6.300 Signal Processing

Week 2, Lecture A:
Continuous-Time Fourier Series (Trig Form)

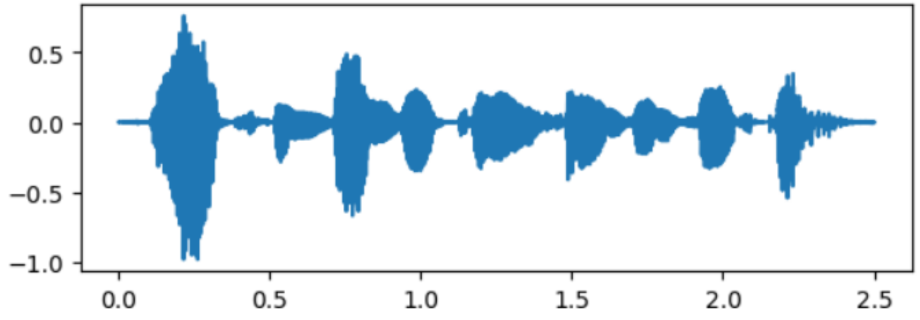
- Fourier Series
- Convergence of Fourier Series
- Symmetry of Fourier Series

Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

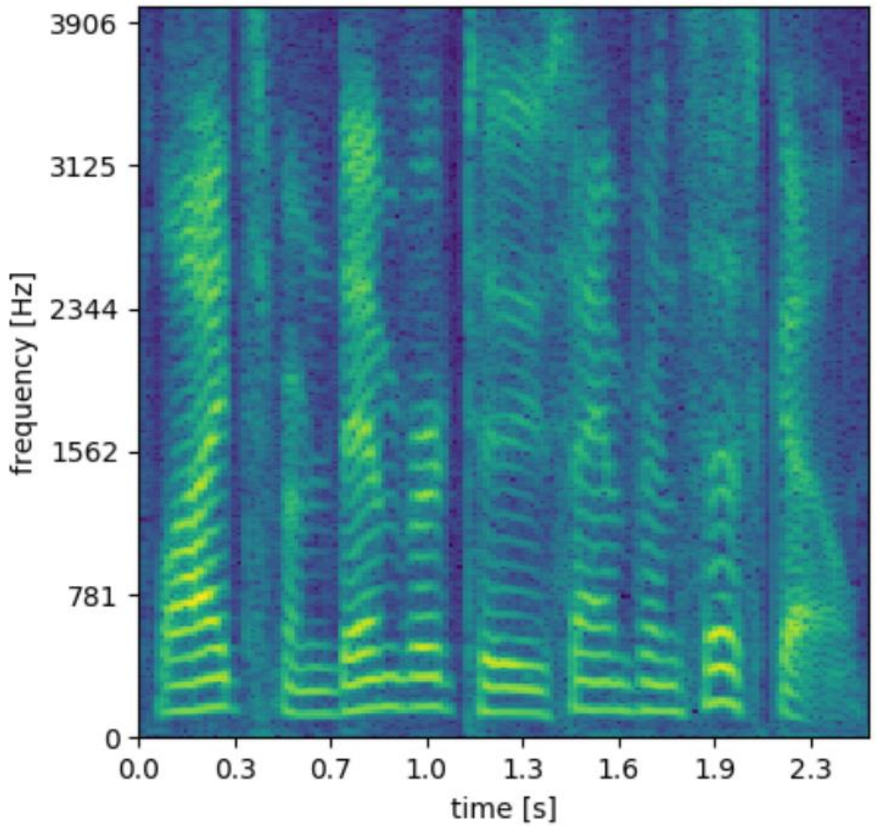
Last time: Two different ways of looking at a signal

- E.g. Two representations of a speech signal:

