6.300 Signal Processing

Week 1, Lecture B:
Signal Processing

- Overview of the subject
- Signals: Definitions, examples, and operations
- Time and Frequency Representations
- Fourier Series

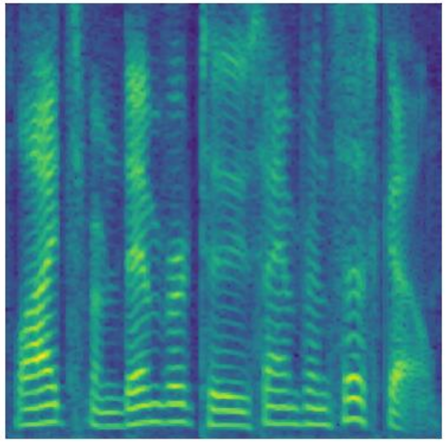
Lecture slides are available on CATSOOP:
https://sigproc.mit.edu/fall24

What is 6.300?

- 6.300 is about signal processing.
- What is a signal?
  - A signal is a function that conveys information
- What is signal processing?
  - Identifying signals in physical, mathematical, computation contexts
  - Analyzing signals to understand the information they contain
  - Manipulating signals to modify the information they contain

At the end of this class

- Learn to identify signals in physical, mathematical, computation contexts
- Signals are functions that contain and convey information.
- Examples:
  - medical (EKG, EEG, MRI, OCT)
  - speech signals
  - music
  - images
  - video
  - seismic signals

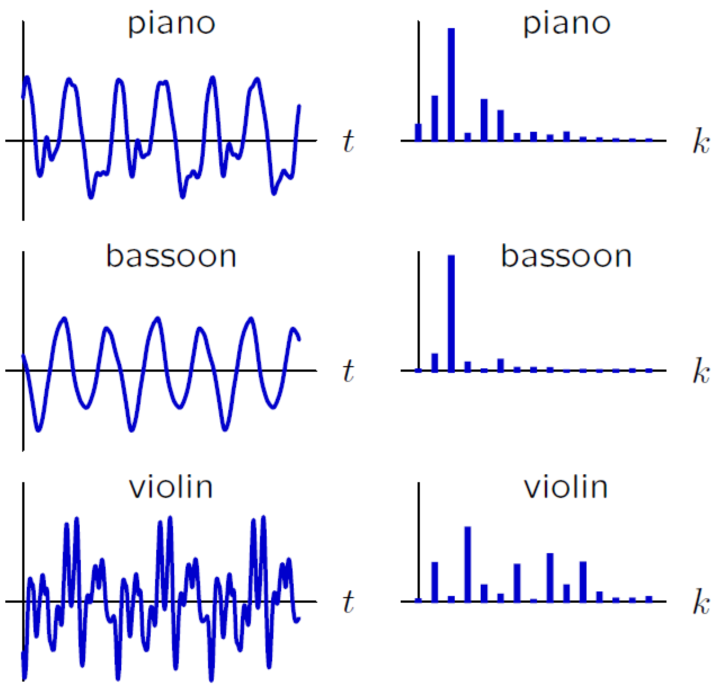


At the end of this class

- Analyzing signals to understand the information they contain
- Learn to think of signals in frequency domain (in addition to time, space, …)
  - Mathematical analysis and physical understanding

Music analysis

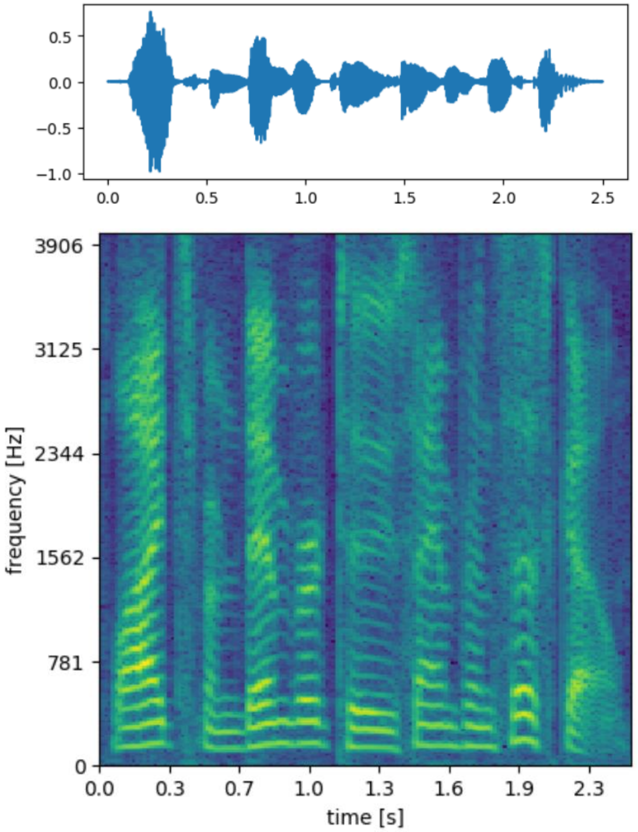
Time domain | Frequency domain



Speech processing

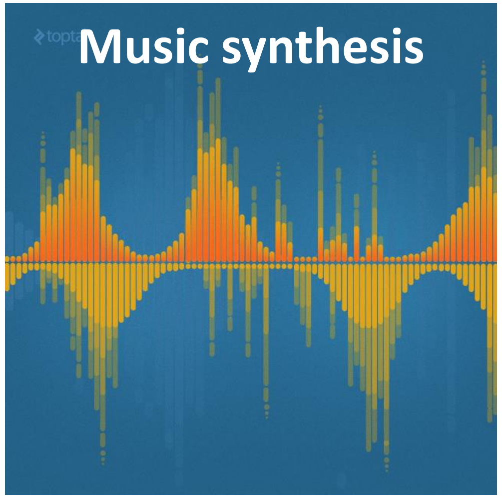
Time domain

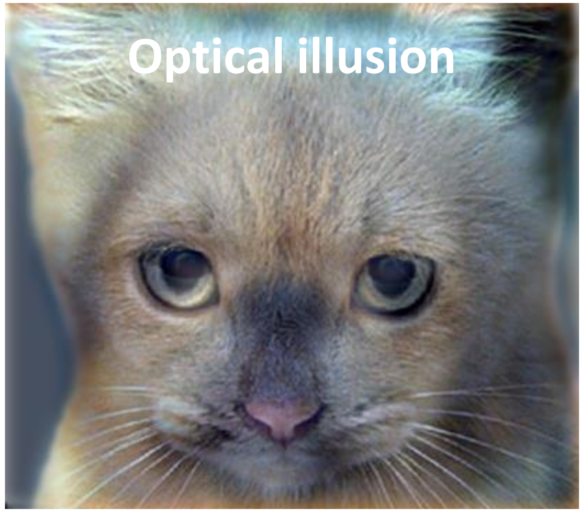
Time-frequency domain



At the end of this class

- Learn to manipulate signals to modify the information they contain
- Learn to apply signal processing theories to real-life applications
  - Problem formulation, design, coding
  - Music, speech, photography, video streaming, astronomy, biomedicine…





Image/video compression

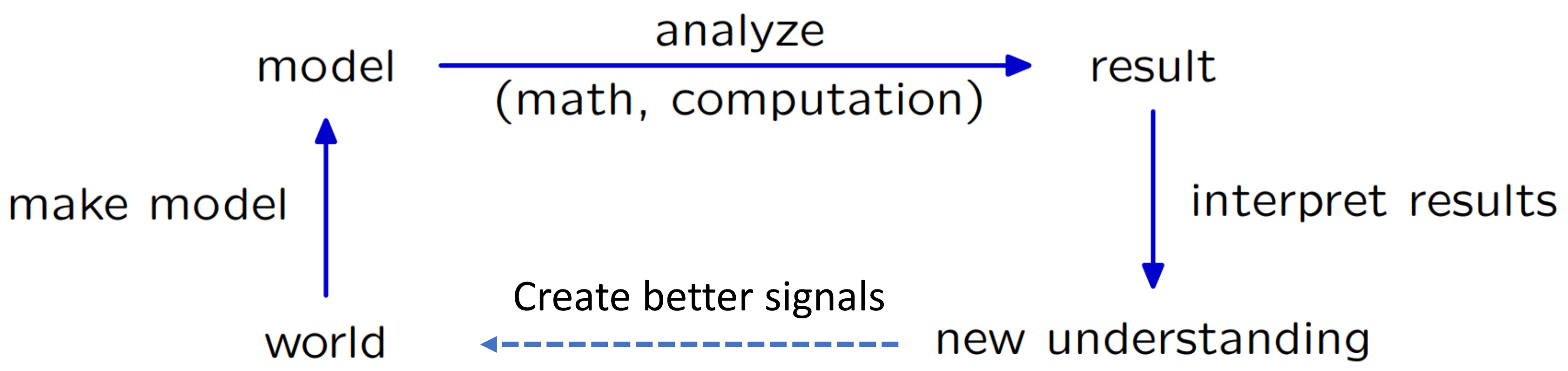
Motion artifacts

Image restoration

Signal Processing

Signal Processing is widely used in science and engineering to ...

