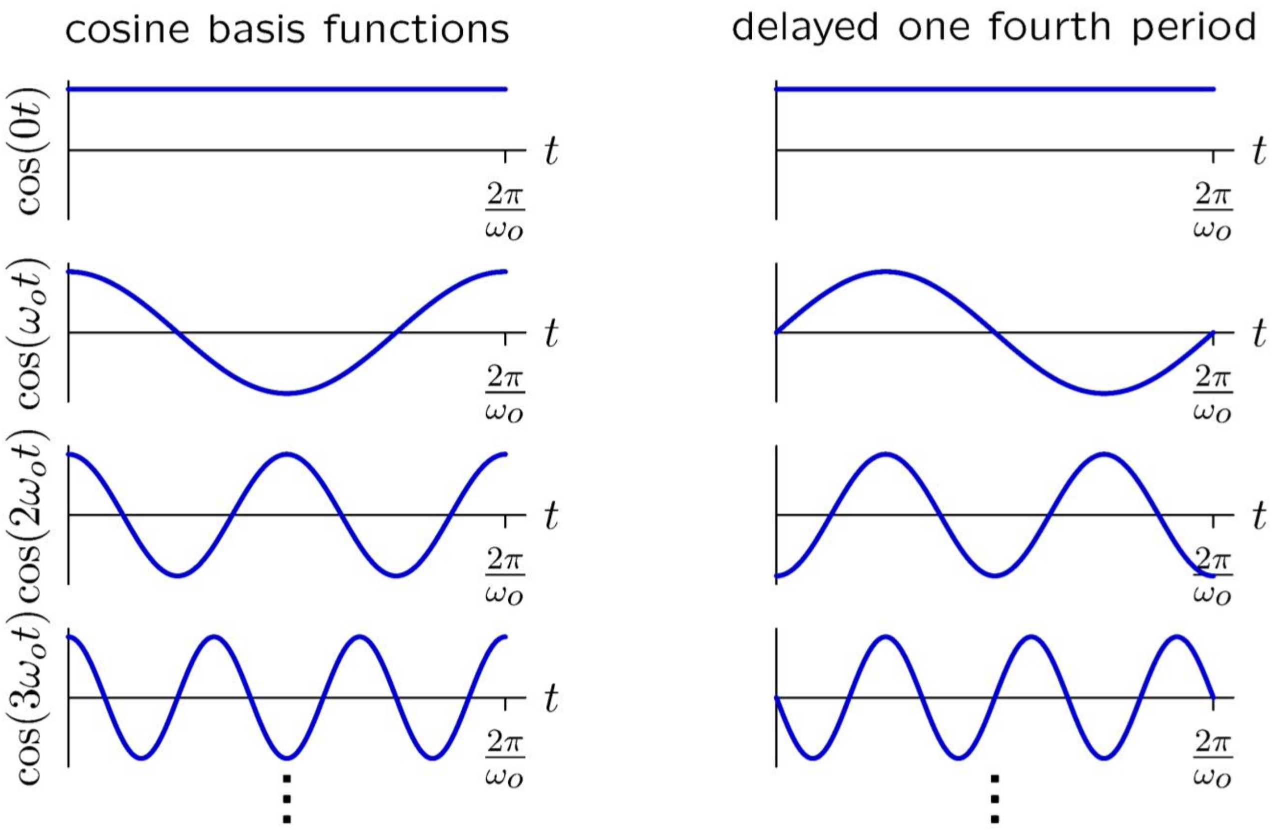


Half-period shift inverts odd harmonics. No effect on even harmonics.

Quarter-period shift

Shifting by $T/4$ is **even more complicated**.

$$f(t - T/4) = \sum_{k=0}^{\infty} c_k \cos\left(k\omega_o(t - T/4)\right) + \sum_{k=1}^{\infty} d_k \sin\left(k\omega_o(t - T/4)\right)$$



$$\cos(\omega_o t) \rightarrow \sin(\omega_o t); \quad \cos(2\omega_o t) \rightarrow -\cos(2\omega_o t); \quad \cos(3\omega_o t) \rightarrow -\sin(3\omega_o t)$$

Eighth-period shift

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t)$$

and c_k''' and d_k''' represent those for an eighth-period delay.

$$g(t) = f(t - T/8) = c_0 + \sum_{k=1}^{\infty} c_k''' \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k''' \sin(k\omega_o t)$$

$$c_k''' = \begin{cases} c_k & \text{if } k = 0, 8, 16, 24, \ldots \\ \frac{\sqrt{2}}{2}(c_k + d_k) & \text{if } k = 1, 9, 17, 25, \ldots \\ d_k & \text{if } k = 2, 10, 18, 26, \ldots \\ \frac{\sqrt{2}}{2}(-c_k + d_k) & \text{if } k = 3, 11, 19, 27, \ldots \\ -c_k & \text{if } k = 4, 12, 20, 28, \ldots \\ \frac{\sqrt{2}}{2}(-c_k - d_k) & \text{if } k = 5, 13, 21, 29, \ldots \\ -d_k & \text{if } k = 6, 14, 22, 30, \ldots \\ \frac{\sqrt{2}}{2}(c_k - d_k) & \text{if } k = 7, 15, 23, 31, \ldots \end{cases} \qquad d_k''' = \ldots$$

Properties of CTFS: Time Shift

- Consider $y(t) = x(t - t_0)$, where x is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Properties of CTFS: Time Derivative

- Consider $y(t) = \frac{d}{dt}x(t)$, where $x(t)$ and $y(t)$ are periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$?

Real-valued periodic signal

If $f(t)$ is real valued periodic signal, what properties do its CTFS coefficients $F[k]$ have?

Properties of CTFS: Time flip(reversal)

- Consider $y(t) = x(-t)$, where $x(t)$ is periodic in T. What are the CTFS coefficients $Y[k]$, in terms of $X[k]$? **Participation question for Lecture**

Summary

- Complex numbers
- Complex exponentials and their relation to sinusoids
- Analysis and synthesis with complex exponentials
- Various properties of CTFS (using complex exponential form)

We will now go to 4-370 for recitation & common hour