Time domain



"Frequency" domain



- Today: we will focus on Continuous-time Fourier series

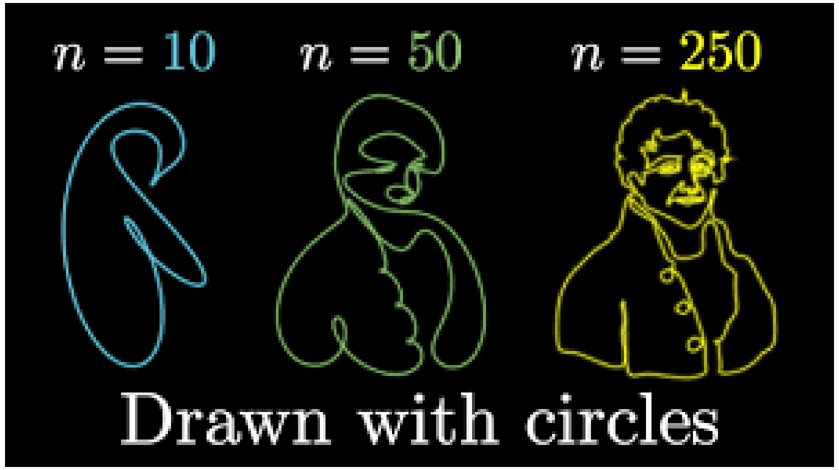
Fourier Series

Series: representing a signal as a sum of simpler signals.

- Taylor or Maclaurin's series

Function	Maclaurin Series
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
$\sin x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
$\cos x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$ (if $-1 < x < 1$)
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (if $-1 < x \le 1$)

- Draw only with circles



- Fourier series are sums of harmonically related sinusoids:

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)\right)$$

Why focus on Fourier Series

- What’s so special about sines and cosines?

Why focus on Fourier Series

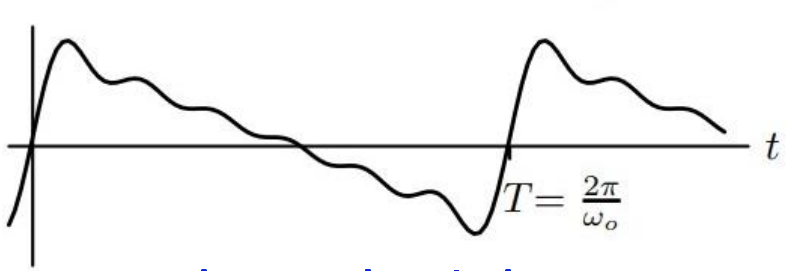
- Sines and cosines have interesting mathematical properties – orthogonality.
- Sines and cosines also play important roles in physics – especially the physics of waves.