- model some aspect of the world,
- analyze the model,
- interpret results to gain a new or better understanding.



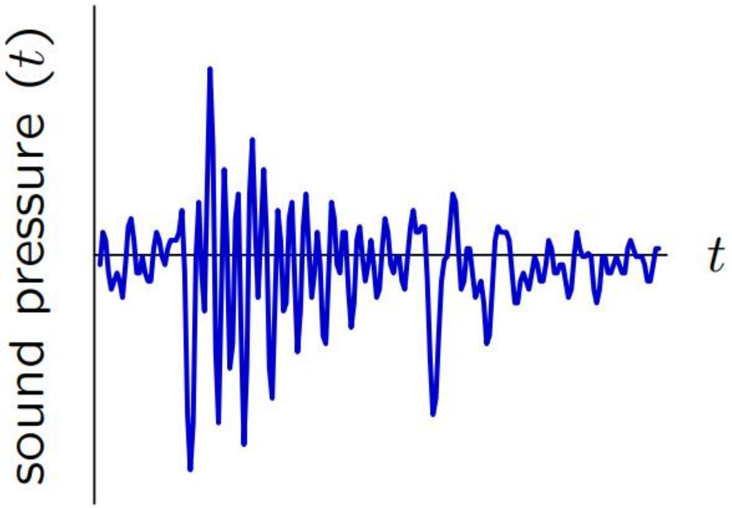
Signal Processing provides a common language across disciplines.

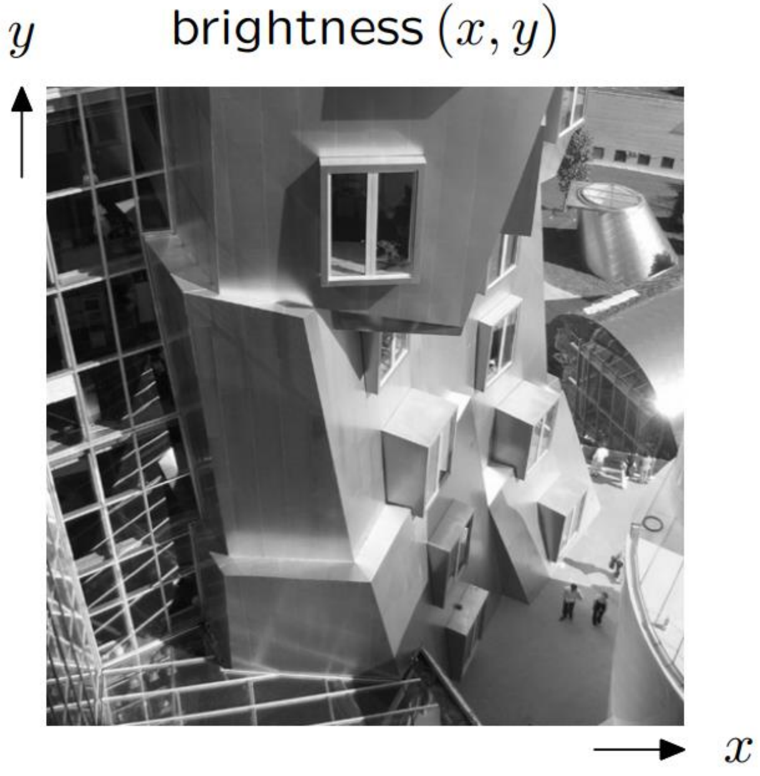
Get the most out of 6.300!

- Course website: CAT-SOOP (detailed policies).
- Lecture: TR2 (3-270)
  - Live questions in lecture (5% graded based on effort or weigh into final exam)
- Recitation: TR3 (32-141)
  - Live questions in recitation (5% graded based on effort or weigh into final exam)
- Piazza: **Only** for logistic questions
- Common-room hours: Monday-Friday 4-5pm, Monday & Wednesday 7-9pm
- Homework: posted Thursdays at 4pm; Lab check-in due the following Mondays at 9pm; Pset due the following Wednesdays 10pm
  - Psets: focus on developing problem solving skills – simple computational extensions to real-world data (15%). Drop one lowest-scored Psets.
  - Labs: focus on applications of 6.300 to real-world problems – more open-ended, multiple approaches, multiple solutions (5%+10%). **Start early!**
  - Two quizzes and a final (15% +20%+35% or 15% +20%+25%+10%)

Signals: independent variable

- Signals are functions that contain and convey information.
- Questions: Independent vs. dependent variable?

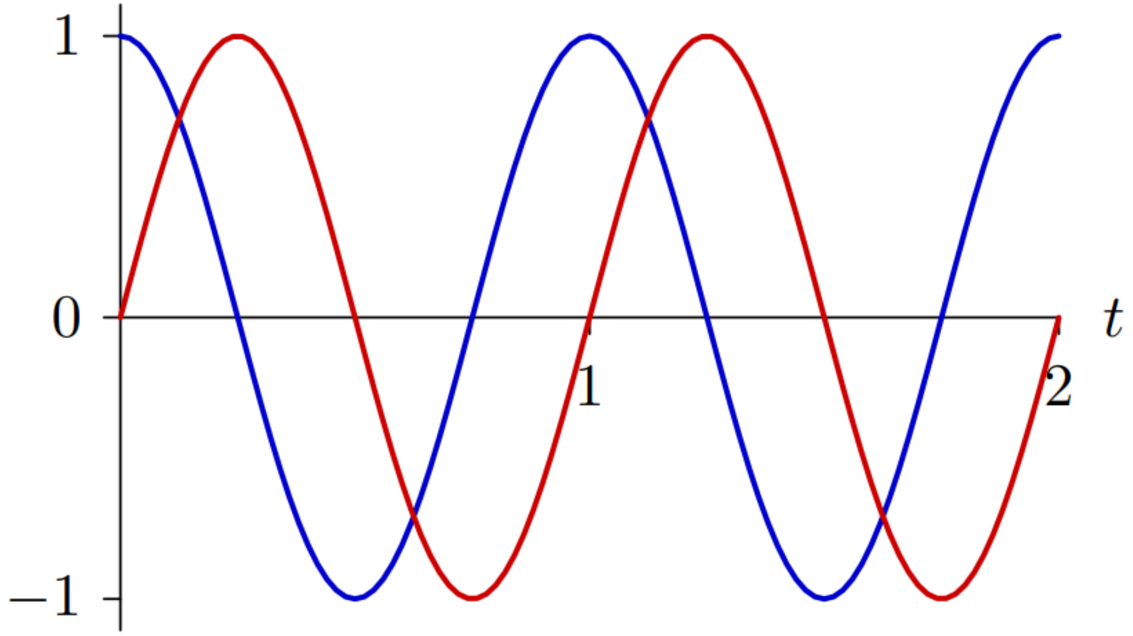




Signals: dependent variable

- Dependent variable can be real, imaginary, or complex-valued

$$x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$$



- Why complex?

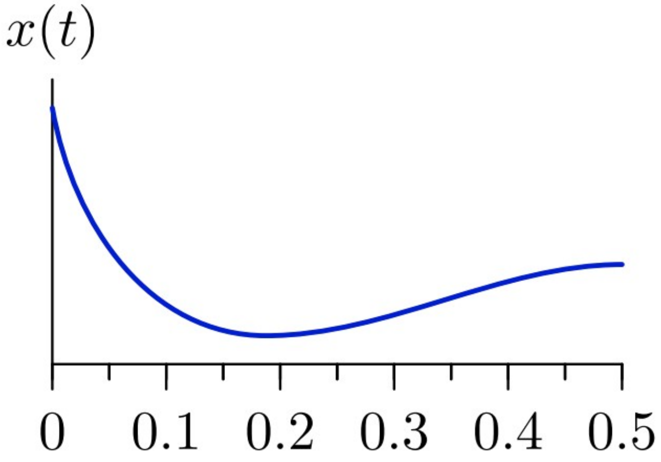
Signals: Continuous vs. Discrete

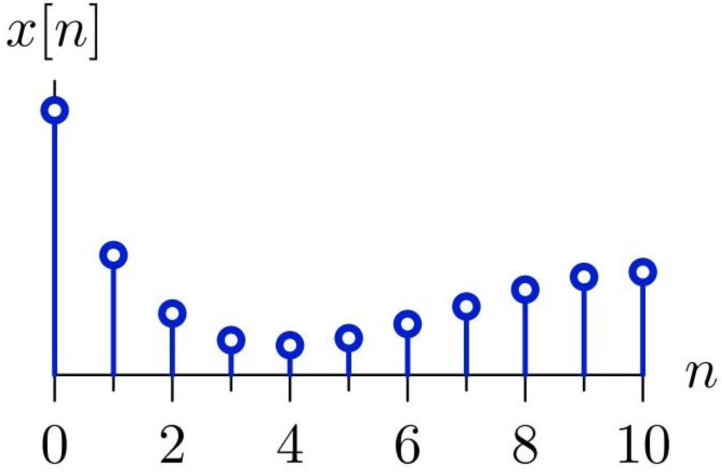
Physical signals are often of continuous domain:

