6.300 Signal Processing

Week 2, Lecture A:
Continuous-Time Fourier Series (Trig Form)

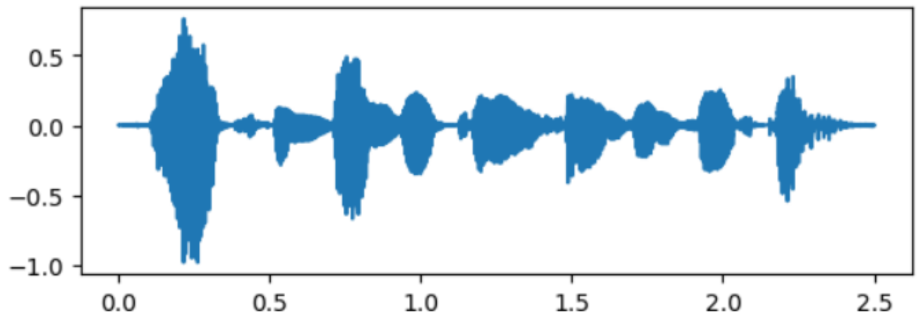
- Fourier Series
- Convergence of Fourier Series
- Symmetry of Fourier Series

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

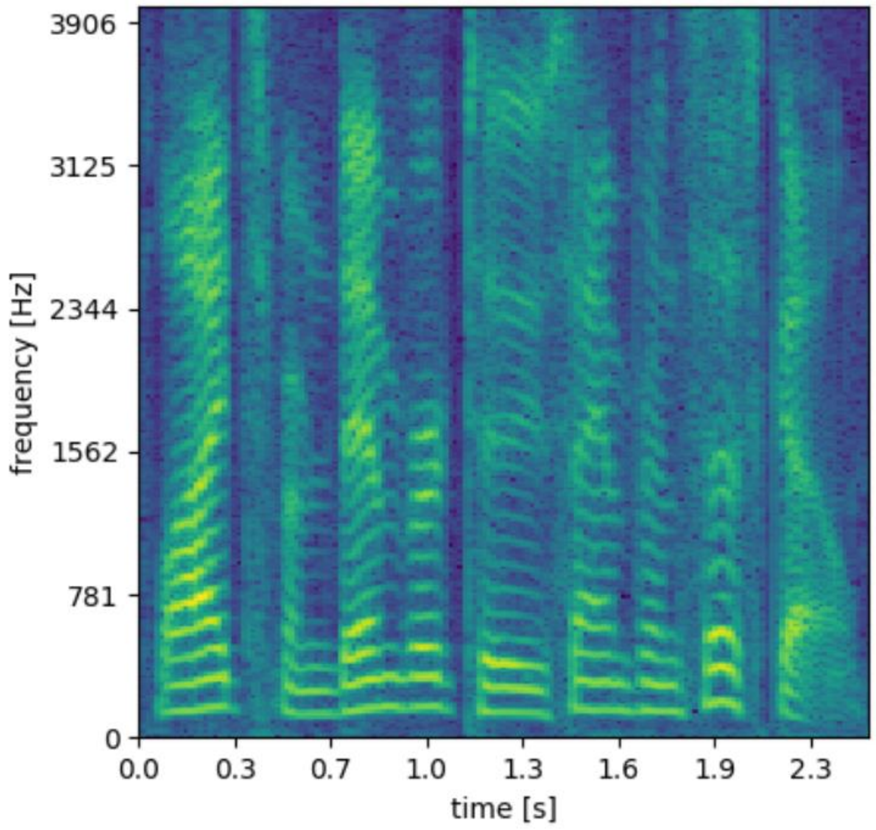
Last time: Two different ways of looking at a signal

- E.g. Two representations of a speech signal:

Time domain



"Frequency" domain



- Today: we will focus on Continuous-time Fourier series

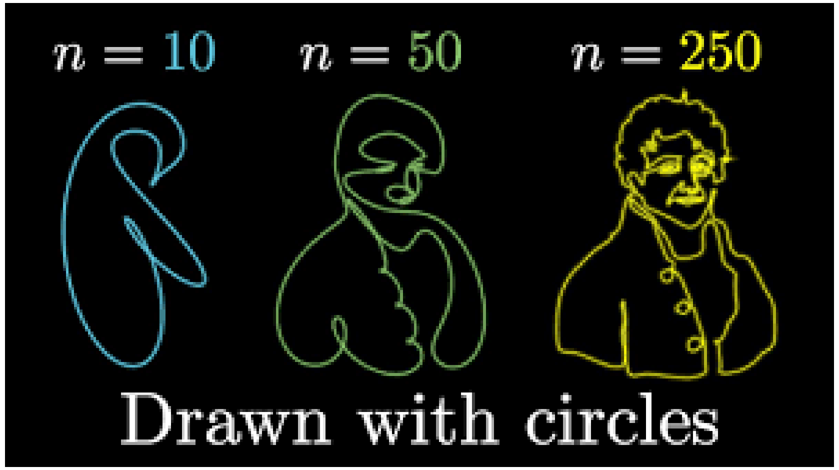
Fourier Series

Series: representing a signal as a sum of simpler signals.

- Taylor or Maclaurin's series

Function	Maclaurin Series
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
$\sin x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
$\cos x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$ (if $-1 < x < 1$)
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (if $-1 < x \le 1$)

- Draw only with circles



- Fourier series are sums of harmonically related sinusoids:

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

Why focus on Fourier Series

- What's so special about sines and cosines?
  - ➢ Sinusoidal functions have interesting mathematical properties.
    - ➢ Harmonically related sinusoids are orthogonal to each other over [0, T]
- Average over a period:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) dt = 0 \qquad \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) dt = \begin{cases} T & if\ k=0 \\ 0 & otherwise \end{cases}$$

- Orthogonality of the basis functions:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = 0$$

k and m are positive integers

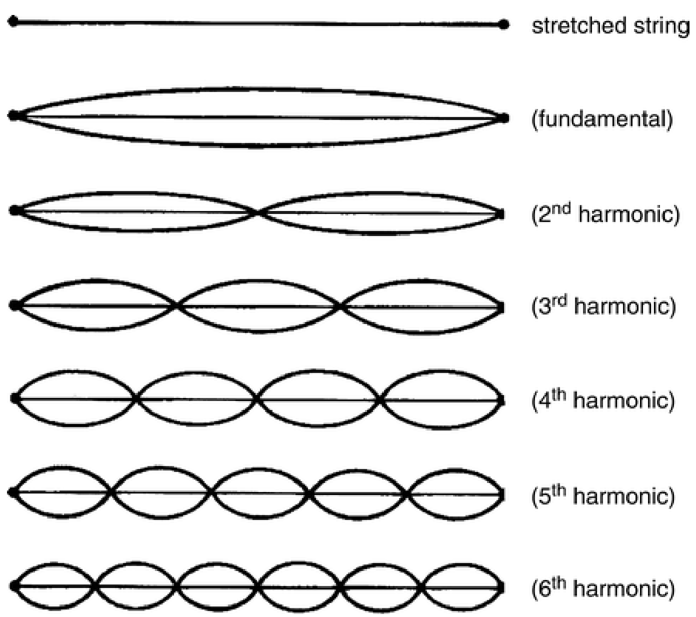
$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k=m, \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \sin\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k=m, \\ 0 & \text{otherwise} \end{cases}$$

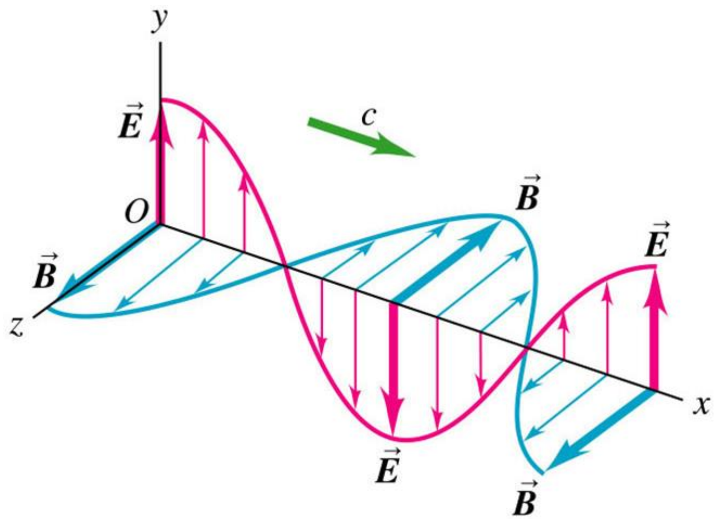
Why focus on Fourier Series

- Sines and cosines have interesting mathematical properties – orthogonality.
- Sines and cosines also play important roles in physics – especially the physics of waves.