Last time: Express periodic signals as a sum of sinusoids

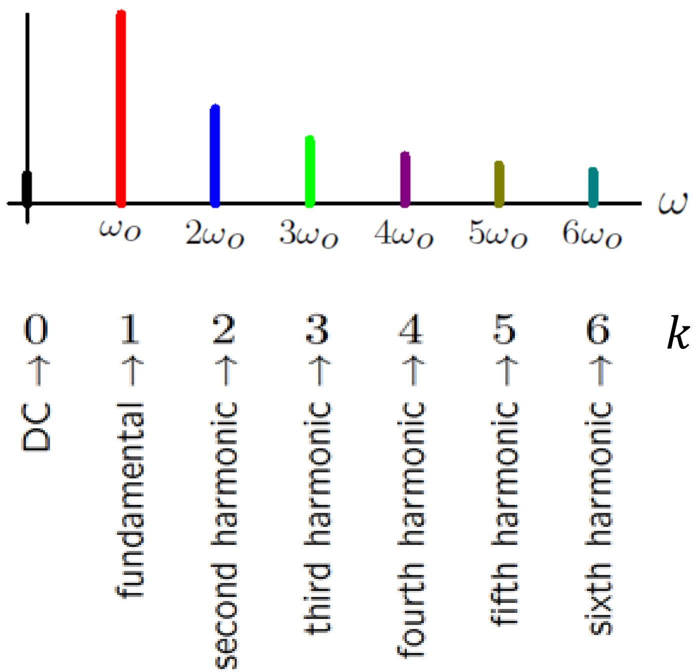
Periodic signal: $f(t) = f(t+T)$

CTFS: $f(t) \rightarrow c_k, d_k$

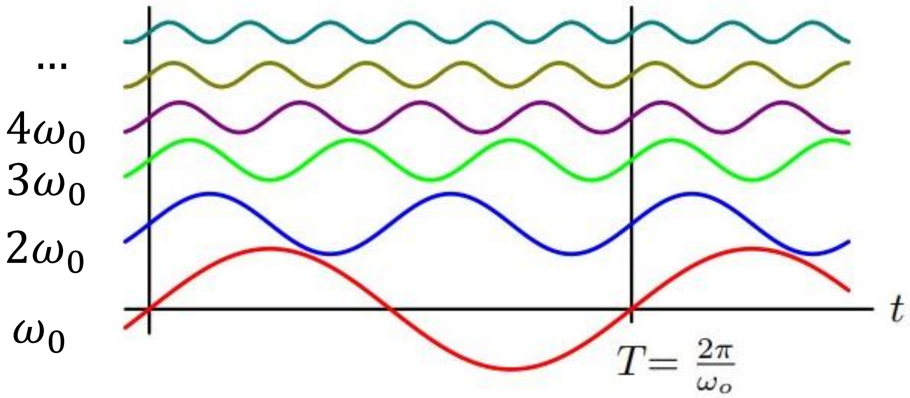
Weights c_k for $\cos(k\omega_0 t)$



- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$



Basis function $\cos(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

Decomposition:

$$f(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)\right)$$

Continuous-Time Fourier Series (CTFS) Trig Form

- **Synthesis equation**

- **Analysis equation**

Check yourself!

- What are the Fourier series coefficients associated with the following signal?

$$f(t) = 0.8\sin(6\pi t) - 0.3\cos(6\pi t) + 0.75\cos(12\pi t)$$

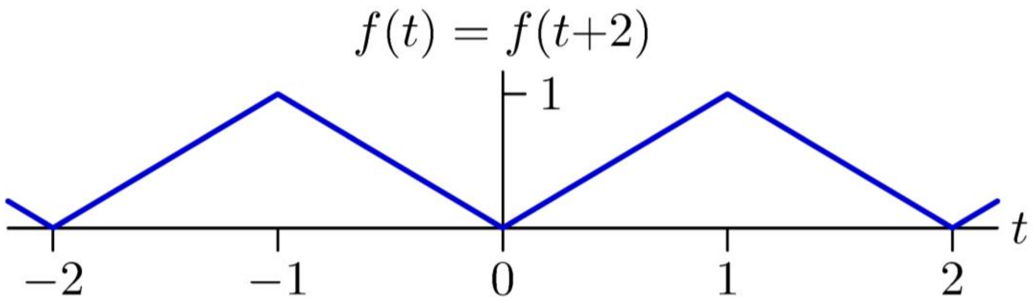
$$\omega_o = \ ?$$

$$c_k = ?$$

$$d_k = ?$$

Example of synthesis

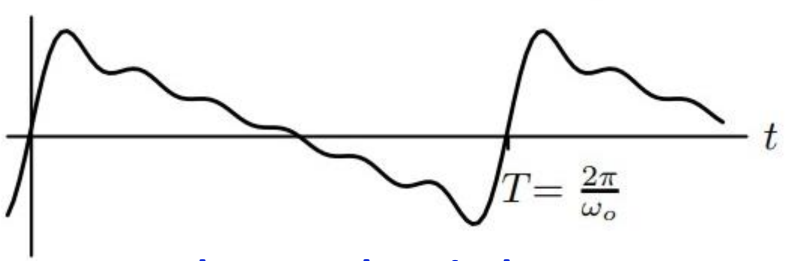
Find the Fourier series coefficients for the following triangle wave:



Can Fourier Series approximate any periodic signals?