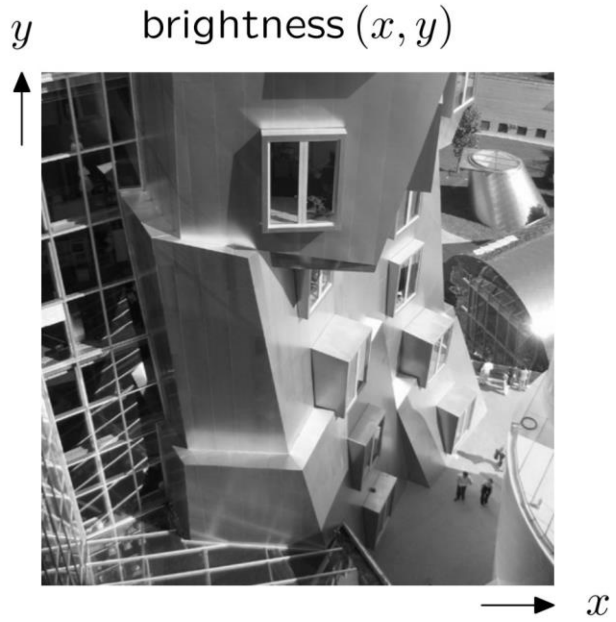
- continuous time (in seconds)
- continuous spatial coordinates (in meters)

Computations manipulate functions of discrete domain:

- discrete time (in samples)
- discrete spatial coordinates (in pixels)



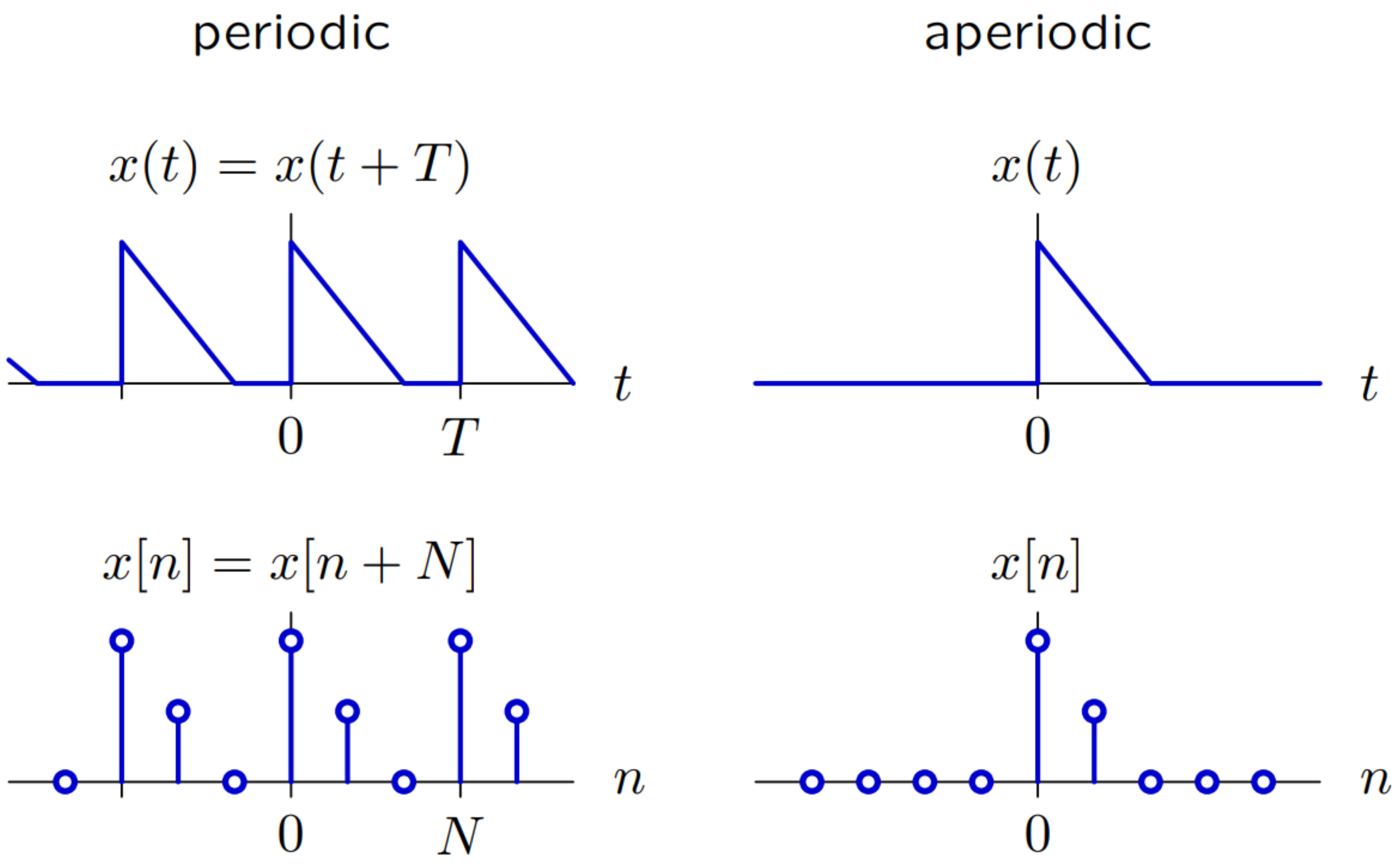




Examples?

Signals: Periodic vs Aperiodic

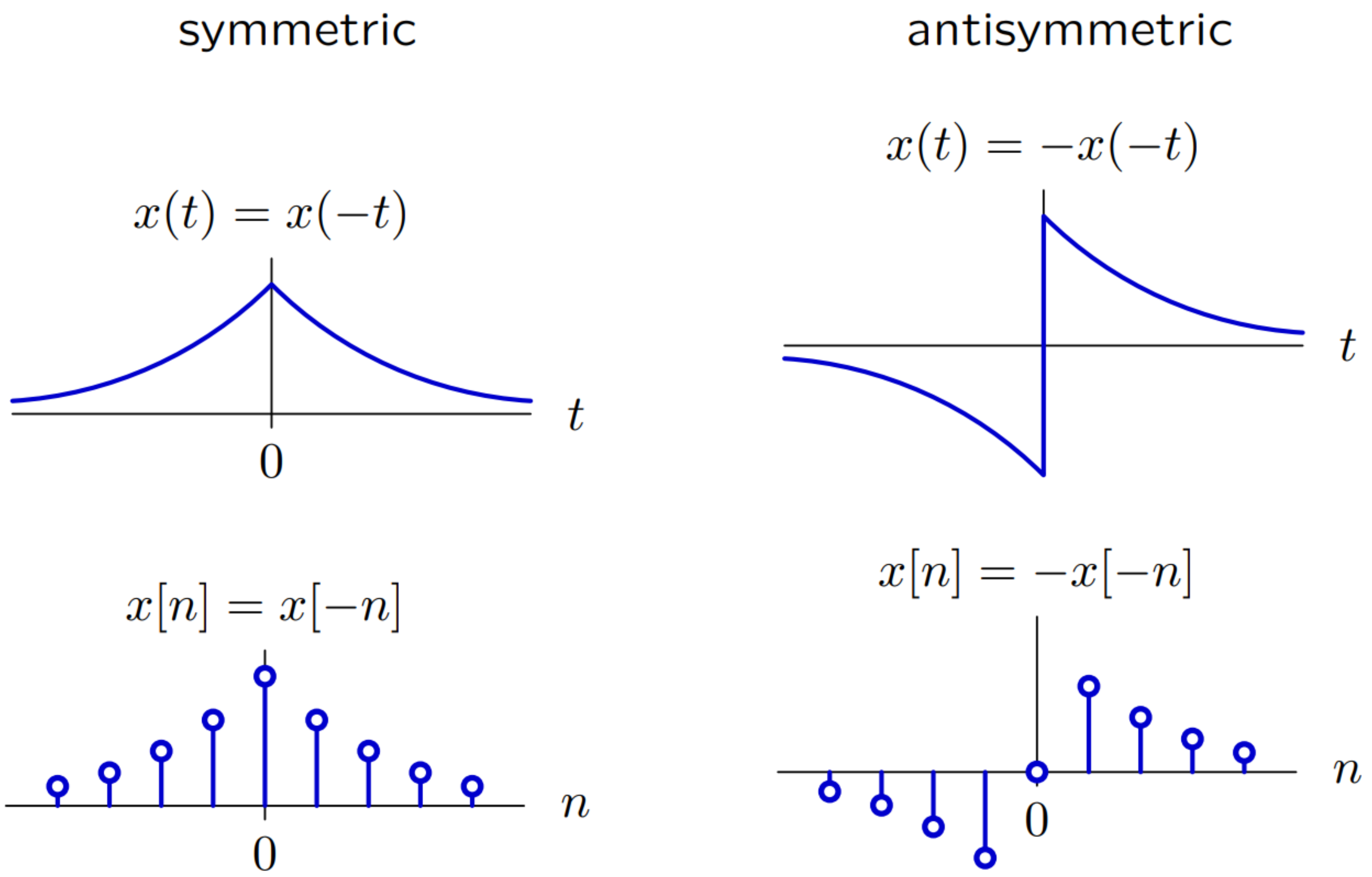
- Periodic signals consist of repeated cycles (periods). Important for analysis later.