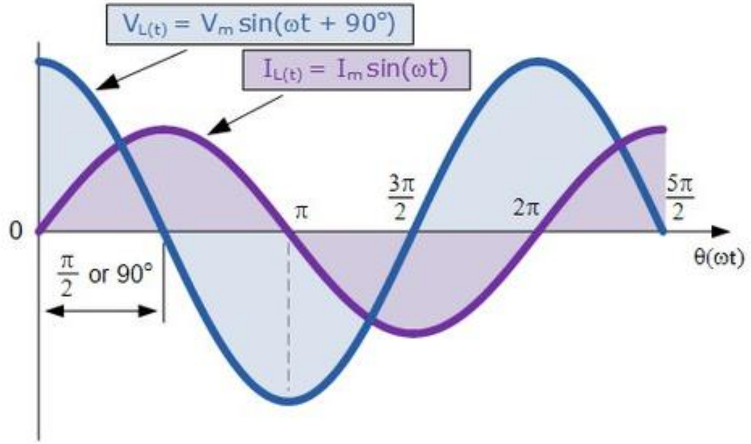
Vibrating string



Light waves



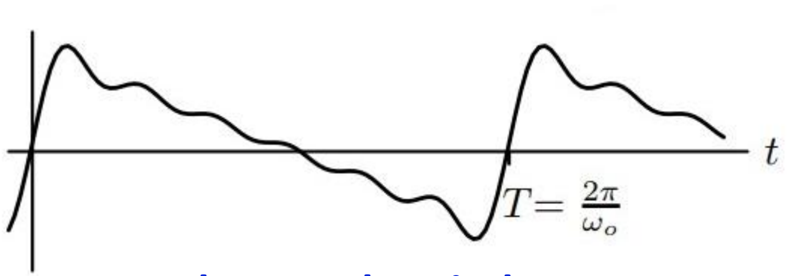
Electrical waves



Last time: Express periodic signals as a sum of sinusoids

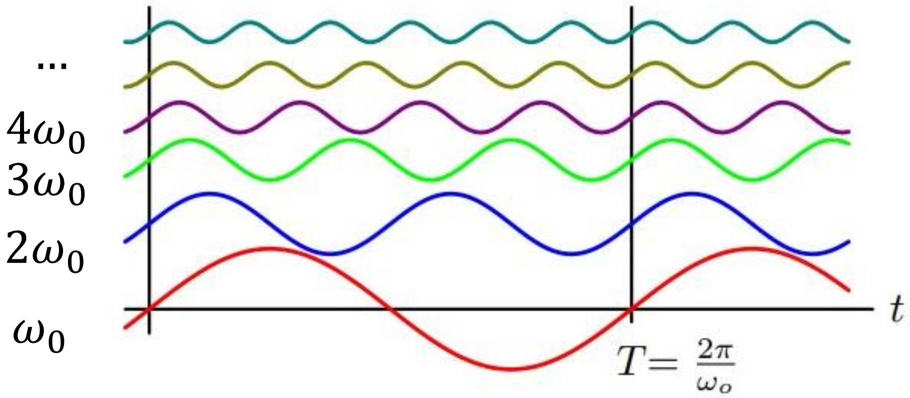
Periodic signal: $f(t) = f(t+T)$

CTFS: $f(t) \rightarrow c_k, d_k$



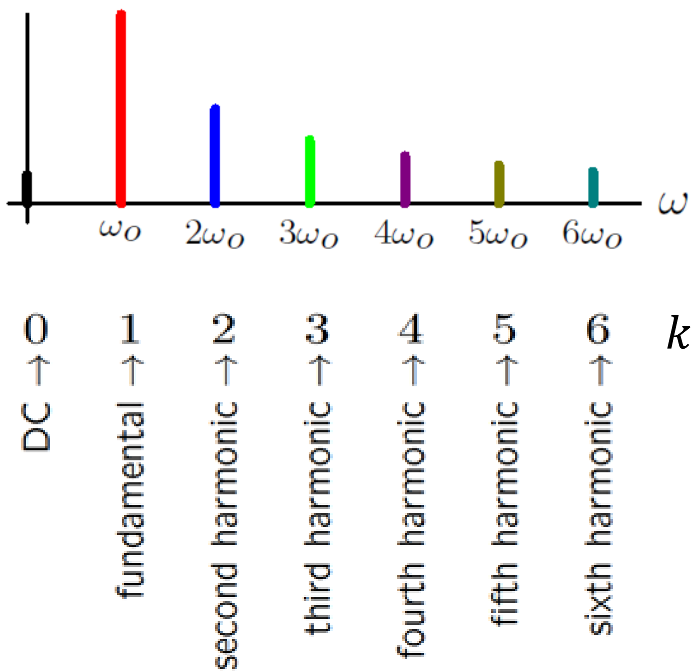
- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$

Basis function $\cos(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

Weights c_k for $\cos(k\omega_0 t)$



Decomposition:

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)\right)$$

Continuous-Time Fourier Series (CTFS) Trig Form

- **Synthesis equation**

Fourier series are weighted sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

where $\omega_o = 2\pi/T$ represents the fundamental frequency.

- **Analysis equation**

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt;\;\; k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt;\;\; k = 1, 2, 3, \ldots$$

Check yourself!

- What are the Fourier series coefficients associated with the following signal?

$$f(t) = 0.8\sin(6\pi t) - 0.3\cos(6\pi t) + 0.75\cos(12\pi t)$$

$$\omega_o = \ ?$$

$$c_k = ?$$

$$d_k = ?$$

Check yourself!

- What are the Fourier series coefficients associated with the following signal?

$$f(t) = 0.8\sin(6\pi t) - 0.3\cos(6\pi t) + 0.75\cos(12\pi t)$$

$$\omega_o = 6\pi$$

$$c_0 = 0$$

$$c_1 = -0.3$$

$$c_2 = 0.75$$

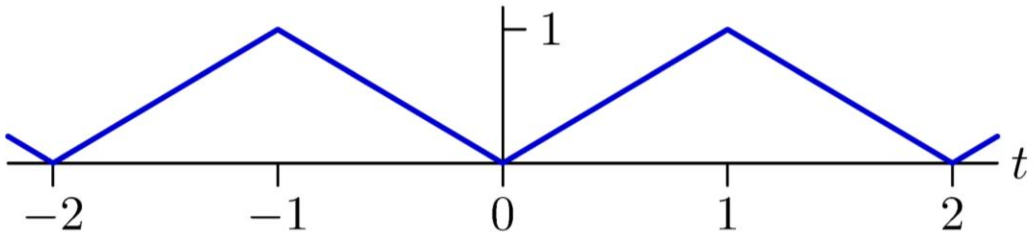
$$d_1 = 0.8$$

All the other c_k 's and d_k 's are zero.

Example of synthesis

Find the Fourier series coefficients for the following triangle wave:

$$f(t) = f(t+2)$$



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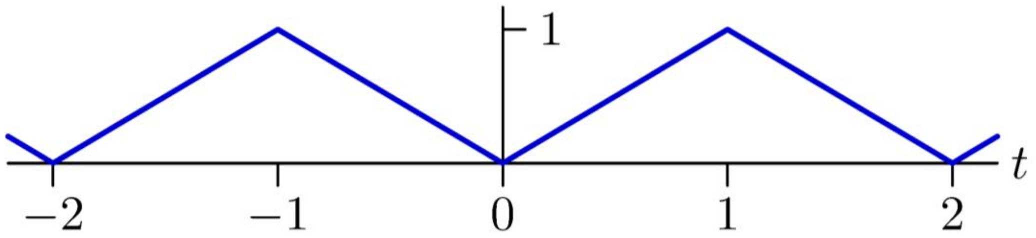


$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt = 2\int_0^1 t\cos(\pi kt)\,dt = \begin{cases} -\frac{4}{\pi^2k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$$d_k = 0 \quad \text{(by symmetry)}$$

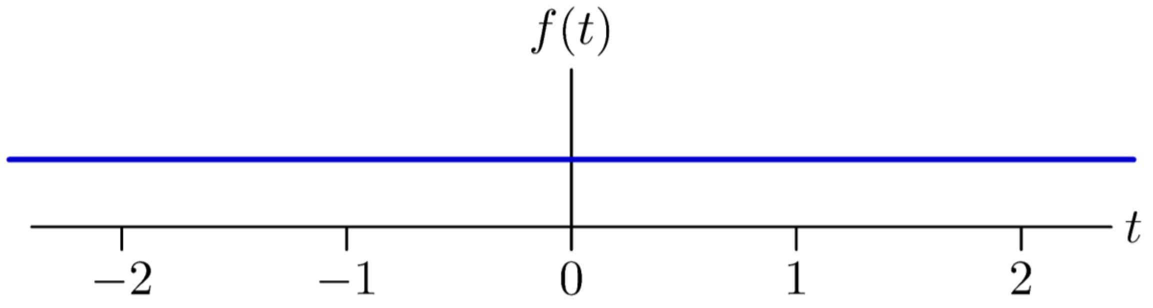
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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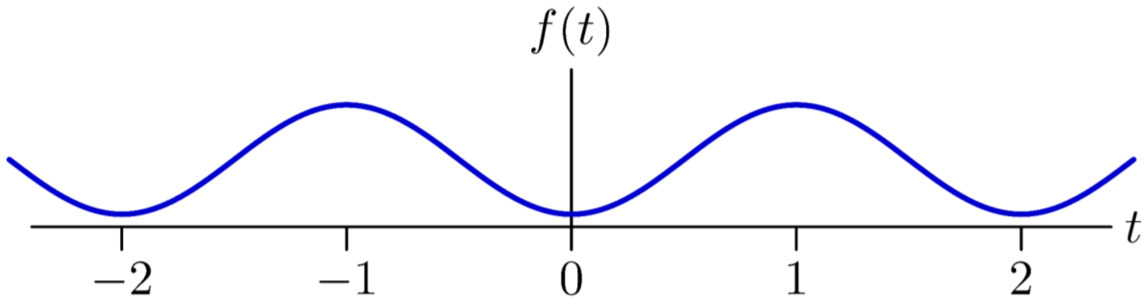
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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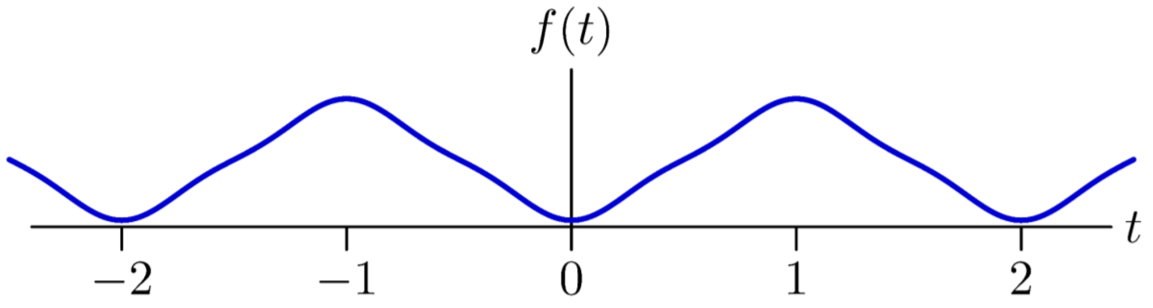

Example of synthesis

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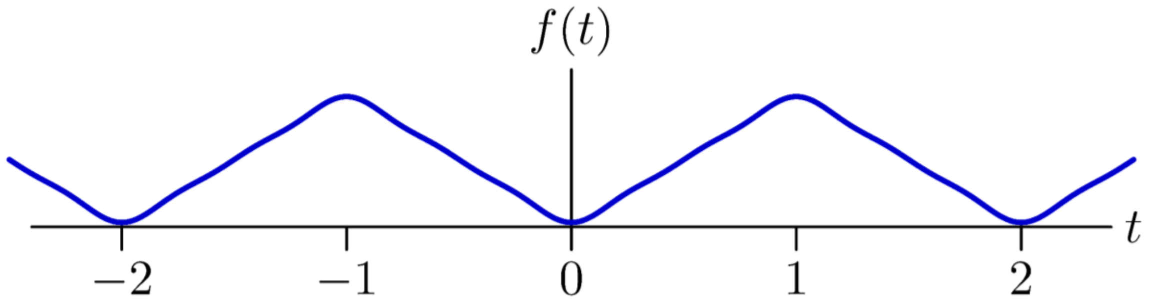
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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Generate $f(t)$ from the Fourier coefficients in the previous slide.

