

6.300 Signal Processing

Week 1, Lecture B: Signal Processing

- Overview of the subject
- Signals: Definitions, examples, and operations
- Time and Frequency Representations
- Fourier Series

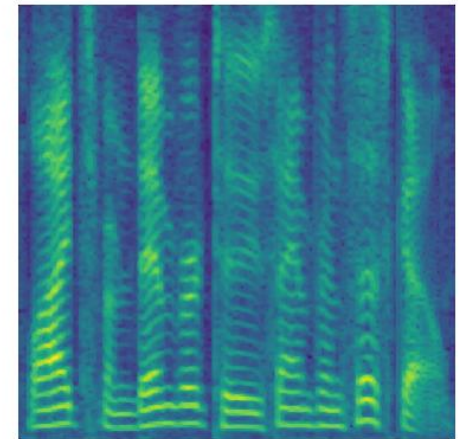
Lecture slides are available on CATSOOP:
<https://sigproc.mit.edu/fall24>

What is 6.300?

- 6.300 is about signal processing.
- What is a signal?
 - A signal is a function that conveys information
- What is signal processing?
 - Identifying signals in physical, mathematical, computation contexts
 - Analyzing signals to understand the information they contain
 - Manipulating signals to modify the information they contain

At the end of this class

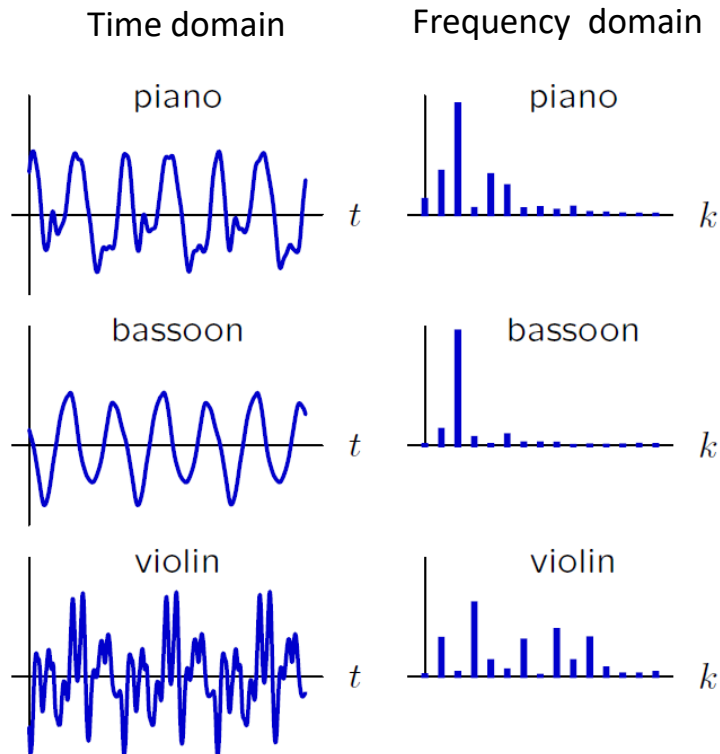
- Learn to identify signals in physical, mathematical, computation contexts
- Signals are **functions** that contain and convey **information**.
- Examples:
 - medical (EKG, EEG, MRI, OCT)
 - speech signals
 - music
 - images
 - video
 - seismic signals