Examples?

Signals: Symmetric vs Antisymmetric

- Signals can be symmetric or antisymmetric, or neither symmetric/antisymmetric at all!

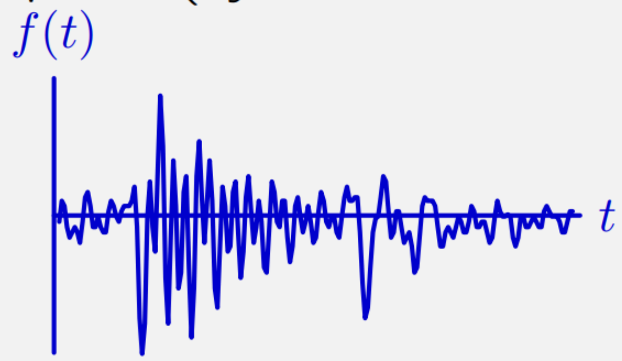


Important for analysis and intuition building later

Check yourself

- Before listening to the manipulated signals, can you think what should f(2t), -f(t) and 1/3f(t) look and sound like?

Computer generated speech (by Robert Donovan)



Listen to the following four manipulated signals:

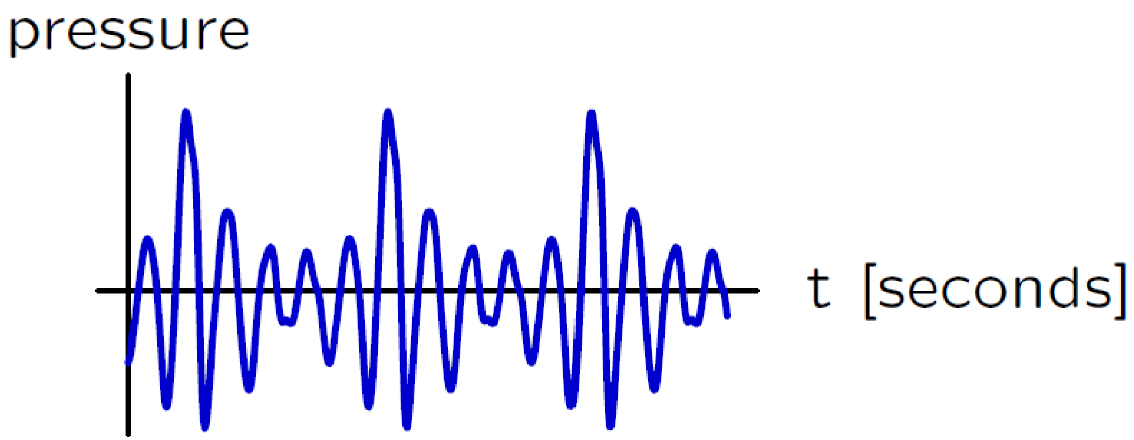
$$f_1(t),\ f_2(t),\ f_3(t),\ f_4(t).$$

How many of the following relations are true?

- $f_1(t) = f(2t)$ √
- $f_2(t) = -f(t)$ X
- $f_3(t) = f(2t)$ X
- $f_4(t) = \frac{1}{3} f(t)$ √

Music sounds as signals

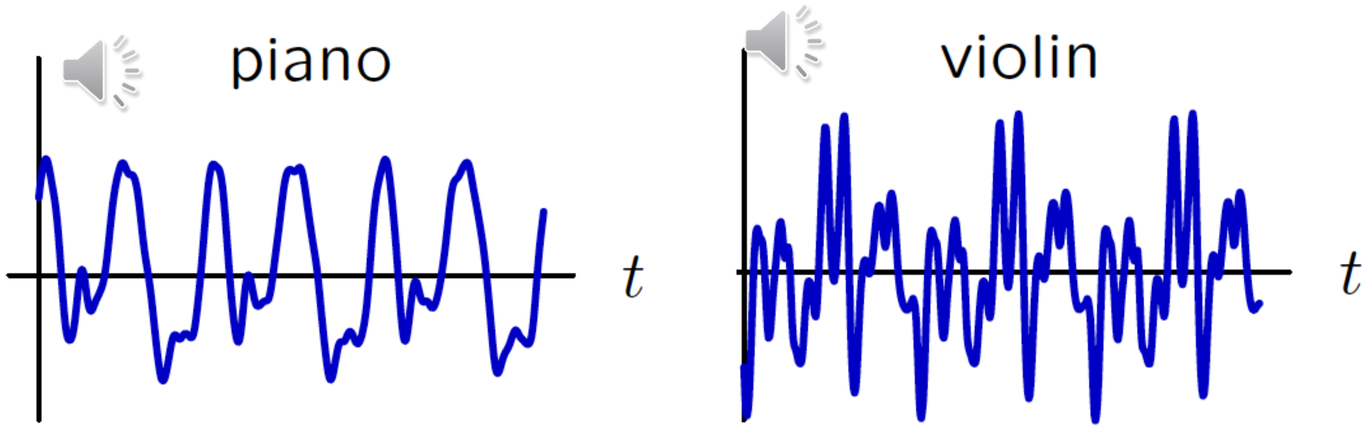
- Signals are functions that contain and convey information
- Example: a musical sound can be represented as a function of time.



- Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Music sounds as signals of time

- Even though these sounds have the same pitch, they sound different.