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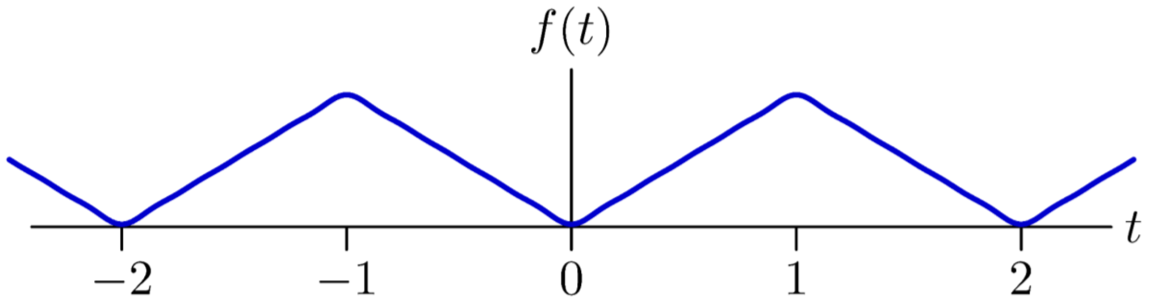
$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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Generate $f(t)$ from the Fourier coefficients in the previous slide.

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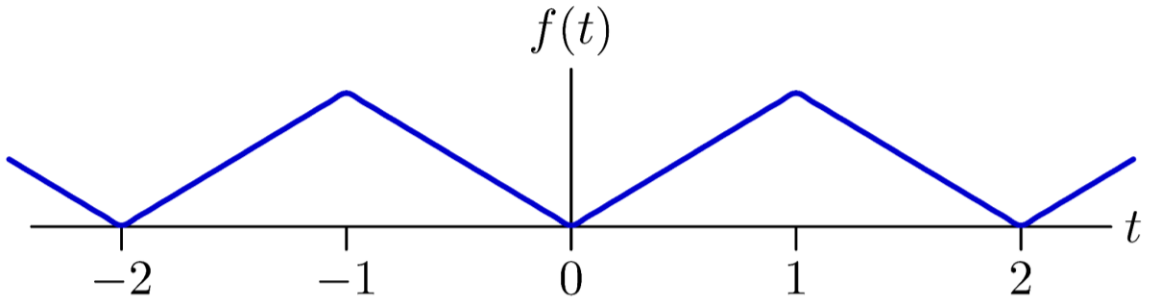
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



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Generate $f(t)$ from the Fourier coefficients in the previous slide.

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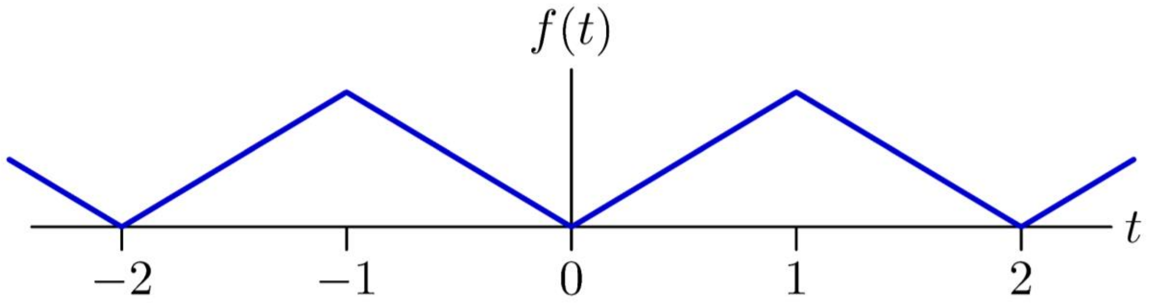
$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

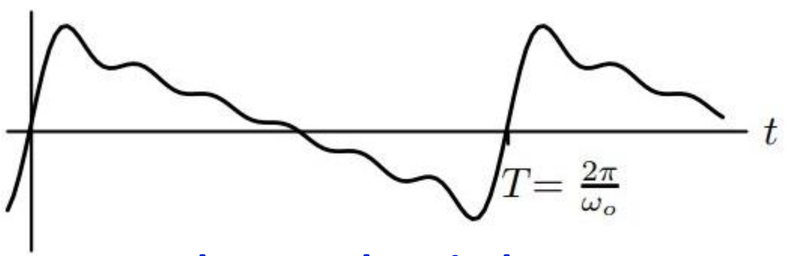


The synthesized function approaches original as number of terms increases.

Can Fourier Series approximate any periodic signals?

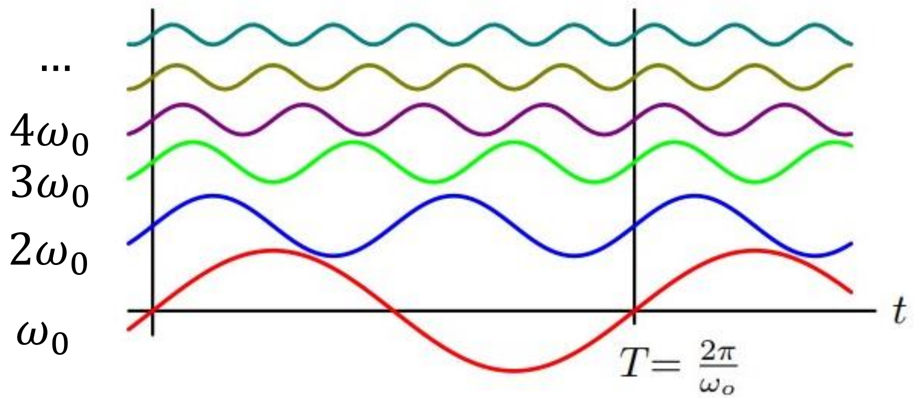
$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

Periodic signal: $f(t) = f(t+T)$



- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$

Basis function $\cos(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

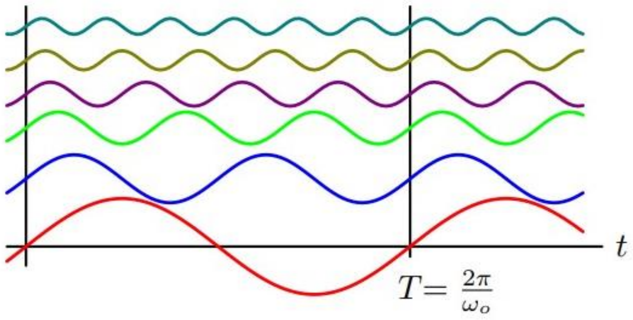
What about discontinuous functions?

The previous example shows that the sum of an infinite number of sinusoids can approximate a piecewise linear function **with discontinuous slope**!

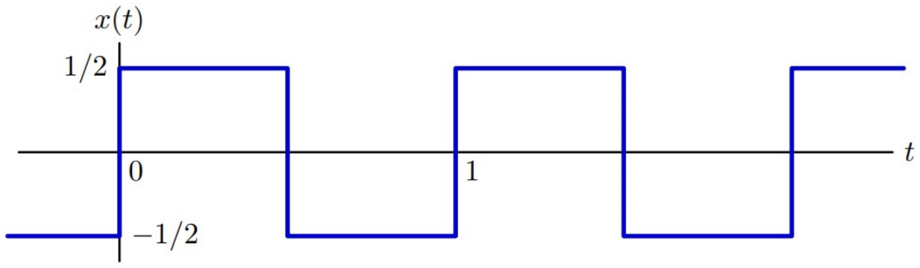
This result is a bit surprising since none of the basis functions have discontinuous slopes.

What about signals **with discontinuous values**?

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$



Continuous → discontinuous ?



A debate two hundred years ago…

Fourier defended the idea that such a series is meaningful.

Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

Not a problem

Jean-Baptiste Joseph Fourier

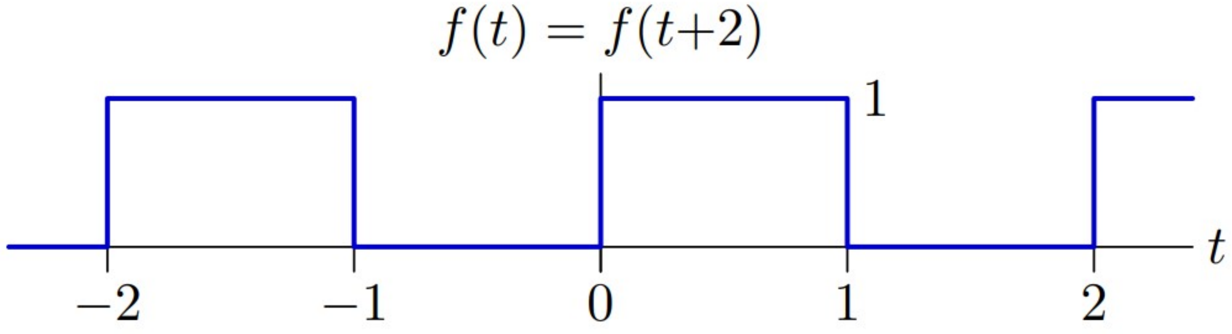
Joseph-Louis Lagrange

No way

Q: What do you think?