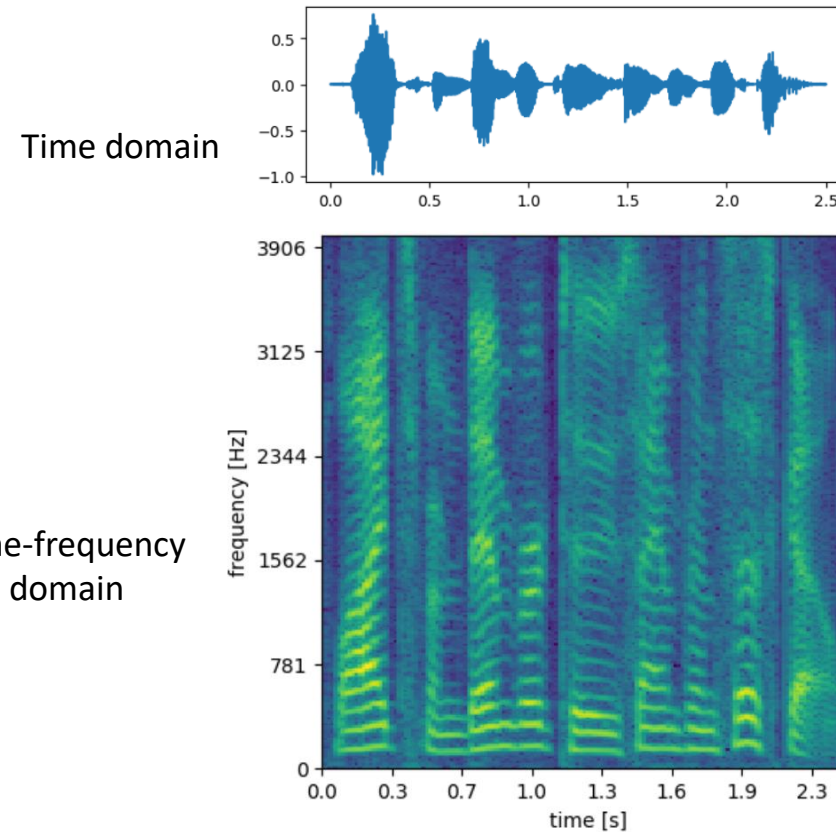
At the end of this class

- Analyzing signals to understand the information they contain
- Learn to think of signals in frequency domain (in addition to time, space, ...)
 - Mathematical analysis and physical understanding

Music analysis

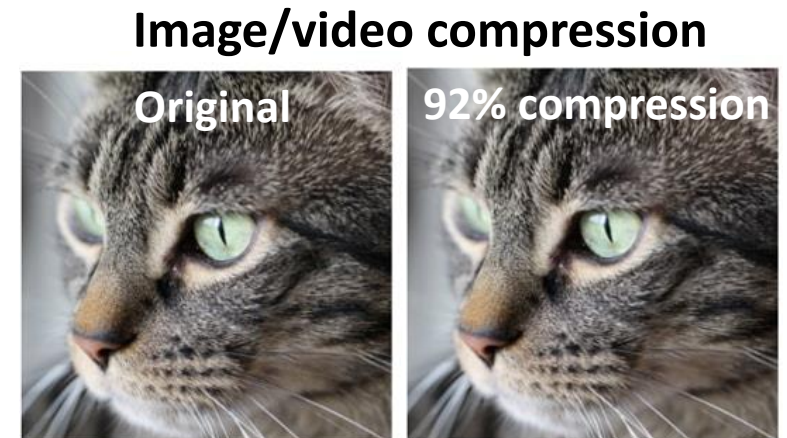
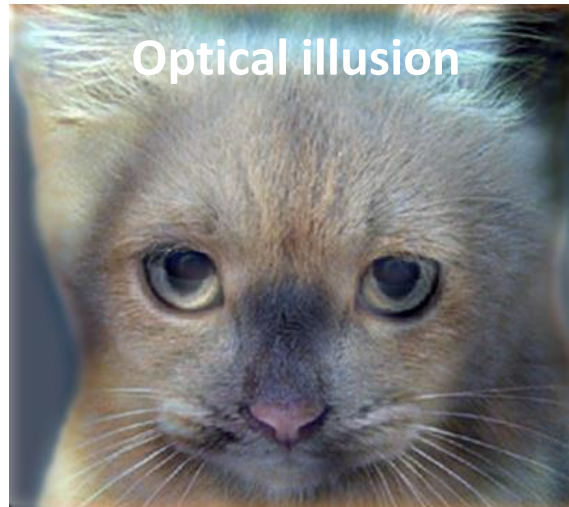
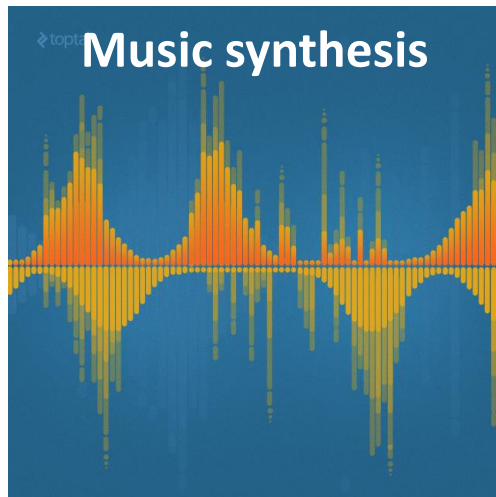


Speech processing



At the end of this class

- Learn to manipulate signals to modify the information they contain
- Learn to apply signal processing theories to real-life applications
 - Problem formulation, design, coding
 - Music, speech, photography, video streaming, astronomy, biomedicine...