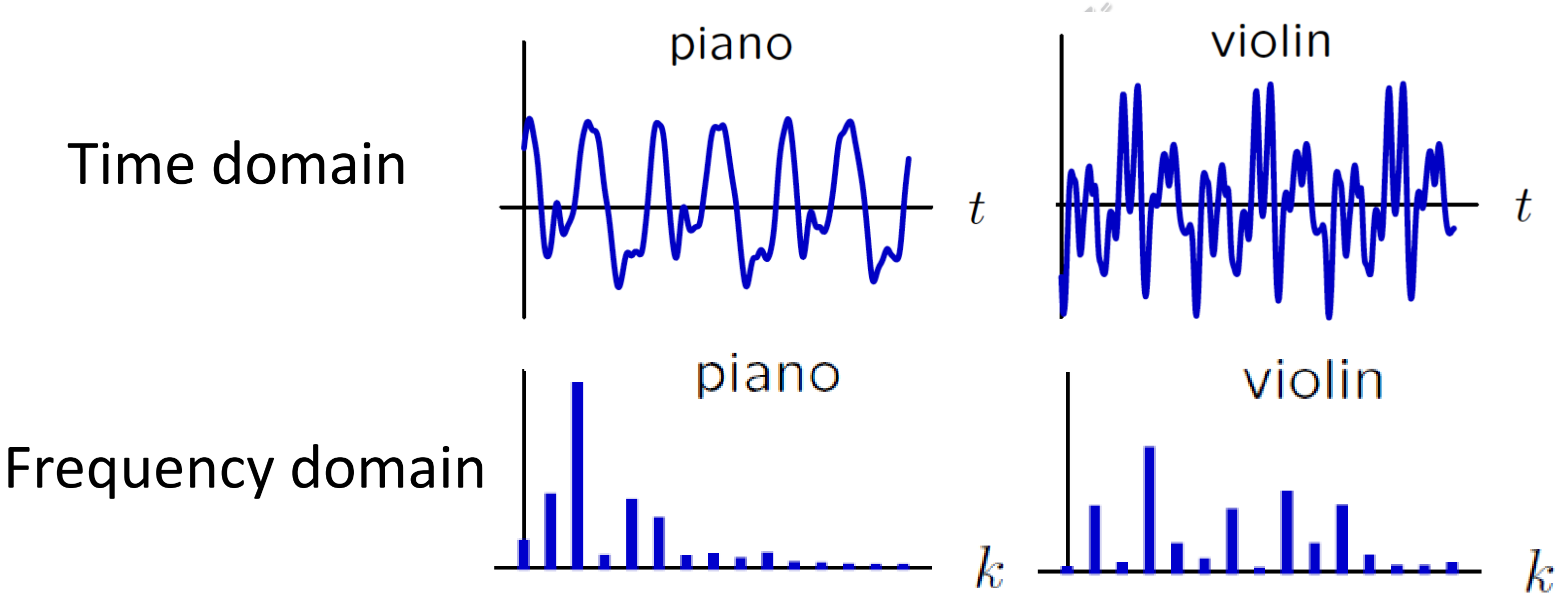


- It's not clear how the differences relate to properties of the signals. (audio clips from http://theremin.music.uiowa.edu)

Music sounds as signals of frequency

- Transform: reveal important properties of the signal (otherwise hidden in time domain)

Same pitch, they sound different. Why?

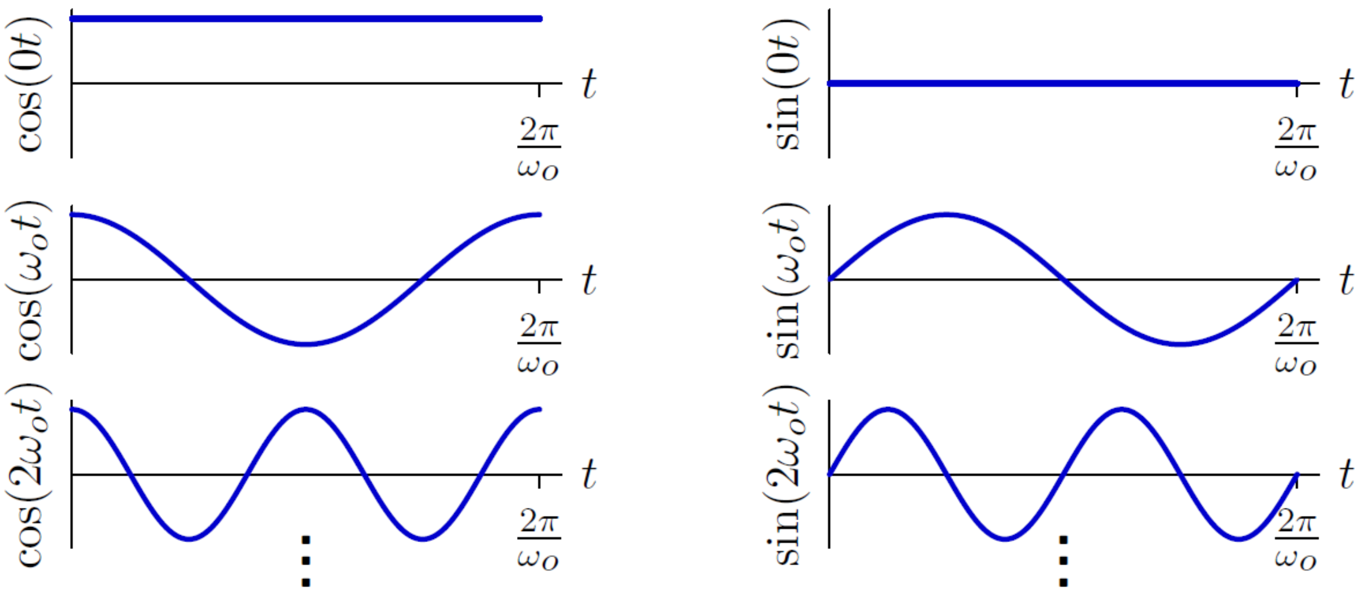


- The harmonic structures of notes from different instruments are different.

Music signals as sum of sinusoids

- How: One way to characterize differences between these signals is express them each as a sum of sinusoids

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t)$$



- Since these sounds are (nearly) periodic, the frequencies of the dominant sinusoids are (nearly) integer multiples of a **fundamental** frequency ω_0

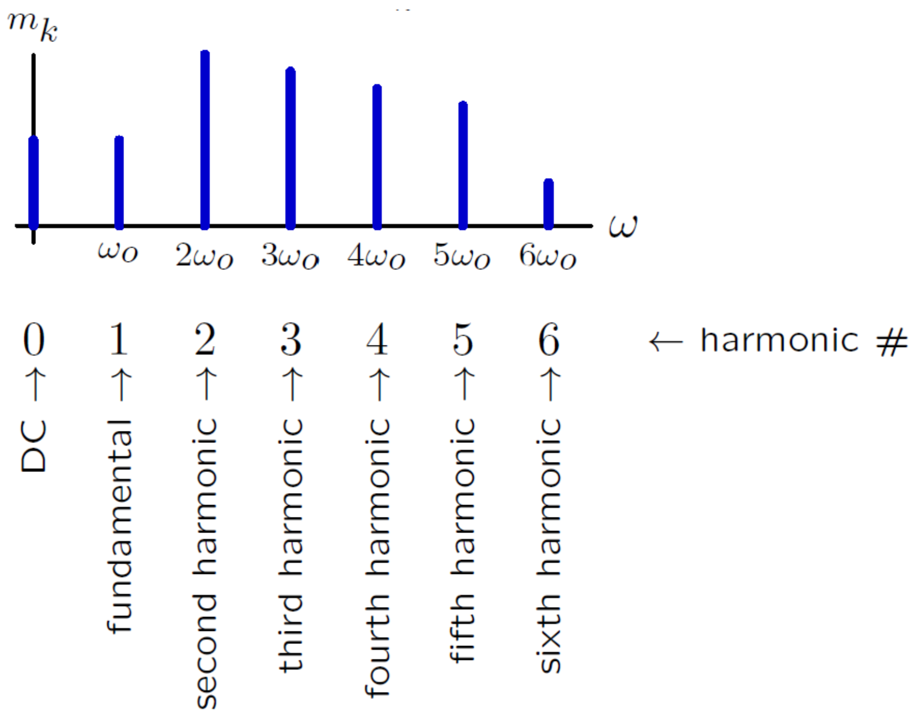
Harmonic structure

- The sum of sinusoids describes the distribution of energy across frequencies

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_o t + d_k \sin k\omega_o t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

where $m_k^2 = c_k^2 + d_k^2$ and $\tan \phi_k = \dfrac{d_k}{c_k}$.

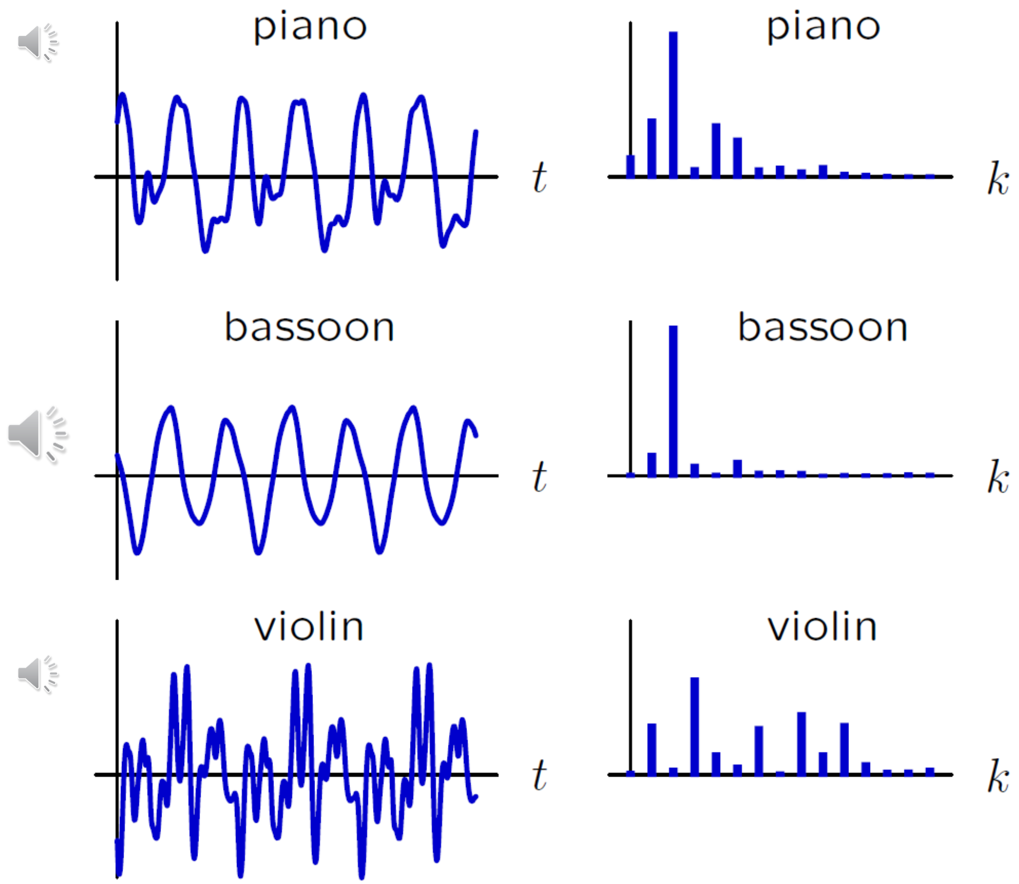
- Transform: signal of continuous time → signal of discrete harmonic numbers



- The distribution represents the **harmonic structure** of the signal.