We can test this idea empirically – using computation

- Find the Fourier series coefficients for the following square wave:



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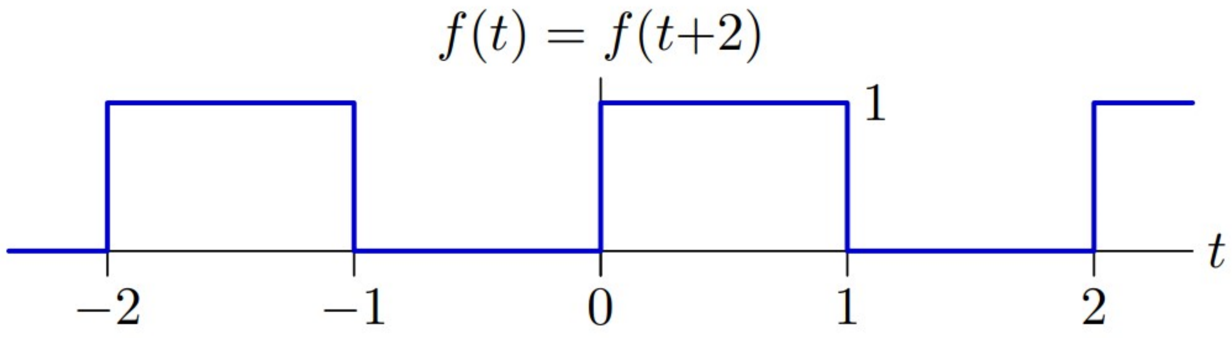

$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

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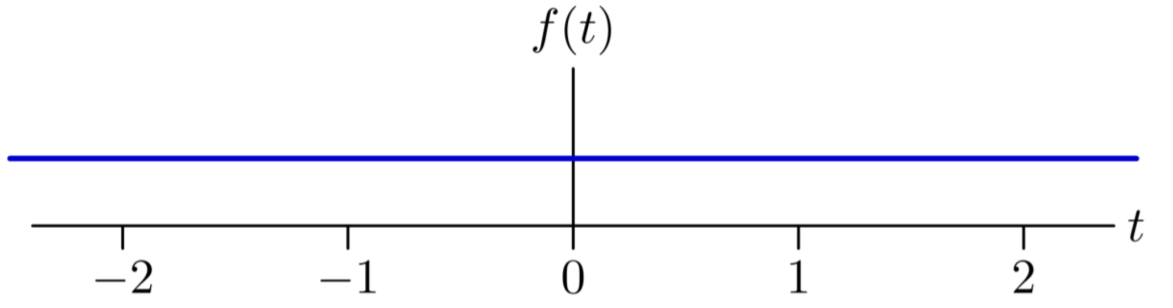

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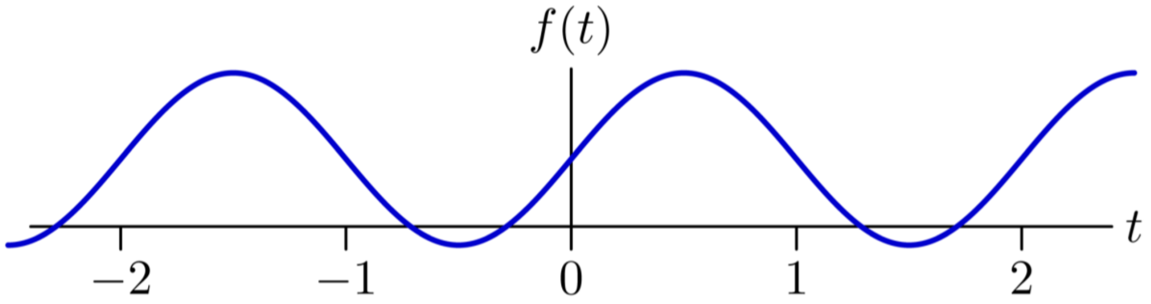

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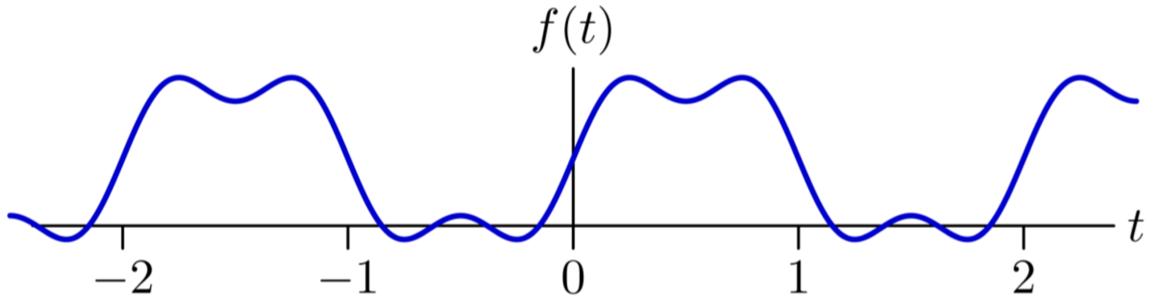

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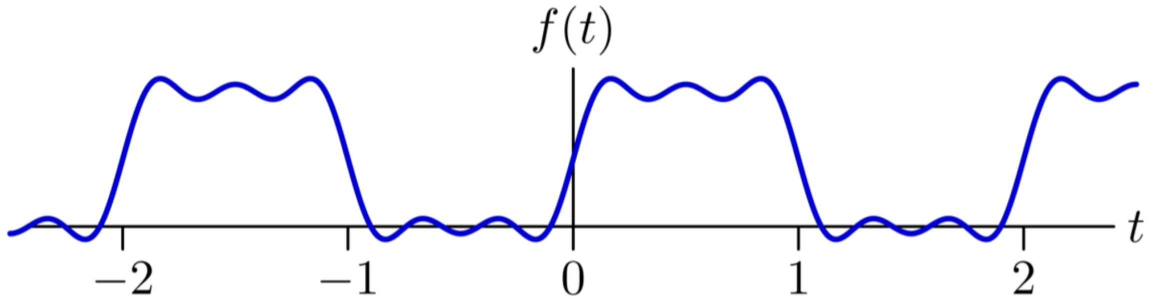

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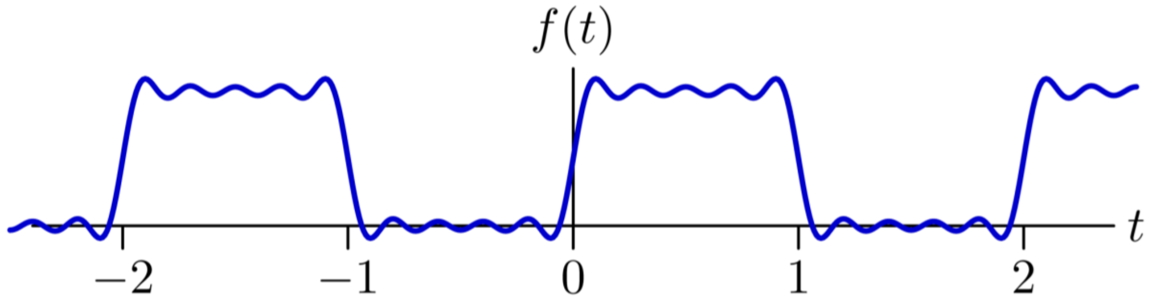

Fourier Synthesis of a Square Wave

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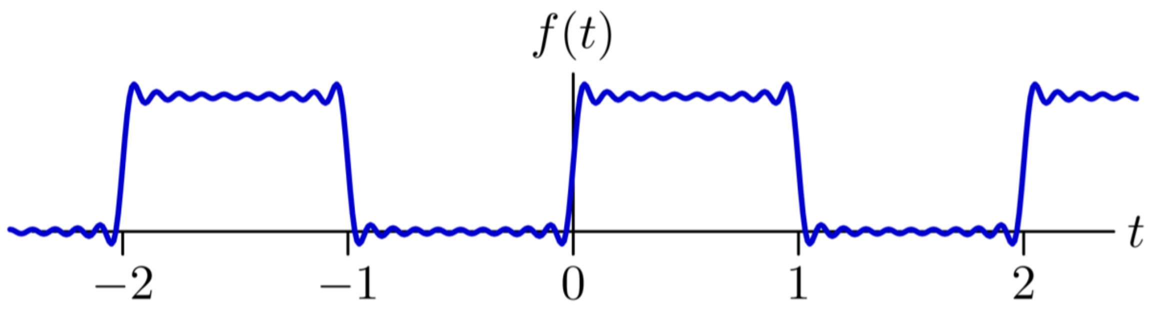

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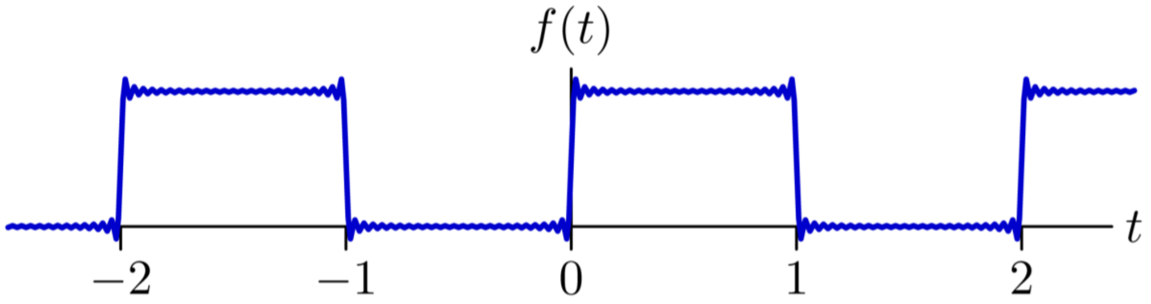
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{49} \frac{2}{k\pi} \sin(k\pi t)$$