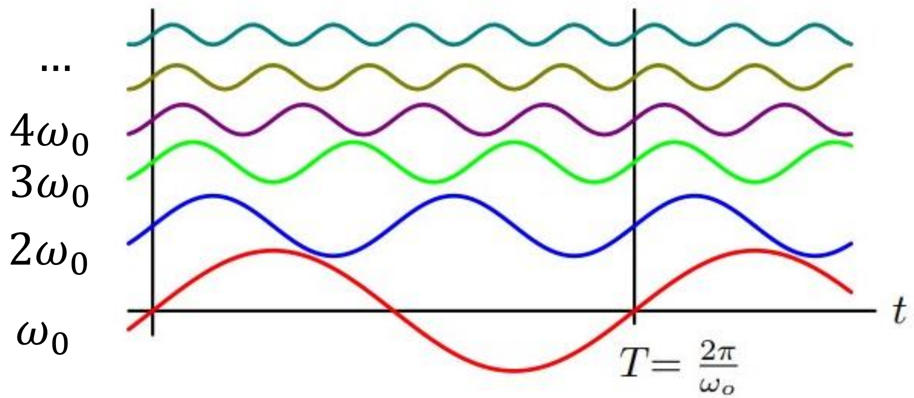
$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

Periodic signal: $f(t) = f(t+T)$



- **Fundamental period:** T
- **Fundamental frequency:** $\omega_0 = \frac{2\pi}{T}$

Basis function $\cos(k\omega_0 t)$



Harmonically related: $\omega = k\omega_0$

What about discontinuous functions?

A debate two hundred years ago…

Fourier defended the idea that such a series is meaningful.
Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

Not a problem

Jean-Baptiste Joseph Fourier

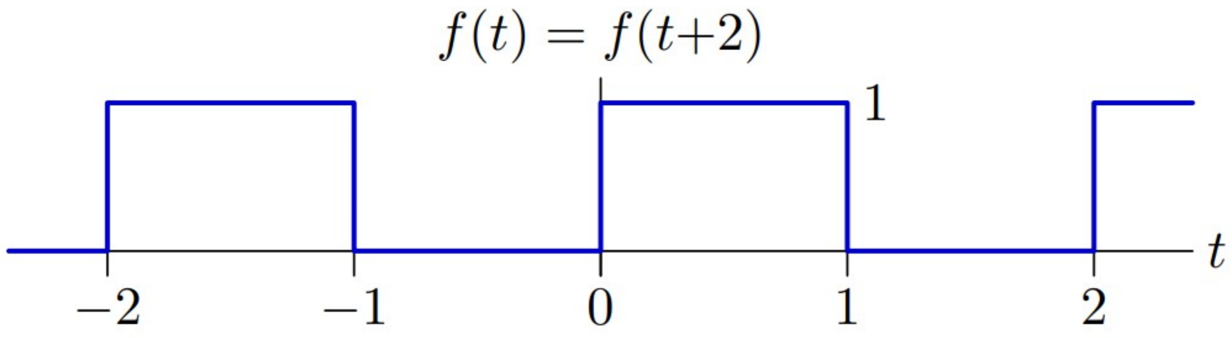
Joseph-Louis Lagrange

No way

Q: What do you think?

We can test this idea empirically – using computation

- Find the Fourier series coefficients for the following square wave:

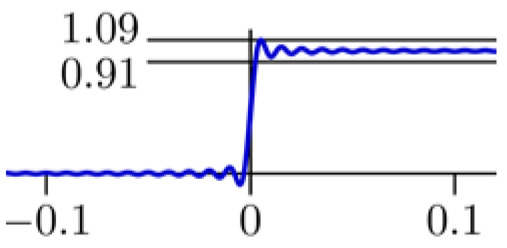


Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \Big(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \Big)$$