Harmonic structure

- The harmonic structures of note from different instruments are different.

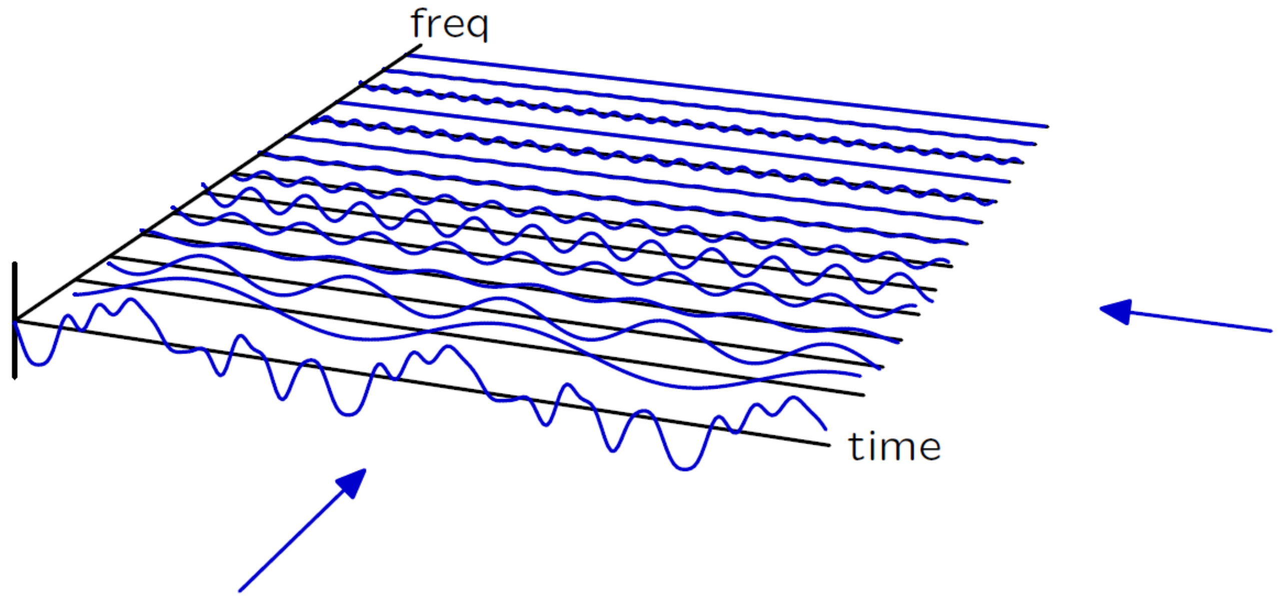


- Some musical qualities are more easily seen in time, others in frequency

Express each signal as a sum of sinusoids

$$f(t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

$$= m_1 \cos(\omega_o t+\phi_1) + m_2 \cos(2\omega_o t+\phi_2) + m_3 \cos(3\omega_o t+\phi_3) + \cdots$$



- Two views: as a function of time and as a function of frequency

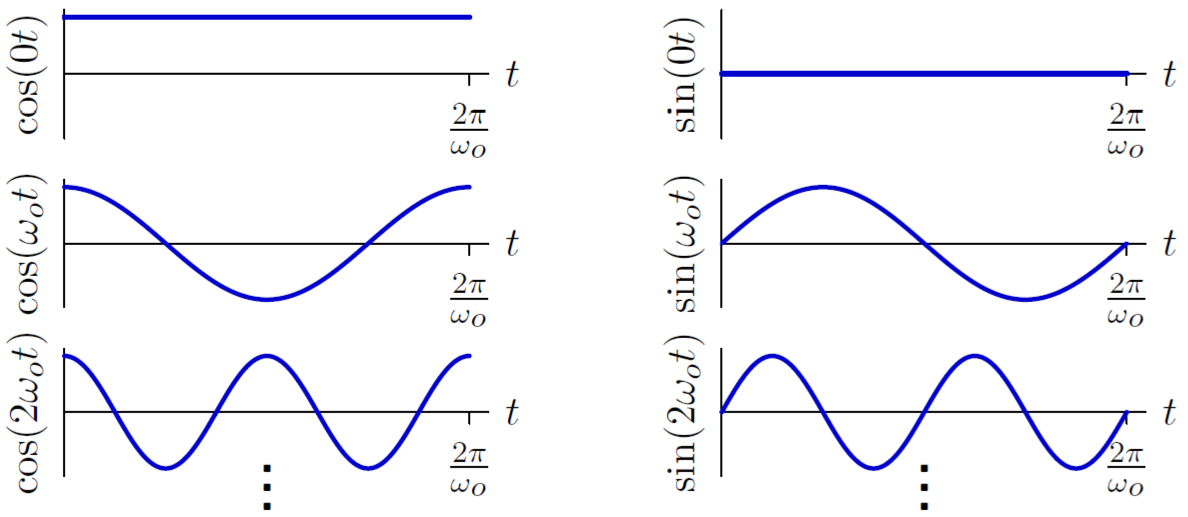
Fourier representations of signals

- Fourier series are sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

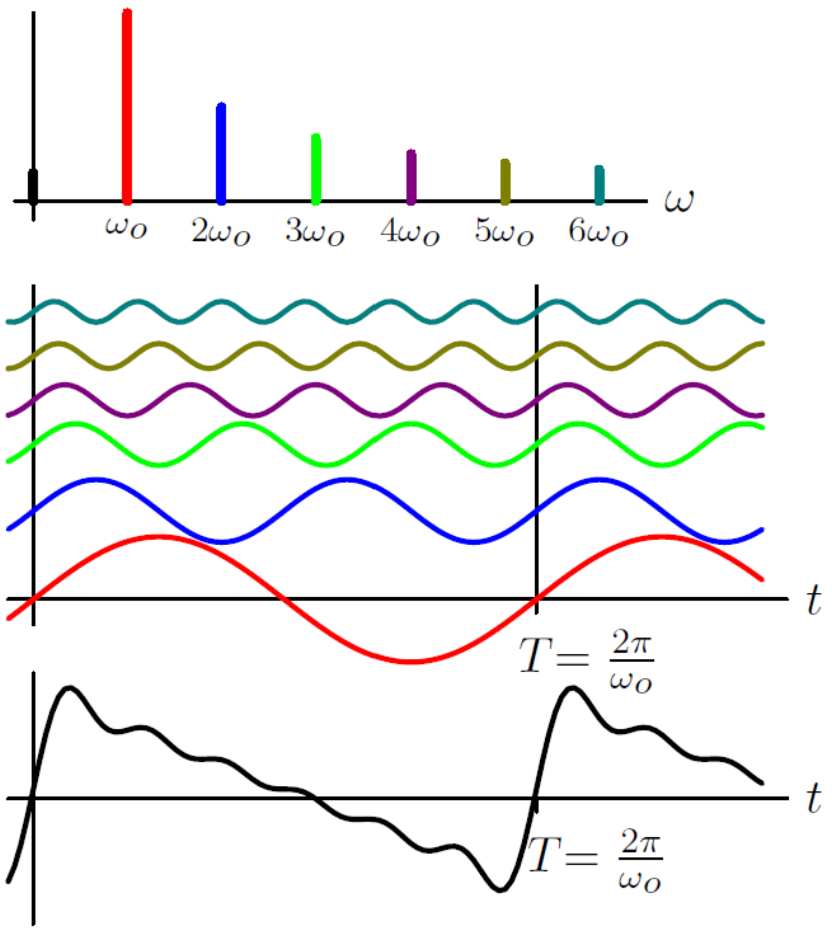
where $\omega_o = 2\pi/T$ represents the fundamental frequency.