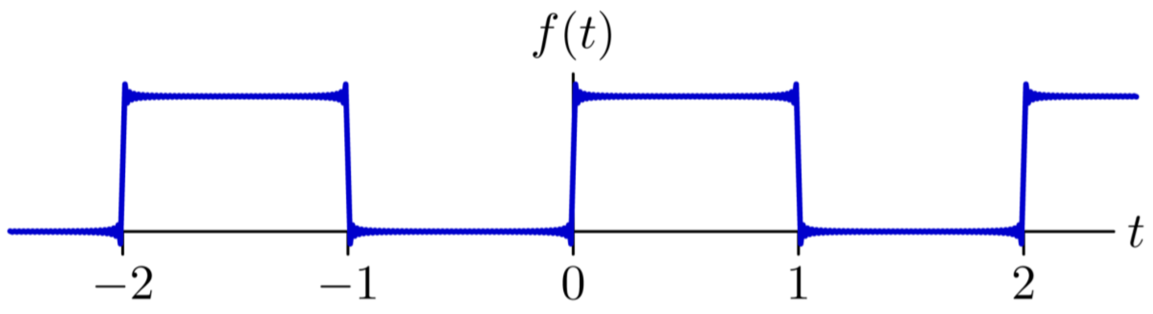


Fourier Synthesis of a Square Wave

- Generate f(t) from the Fourier coefficients in the previous slide:

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$

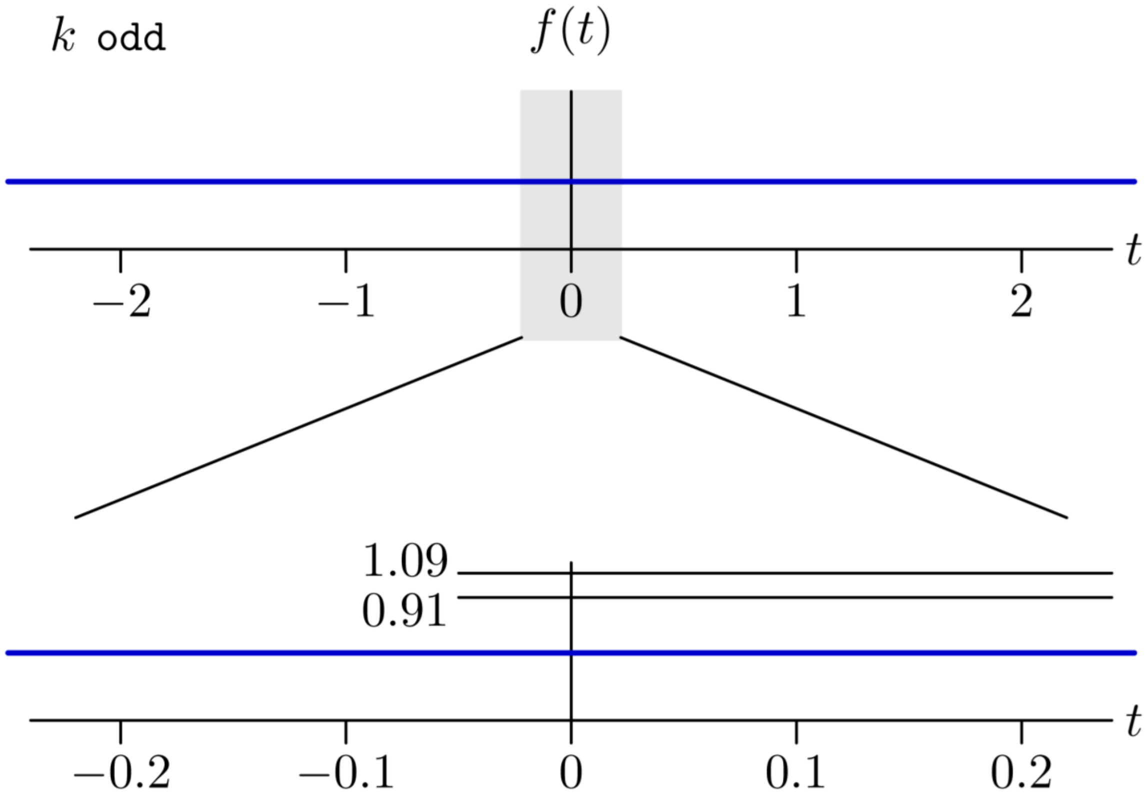


The synthesized function approaches original as number of terms increases.

Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

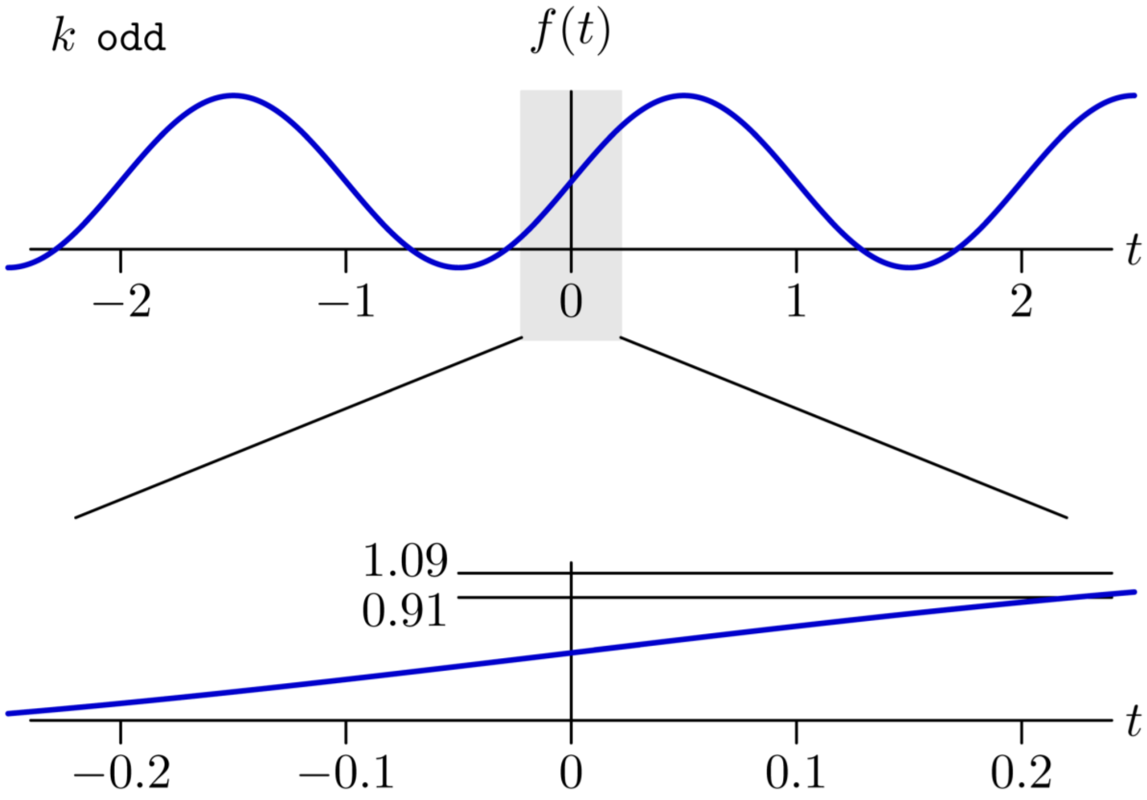
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

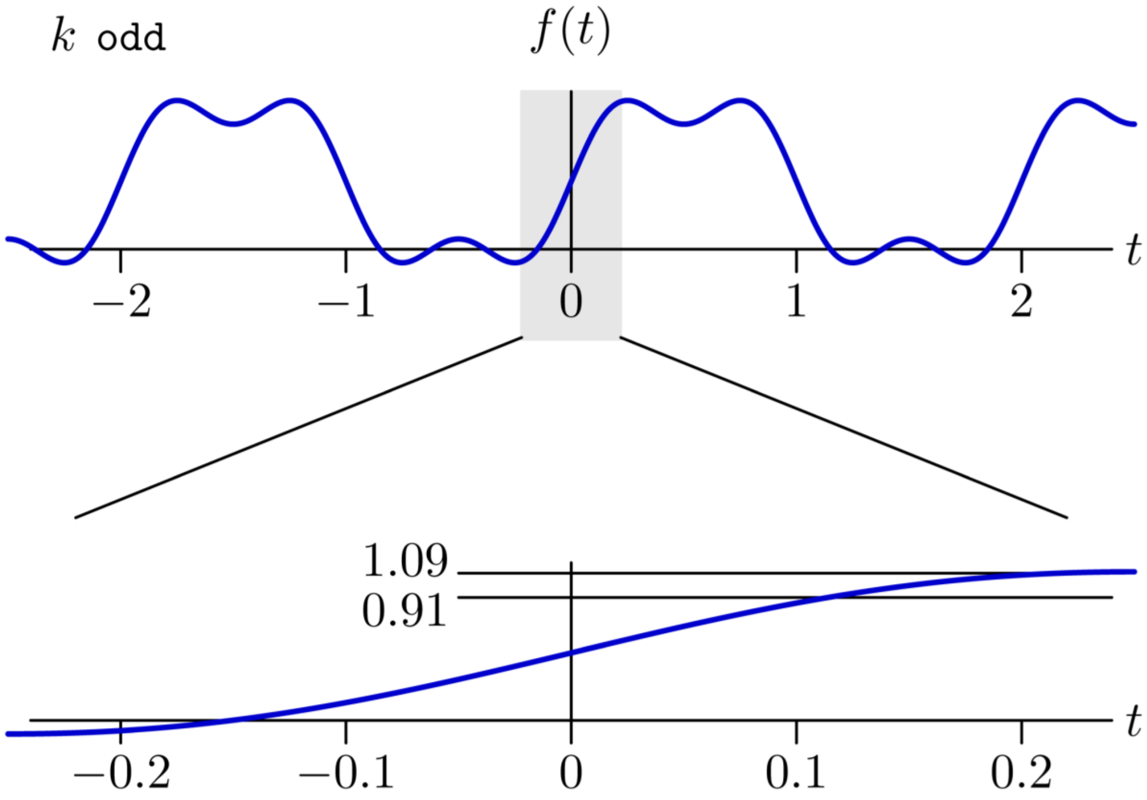
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