As $K \to \infty$,

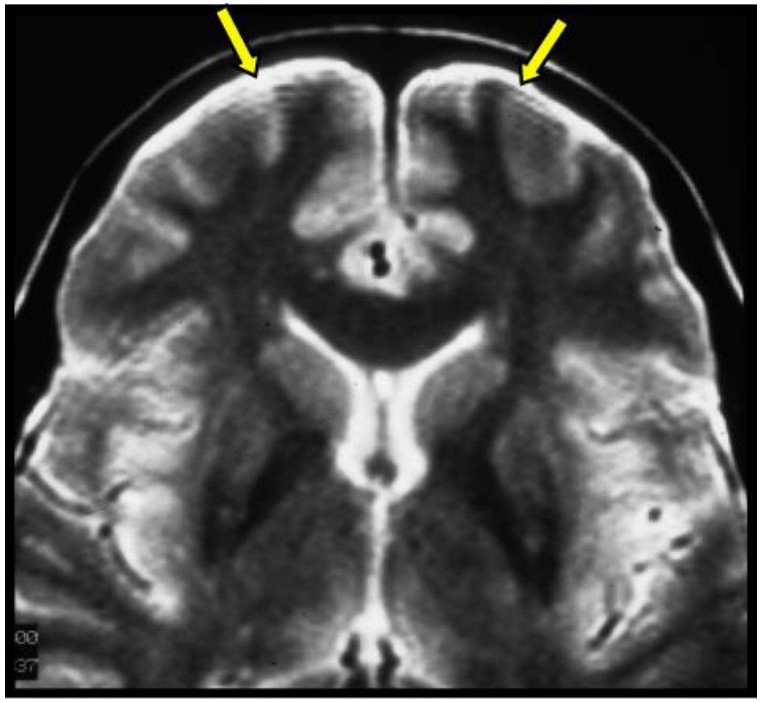
- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

We refer to this type of overshoot as **Gibb's Phenomenon**.

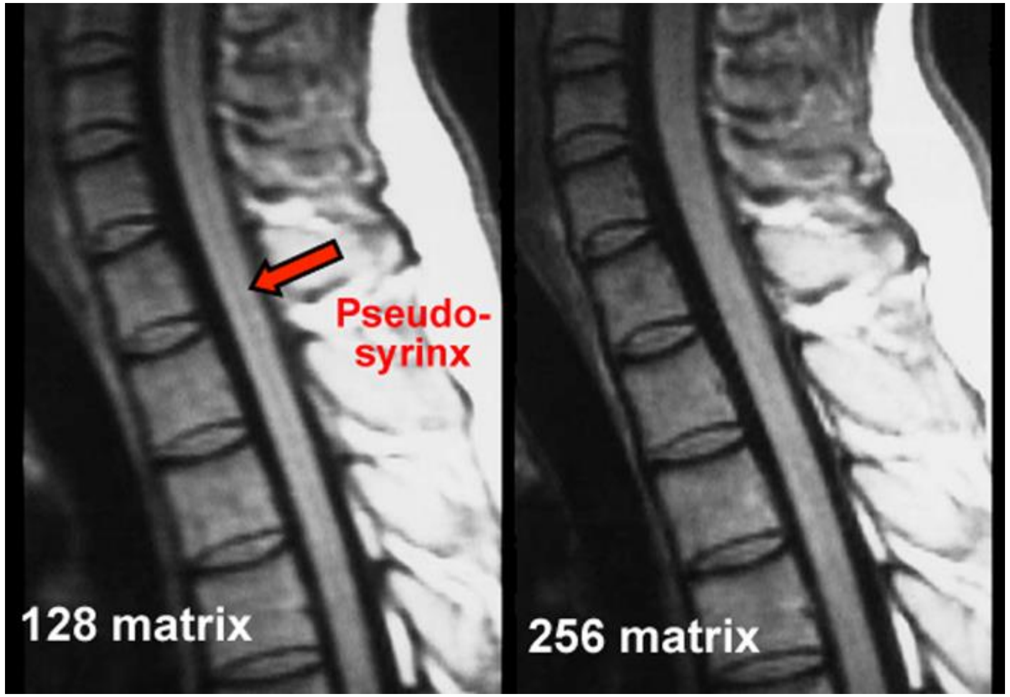
So who was right? Fourier or Lagrange?