Basis functions:



- Q1: Under what conditions can we write $f(t)$ as a Fourier series?
- Q2: How do we find the coefficients $c_k,\ d_k$?

Fourier series can only represent periodic signals



All harmonics of ω_o ($\cos(k\omega_o t)$ or $\sin(k\omega_o t)$) are periodic in $T = 2\pi/\omega_o$.
$\rightarrow$ all sums of such signals are periodic in $T = 2\pi/\omega_o$.
$\rightarrow$ Fourier series can only represent periodic signals.

Fourier series can only represent periodic signals

- Definition: a signal $f(t)$ is **periodic** in T if
  - $f(t) = f(t+T)$ for all t

- Note: if a signal is periodic in T it is also periodic in $2T$, $3T$, ...

- The smallest positive number T_0 for which $f(t) = f(t+T_0)$ for all t is sometimes called the **fundamental period**. **Fundamental** frequency ω_0

- If a signal does not satisfy $f(t) = f(t+T)$ for any value of T, then the signal is **aperiodic**.

$$f(t+T) = \sum_{k=0}^{\infty} \left(c_k \cos\big(k\omega_0(t+T)\big) + d_k \sin\big(k\omega_0(t+T)\big)\right) = f(t)$$

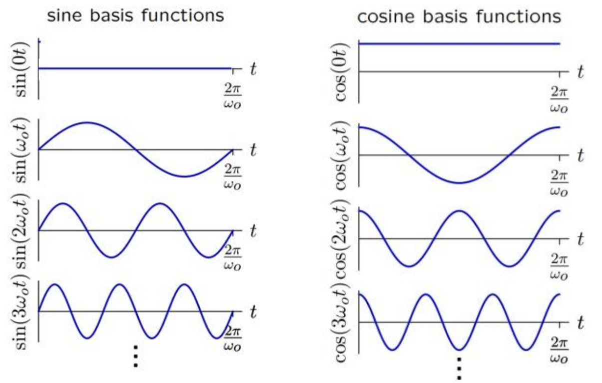
Q2: How do we find the coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos\left(\frac{2\pi k}{T} t\right) + d_k \sin\left(\frac{2\pi k}{T} t\right)\right)$$

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

- How to sift out coefficients?

Preliminaries: Sinusoids



- Average over a period:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T} t\right) dt = 0$$

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T} t\right) dt = \begin{cases} T & if\ k = 0 \\ 0 & otherwise \end{cases}$$

- Orthogonality of the basis functions:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T} t\right) \cos\left(\frac{2\pi m}{T} t\right) dt = 0$$

k and m are positive integers

- Orthogonality of the basis functions:

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T} t\right) \cos\left(\frac{2\pi m}{T} t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T} t\right) \sin\left(\frac{2\pi m}{T} t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

A product of sinusoids can be expressed as sum and difference frequencies.

$$\cos(k\omega_o t)\cos(l\omega_o t) = \frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)$$

$$\sin(k\omega_o t)\cos(l\omega_o t) = \frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)$$

Q2: How do we find the coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos\left(\frac{2\pi k}{T} t\right) + d_k \sin\left(\frac{2\pi k}{T} t\right)\right) \qquad f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

- How to sift out coefficients?
  - Key idea: by multiplying with each basis function and integrate over the period T.

Q: What will happen?

Integrate both sides over T:

$$\int_0^T f(t)\, dt = \int_0^T c_0\, dt + \int_0^T \left(\sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) \right) dt$$

$$= Tc_0 + \sum_{k=1}^{\infty} \left(c_k \int_0^T \cos(k\omega_o t)\, dt + d_k \int_0^T \sin(k\omega_o t)\, dt \right) = Tc_0$$

All but the first term integrates to zero, leaving

$$c_0 = \frac{1}{T} \int_0^T f(t)\, dt.$$

This $k{=}0$ term represents the average ("DC") value.

How do we find c_k

- Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}
\int_0^T f(t)\cos(l\omega_o t)\,dt &= \int_0^T c_0 \cos(l\omega_o t)\,dt \\
&+ \sum_{k=1}^{\infty} \int_0^T c_k \left(\frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)\right) dt \\
&+ \sum_{k=1}^{\infty} \int_0^T d_k \left(\frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)\right) dt
\end{aligned}$$

A product of sinusoids can be expressed as sum and difference frequencies.

$$\cos(k\omega_o t)\cos(l\omega_o t) = \frac{1}{2}\cos((k-l)\omega_o t) + \frac{1}{2}\cos((k+l)\omega_o t)$$

$$\sin(k\omega_o t)\cos(l\omega_o t) = \frac{1}{2}\sin((k-l)\omega_o t) + \frac{1}{2}\sin((k+l)\omega_o t)$$

How do we find c_k

- Isolate the c_l term by multiplying both sides by $\cos(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}
\int_0^T f(t)\cos(l\omega_o t)\,dt &= \int_0^T \cancelto{0}{c_0 \cos(l\omega_o t)}\,dt \\
&+ \sum_{k=1}^{\infty}\int_0^T c_k\left(\frac{1}{2}\cancelto{\frac{T}{2}c_l}{\cos((k-l)\omega_o t)} + \frac{1}{2}\cancelto{0}{\cos((k+l)\omega_o t)}\right)dt \\
&+ \sum_{k=1}^{\infty}\int_0^T d_k\left(\frac{1}{2}\cancelto{0}{\sin((k-l)\omega_o t)} + \frac{1}{2}\cancelto{0}{\sin((k+l)\omega_o t)}\right)dt
\end{aligned}$$

If $k = l$, then $\sin((k-l)\omega_o t = 0$ and the integral is 0.
All of the other d_k terms are harmonic sinusoids that integrate to 0.

The only non-zero term on the right side is $\frac{T}{2}c_l$.
We can solve to get an expression for c_l as

$$c_l = \frac{2}{T}\int_0^T f(t)\cos(l\omega_o t)\,dt$$

Calculating Fourier Coefficients : d_k

- Analogous reasoning allows us to calculate the d_k coefficients, but this time multiplying by $\sin(l\omega_o t)$ before integrating.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right)$$

$$\begin{aligned}\int_0^T f(t) \sin(l\omega_o t)\, dt &= \int_0^T c_0 \sin(l\omega_o t)\, dt \\ &+ \sum_{k=1}^{\infty} \int_0^T c_k \cos(k\omega_o t) \sin(l\omega_o t)\, dt \\ &+ \sum_{k=1}^{\infty} \int_0^T d_k \sin(k\omega_o t) \sin(l\omega_o t)\, dt\end{aligned}$$

A single term remains after integrating, allowing us to solve for d_l as

$$d_l = \frac{2}{T} \int_0^T f(t) \sin(l\omega_o t)\, dt$$

Calculating Fourier Coefficients

- Summarizing . . .

If $f(t)$ is expressed as a Fourier series

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

the Fourier coefficients are given by

$$c_0 = \frac{1}{T} \int_T f(t)\, dt$$

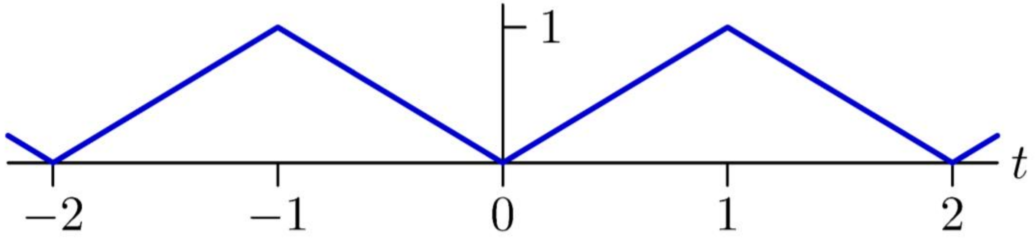
$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t)\, dt; \;\; k = 1, 2, 3, \ldots$$