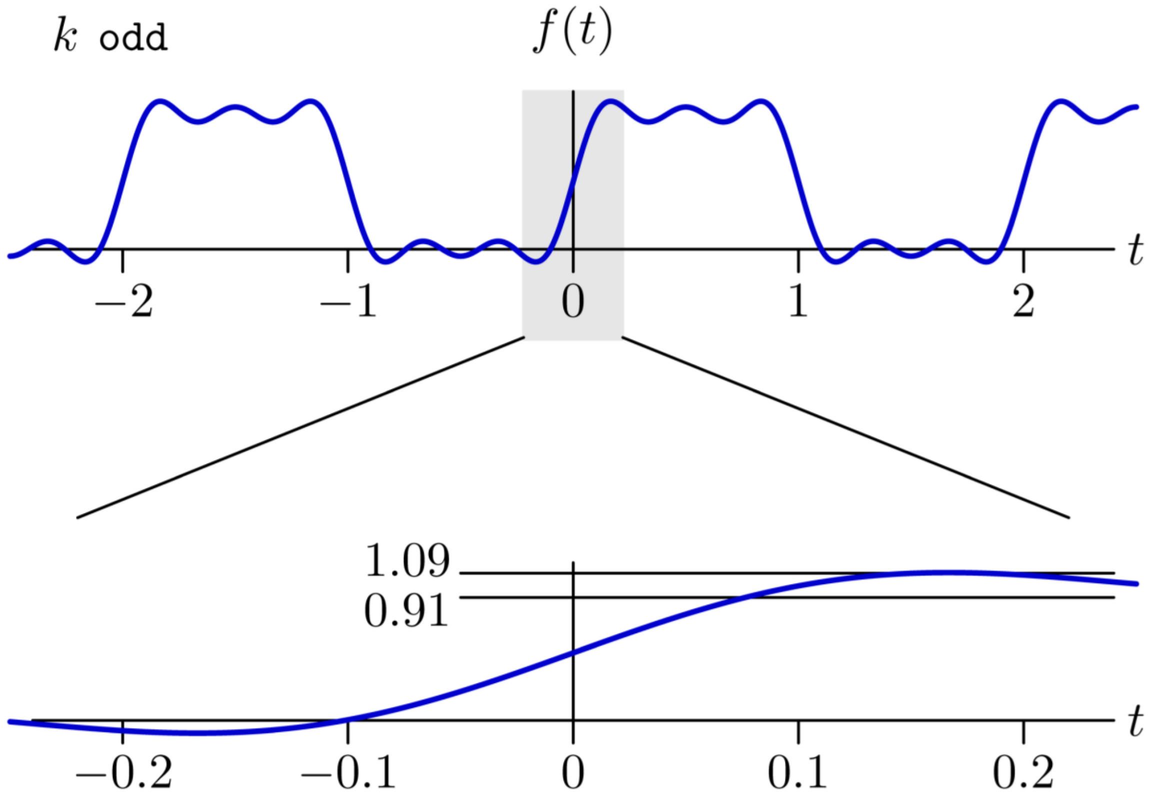
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{3} \frac{2}{k\pi}\sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

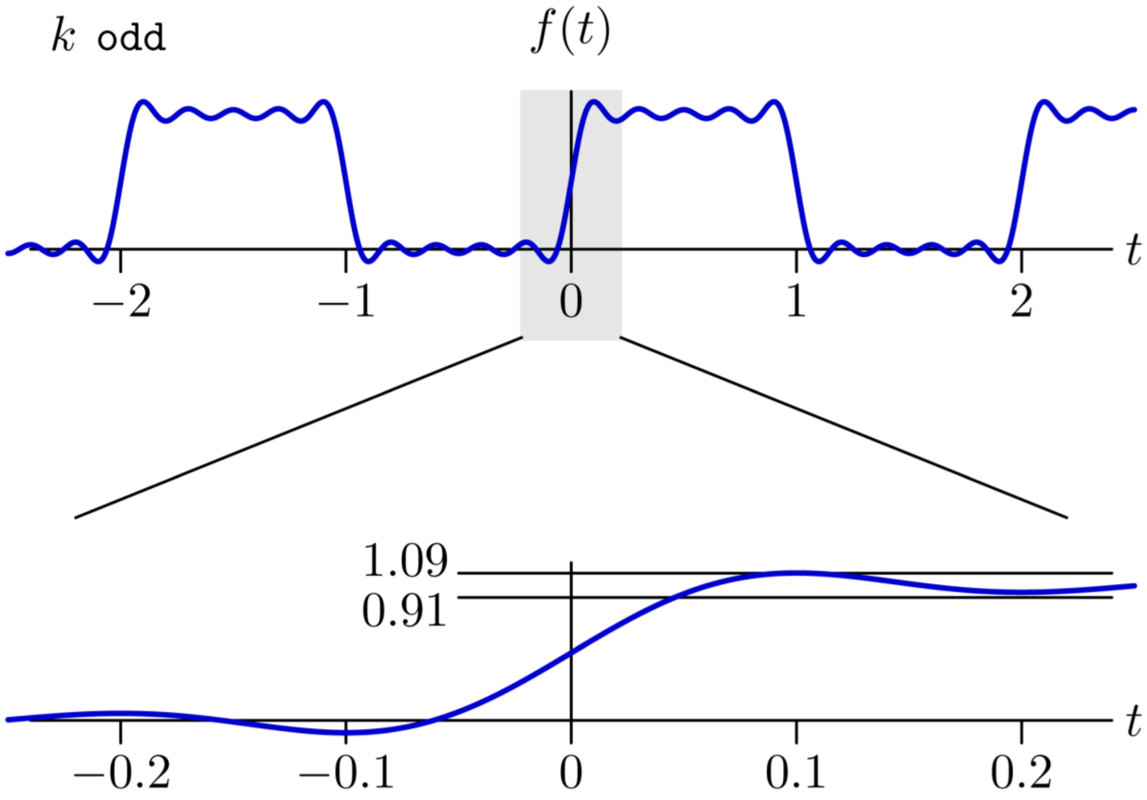
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

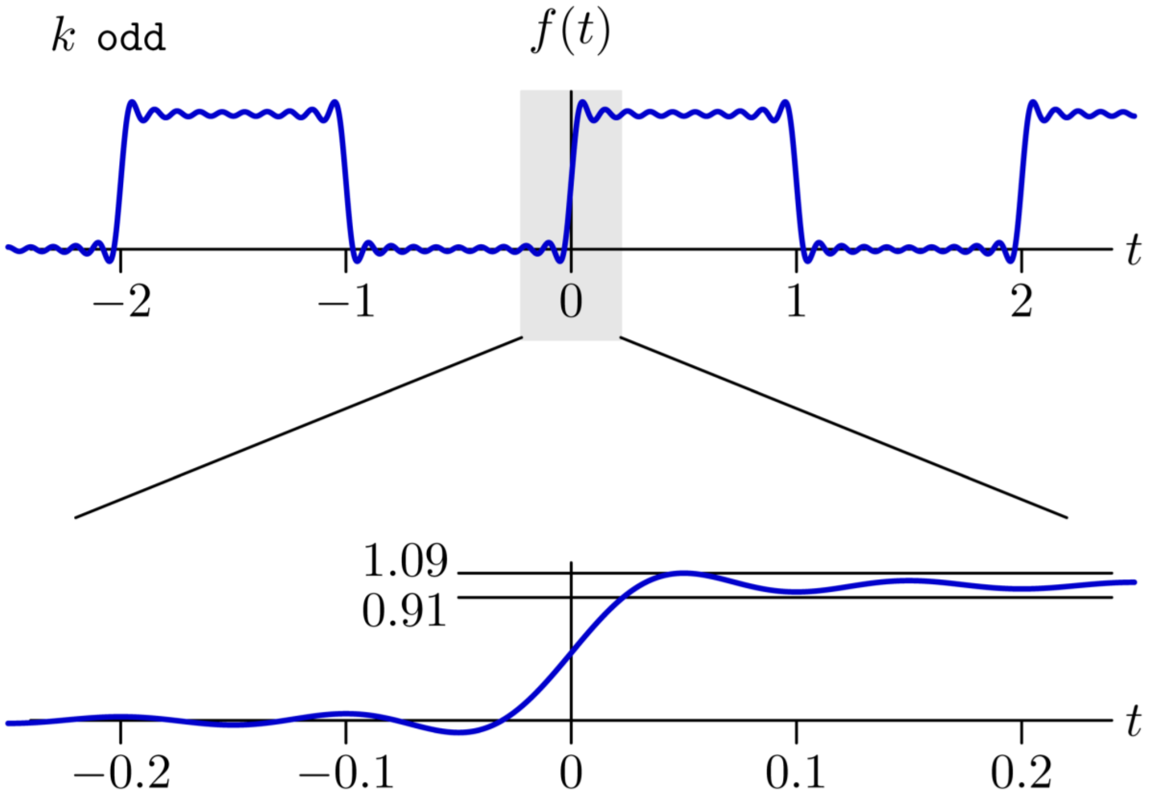
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