Motion artifacts

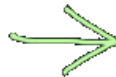
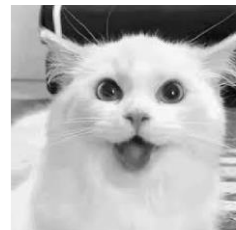


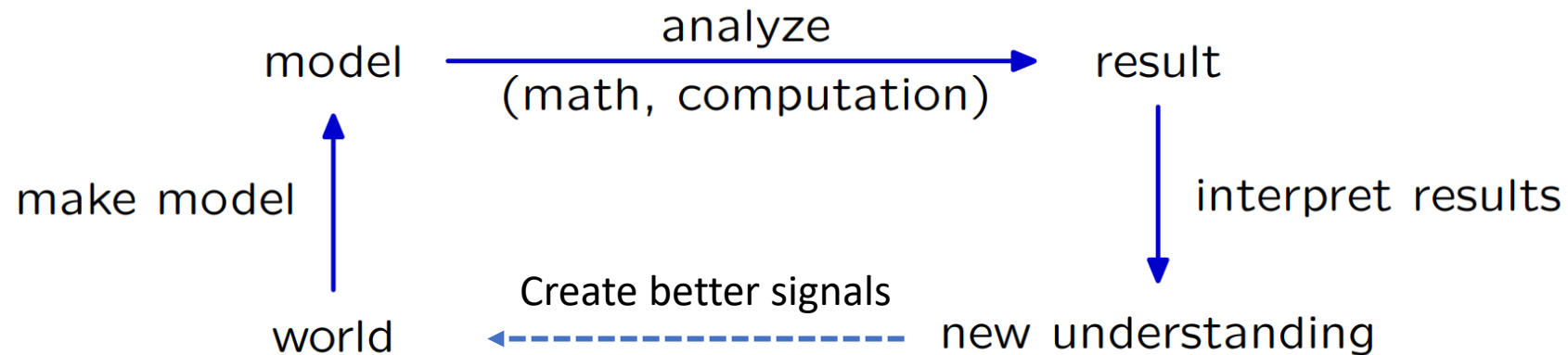
Image restoration



Signal Processing

Signal Processing is **widely used** in science and engineering to ...

- **model** some aspect of the world,
- **analyze** the model,
- **interpret** results to gain a new or better understanding.



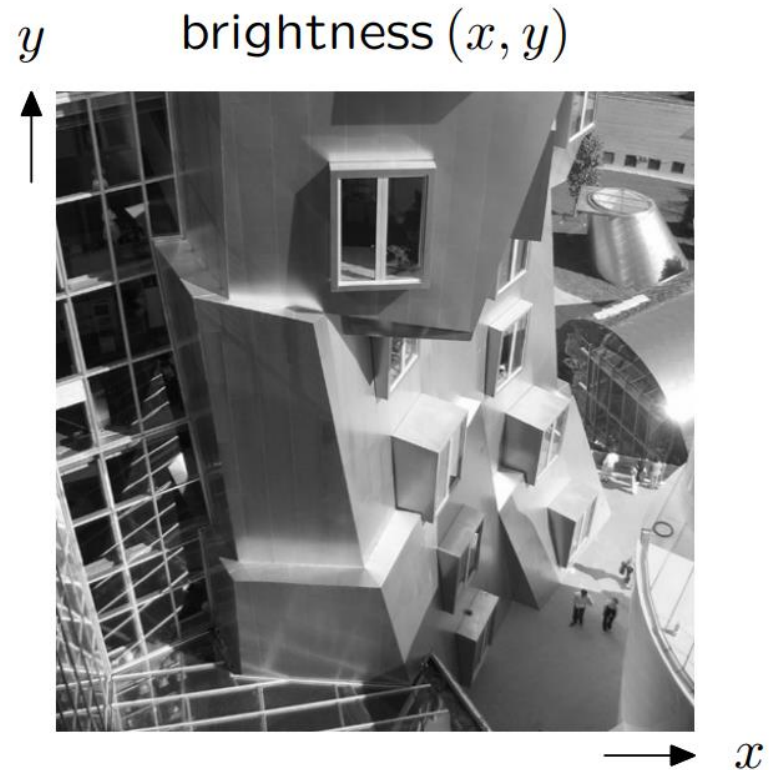
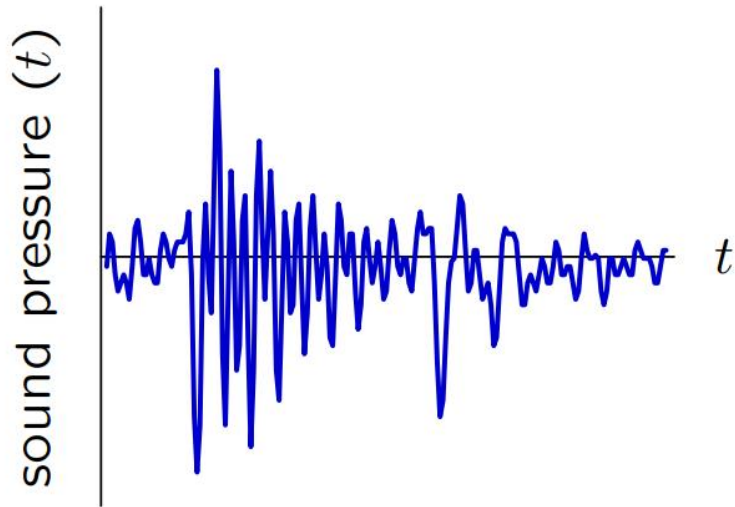
Signal Processing provides a common language across disciplines.