Example of synthesis

Find the Fourier series coefficients for the following triangle wave:

$$f(t) = f(t+2)$$



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt = 2\int_0^1 t\cos(\pi kt)\,dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases}$$

$$d_k = 0 \quad \text{(by symmetry)}$$

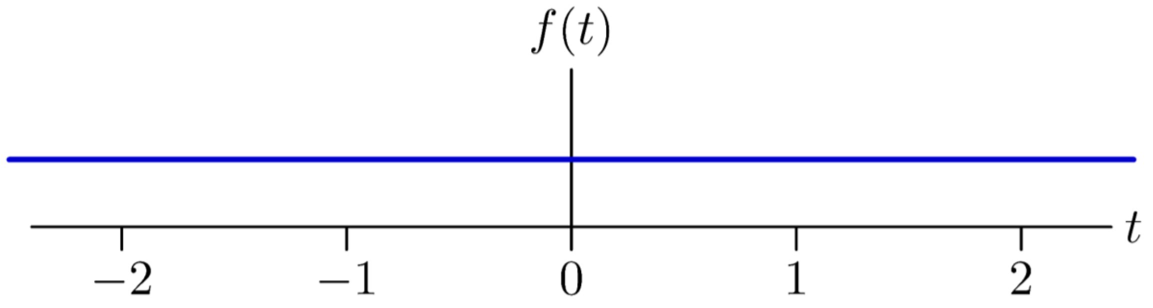
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{0} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



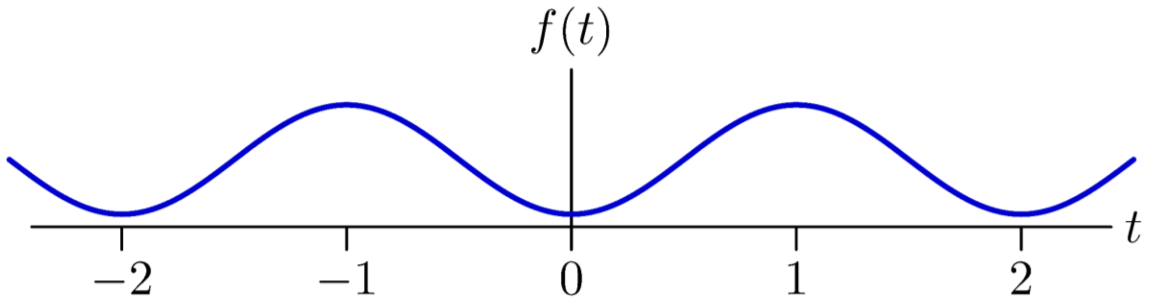
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{1} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



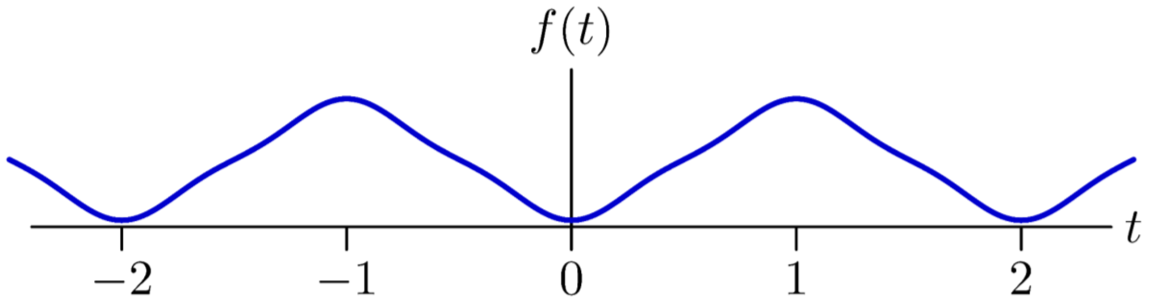
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{3} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



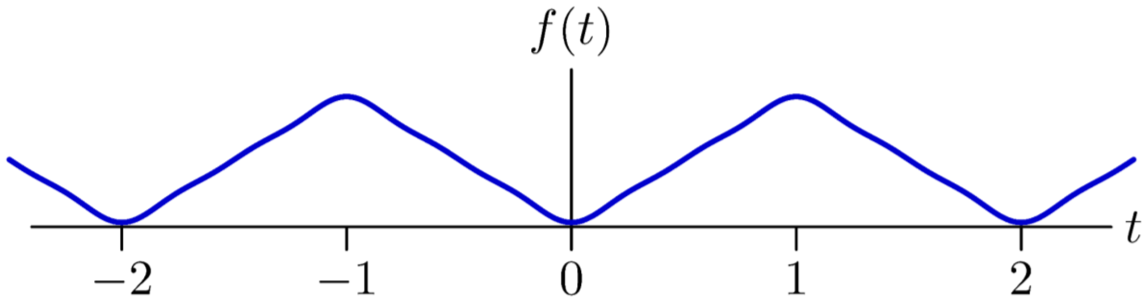
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{5} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



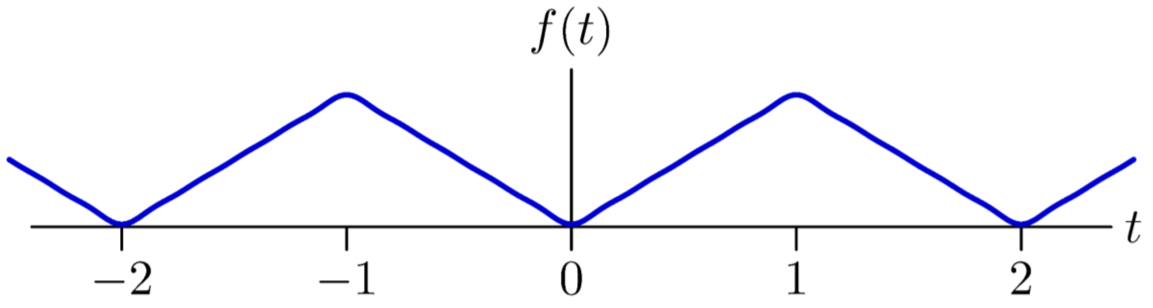
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{9} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



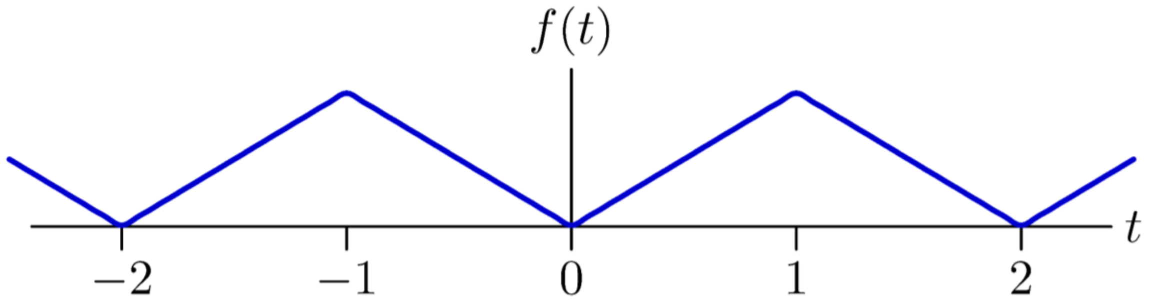
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{19} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



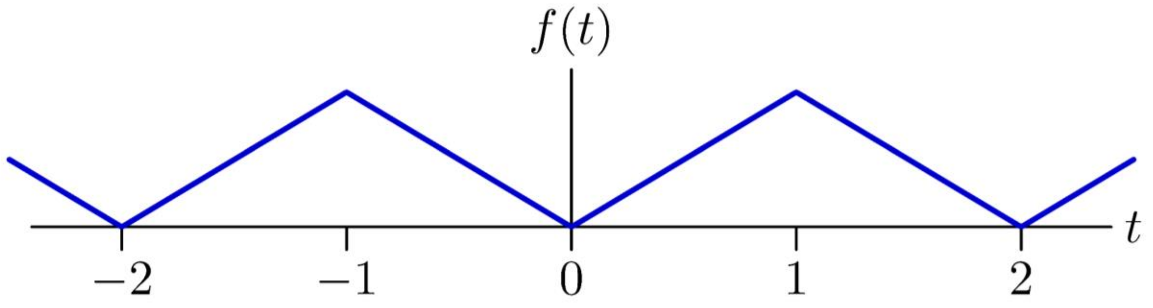
Example of synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



The synthesized function approaches original as number of terms increases.

Summary: Two views of the same signal

The harmonic expansion provides an alternative view of the signal.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) = \sum_{k=0}^{\infty} m_k \cos(k\omega_o t + \phi_k)$$

We can view the musical signal as

- a function of time $f(t)$, or
- as a sum of harmonics with amplitudes m_k and phase angles ϕ_k.

Both views are useful. For example,

- the peak sound pressure is more easily seen in $f(t)$, while
- consonance is more easily analyzed by comparing harmonics.

This type of harmonic analysis is an example of **Fourier Analysis**, which is a major theme of this subject.

Recitation and common-room hours

- Live question for the lecture
    - What’s your favorite type of signal? Try to express it as a function.
- We will go to 32-141 today for recitation & common hour~
- Common room hours this week
    - https://sigproc.mit.edu/fall24/software