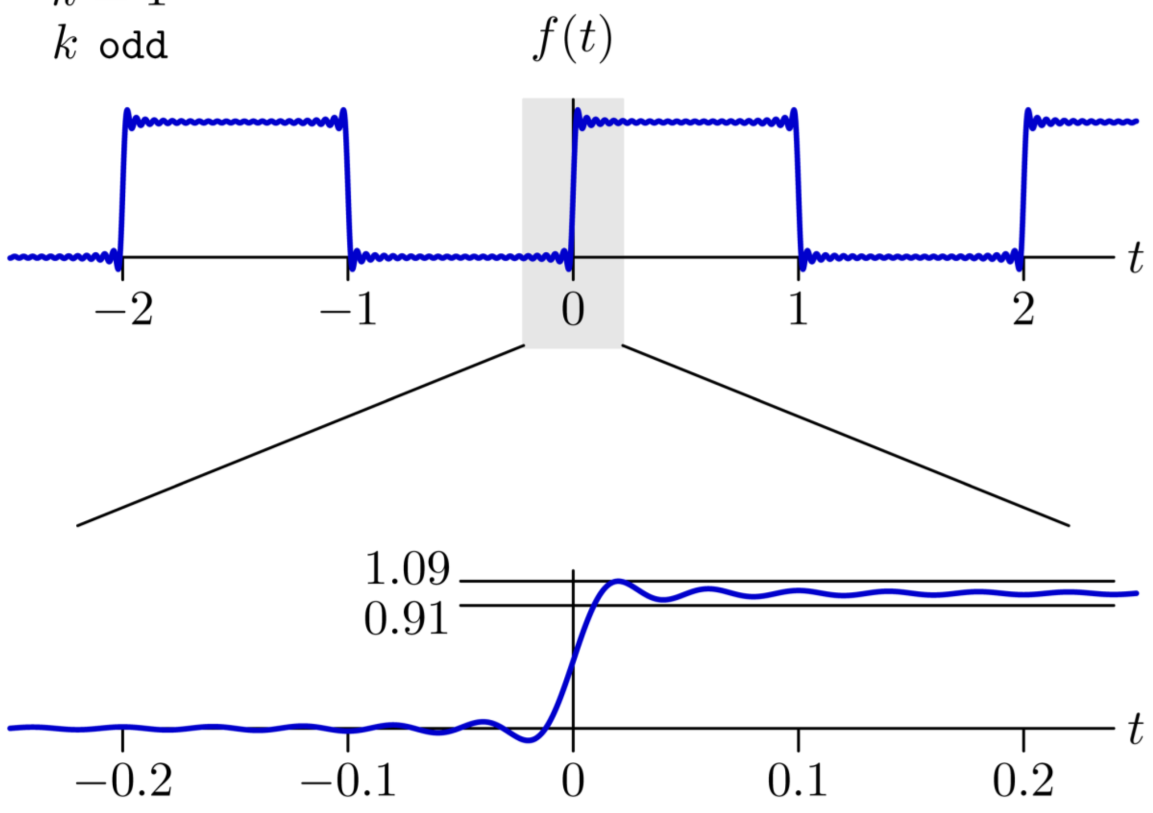
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

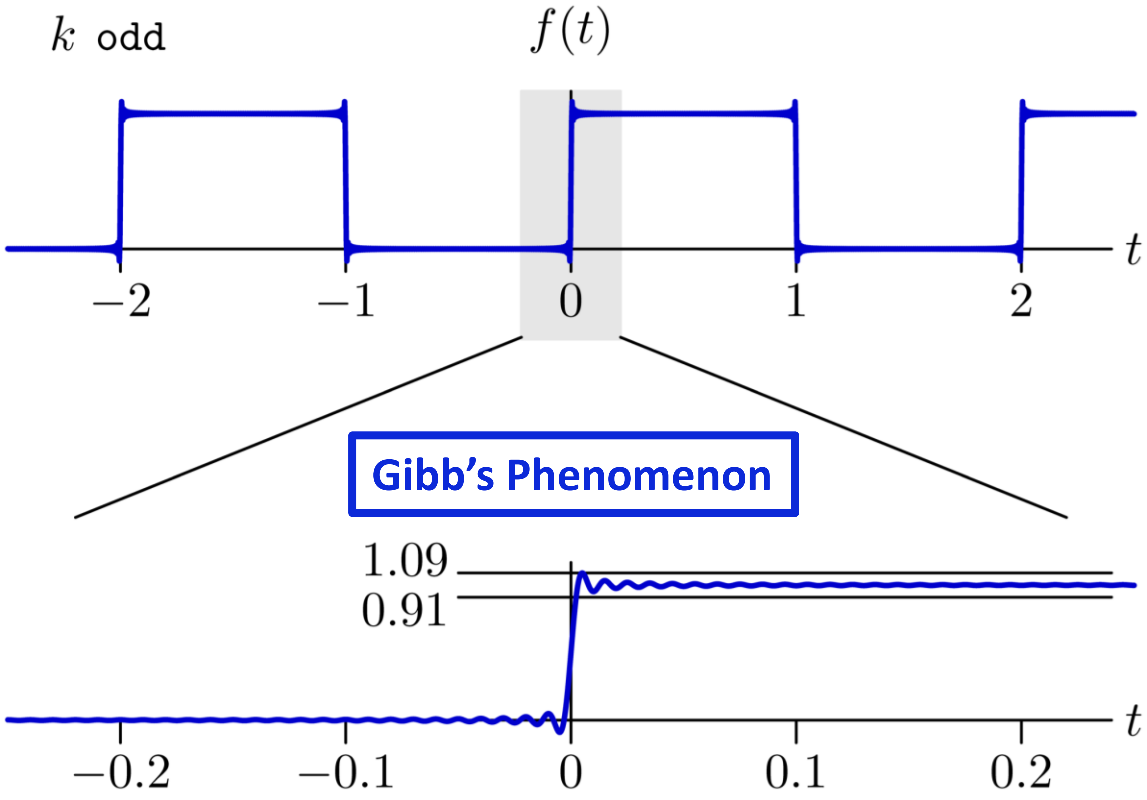
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{49} \frac{2}{k\pi} \sin(k\pi t)$$



Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi}\sin(k\pi t)$$



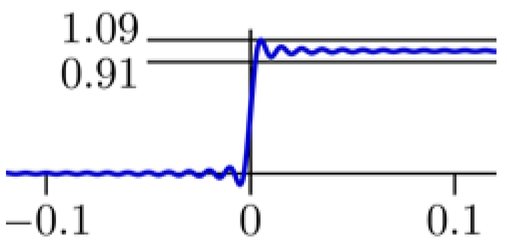
Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$



As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

We refer to this type of overshoot as **Gibb's Phenomenon**.

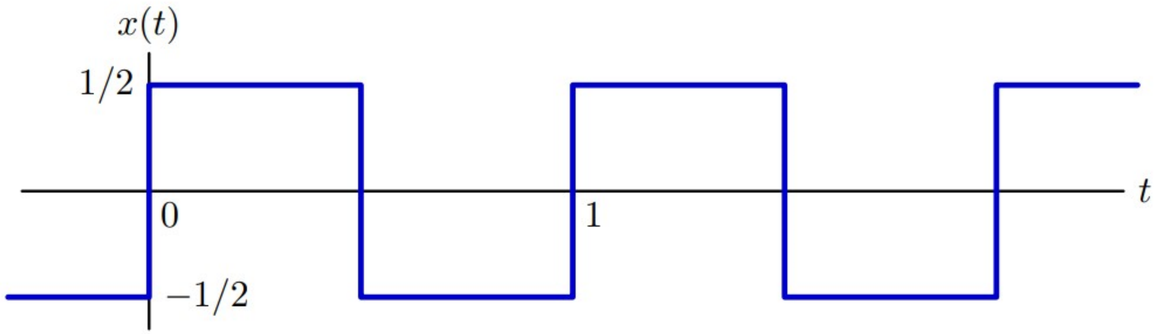
So who was right? Fourier or Lagrange?

Can any periodic signals be represented by Fourier Series

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

Jean-Baptiste Joseph Fourier

Joseph-Louis Lagrange



Who was right? **Participation question for Lecture**

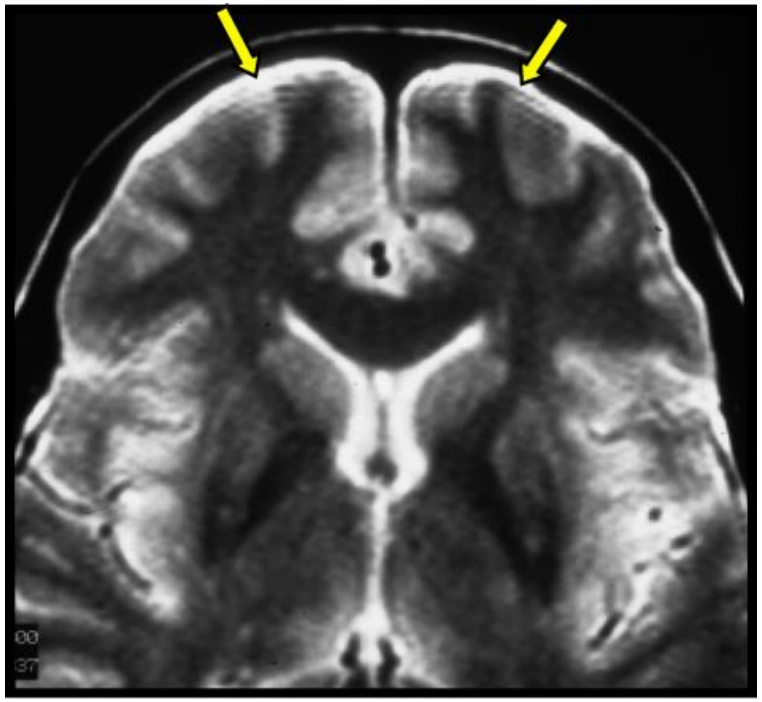
In a way both were right. The series representation of a discontinuous function converges, but no uniformly.