Gibb's Phenomenon

Gibbs artifacts in MRI

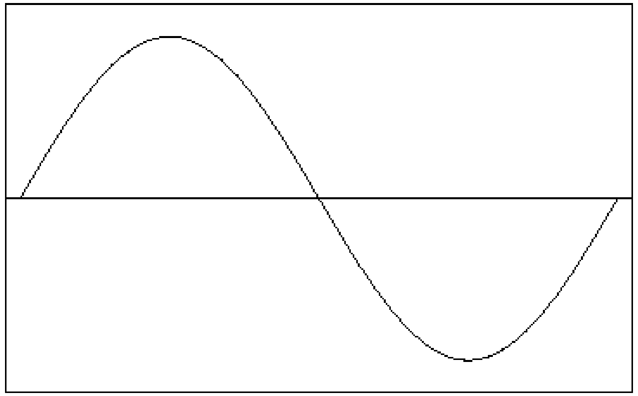


Decreasing artifacts with more frequency components



Q1: Why these happens?

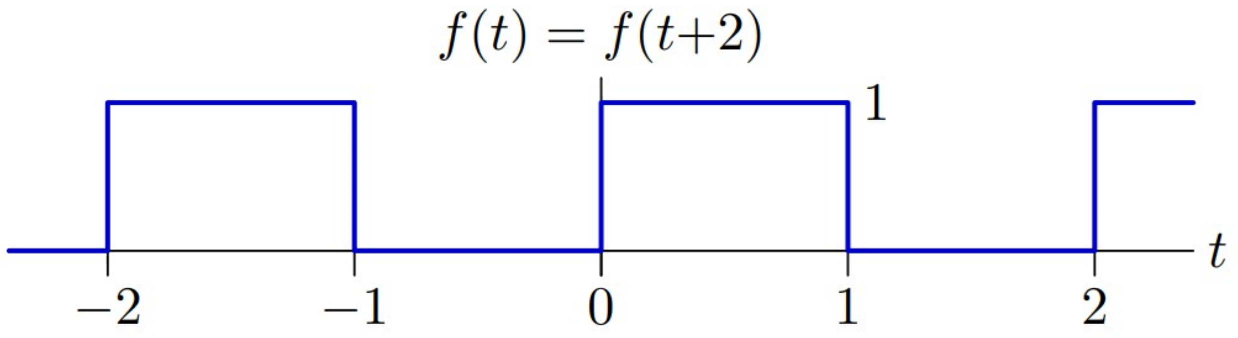
Q2: How to alleviate Gibbs artifacts?



https://mriquestions.com/gibbs-artifact.html

Properties of Fourier Series: Symmetry

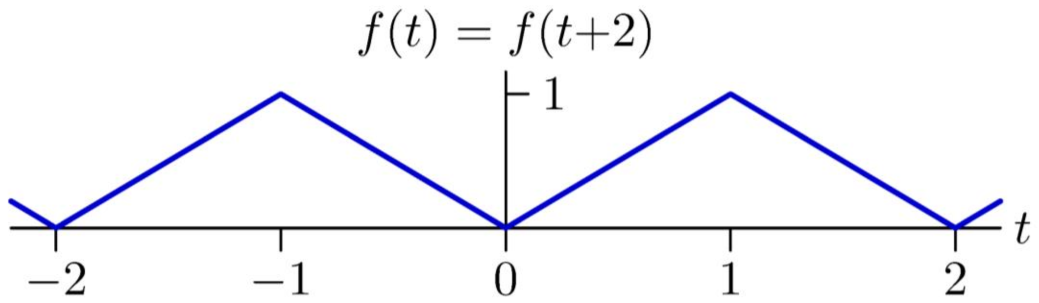
- Find the Fourier series coefficients for the following square wave:



What coefficients are zero and why?

The other example

Find the Fourier series coefficients for the following triangle wave:



Which coefficients are zero?
Which are non-zero?

Summary

- We examined the convergence of Fourier Series
  - Functions with discontinuous slopes
  - Functions with discontinuous values
    - Gibb’s phenomenon.
- We looked at the symmetry properties of Fourier Series

We will now go to 4-370 for recitation & common hour