Dirichlet conditions:

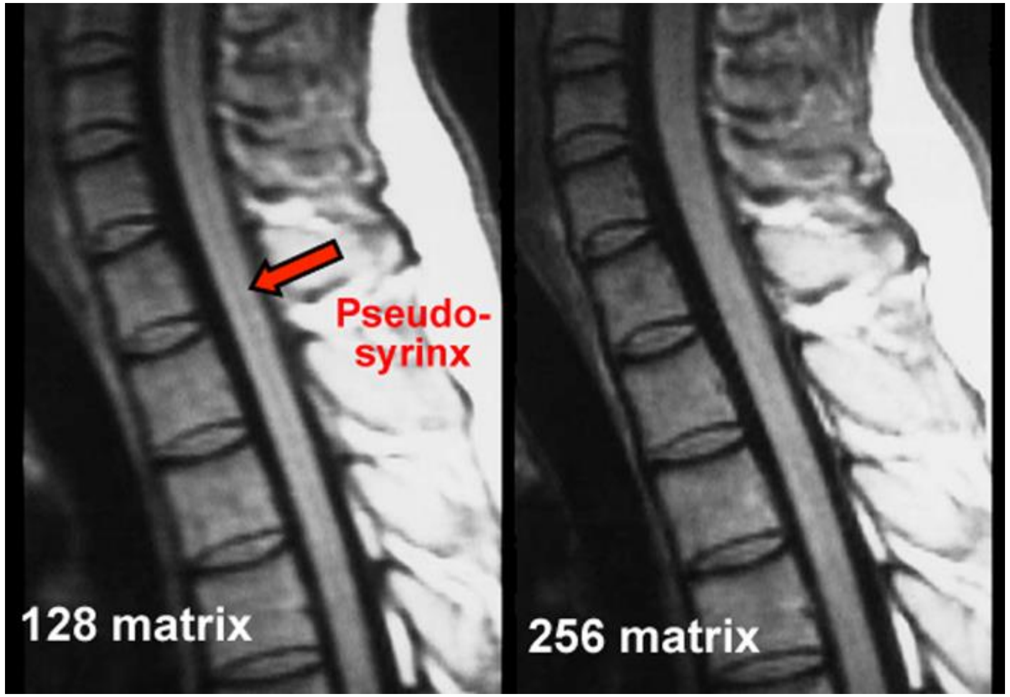
- Over any period, f(t) absolutely integrable;
- In any finite interval of time, f(t) is of bounded variation
- In any finite interval of time, there are only a finite number of discontinuities, each discontinuity is finite

Gibb's Phenomenon

Gibbs artifacts in MRI

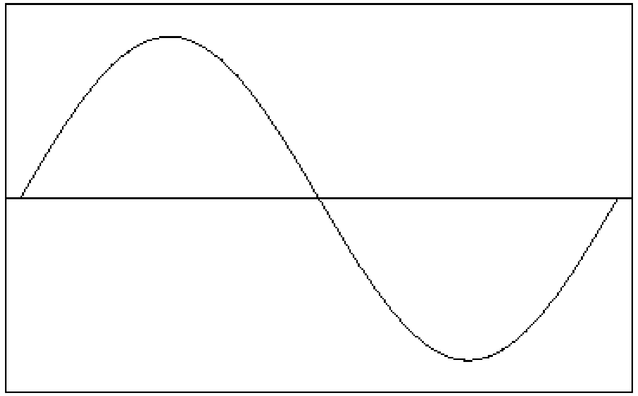


Decreasing artifacts with more frequency components



Q1: Why these happens?

Q2: How to alleviate Gibbs artifacts?

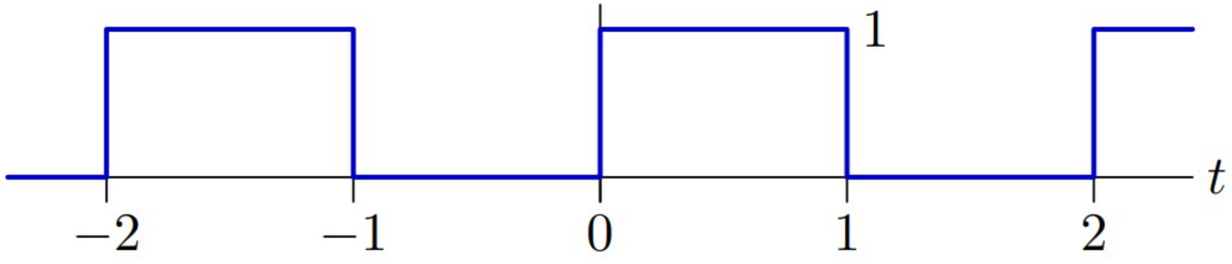


https://mriquestions.com/gibbs-artifact.html

Properties of Fourier Series: Symmetry

- Find the Fourier series coefficients for the following square wave:

$$f(t) = f(t+2)$$



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

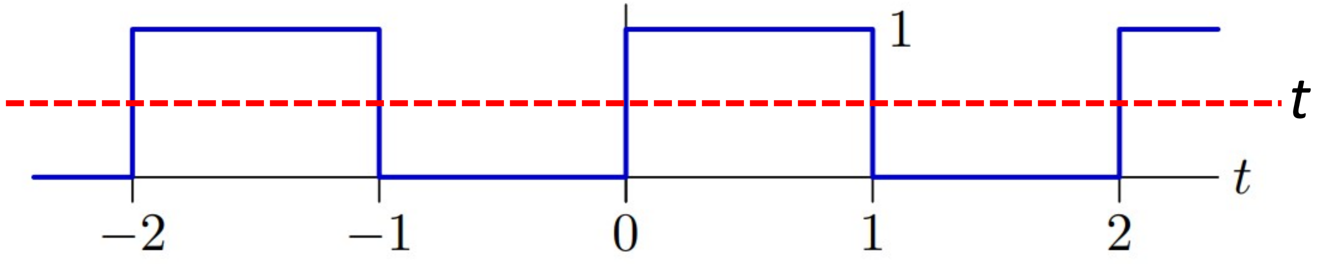
$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \left.\frac{\sin(k\pi t)}{k\pi}\right|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\left.\frac{\cos(k\pi t)}{k\pi}\right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Properties of Fourier Series: Symmetry

- Find the Fourier series coefficients for the following square wave:

$$f(t) = f(t+2)$$



Why are the C_k coefficients zero (except c_0)?

$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \frac{\sin(k\pi t)}{k\pi}\Big|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

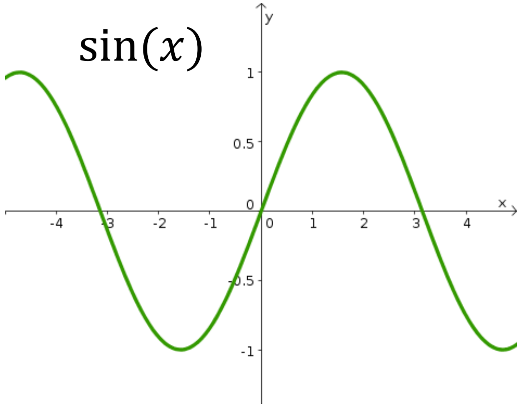
$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\frac{\cos(k\pi t)}{k\pi}\Big|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

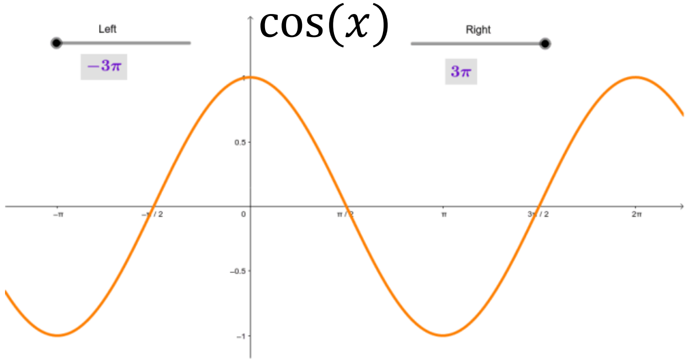
If without $c_0 = \frac{1}{2}$ "DC" part, $f(t)$ is antisymmetric around t=0, thus only having non-zero d_k's

Symmetric and Antisymmetric Parts in CTFS

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$f(-t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) - d_k \sin(k\omega_0 t))$$





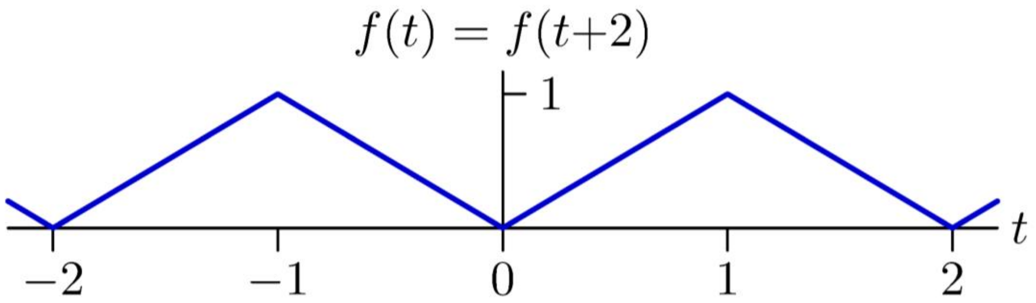
- c_k's (cosines) alone only represent the symmetric part of the signal.
- d_k 's (sines) alone only represent the antisymmetric part of the signal.

$$f_S(t) = \frac{f(t) + f(-t)}{2} \qquad f_A(t) = \frac{f(t) - f(-t)}{2}$$

The symmetric part shows up in the c_k coefficients, and the antisymmetric part shows up in the d_k coefficients.

The other example

Find the Fourier series coefficients for the following triangle wave:



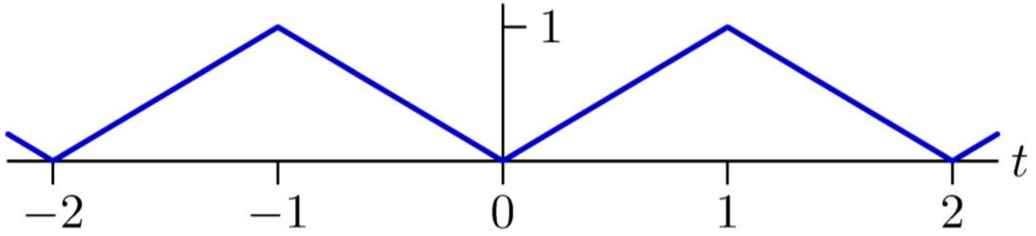
Which coefficients are zero?

Which are non-zero?

The other example

Find the Fourier series coefficients for the following triangle wave:

$$f(t) = f(t+2)$$



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi k t}{T}\,dt = 2\int_0^1 t\cos(\pi k t)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$$d_k = 0 \quad \text{(by symmetry)}$$

Summary

- We examined the convergence of Fourier Series
  - Functions with discontinuous slopes well represented
  - Functions with discontinuous values generate ripples
    - Gibb's phenomenon.
- We looked at the symmetry properties of Fourier Series

We will now go to 4-370 for recitation & common hour