Get the most out of 6.300!

- Course website: CAT-SOOP (detailed policies).
- Lecture: TR2 (3-270)
 - Live questions in lecture (5% graded based on effort or weigh into final exam)
- Recitation: TR3 (32-141)
 - Live questions in recitation (5% graded based on effort or weigh into final exam)
- Piazza: **Only** for logistic questions
- Common-room hours: Monday-Friday 4-5pm, Monday & Wednesday 7-9pm
- Homework: posted Thursdays at 4pm; Lab check-in due the following Mondays at 9pm; Pset due the following Wednesdays 10pm
 - Psets: focus on developing problem solving skills – simple computational extensions to real-world data (15%). Drop one lowest-scored Psets.
 - Labs: focus on applications of 6.300 to real-world problems – more open-ended, multiple approaches, multiple solutions (5%+10%). **Start early!**
 - Two quizzes and a final (15% +20%+35% or 15% +20%+25%+10%)

Signals: independent variable

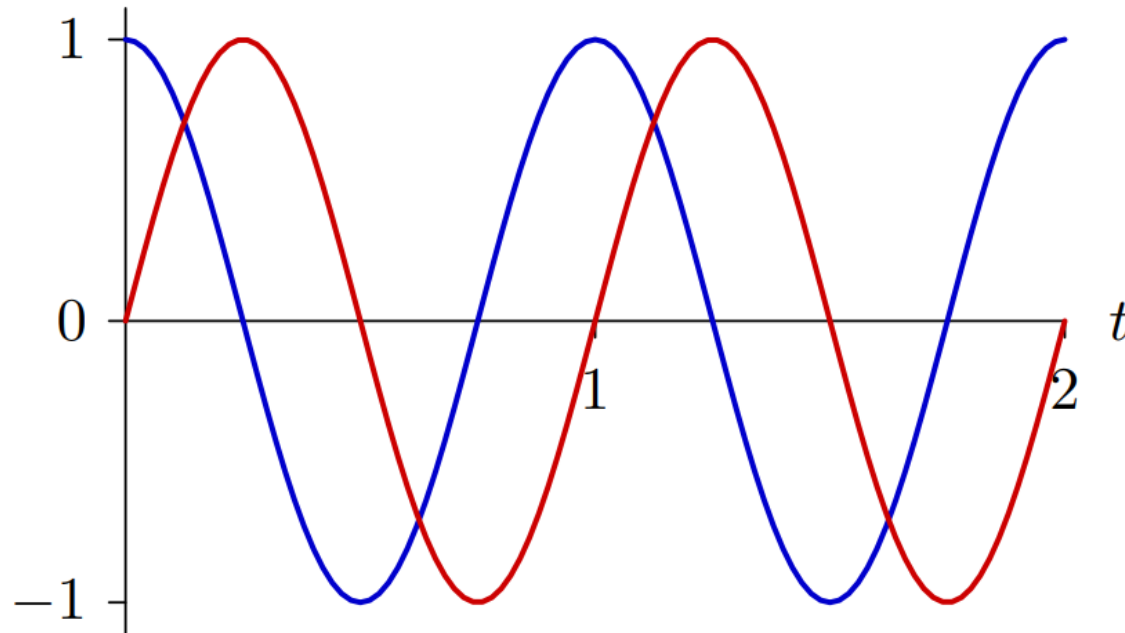
- Signals are functions that contain and convey information.
- Questions: Independent vs. dependent variable?



Signals: dependent variable

- Dependent variable can be real, imaginary, or complex-valued

$$x(t) = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$$



- Why complex?

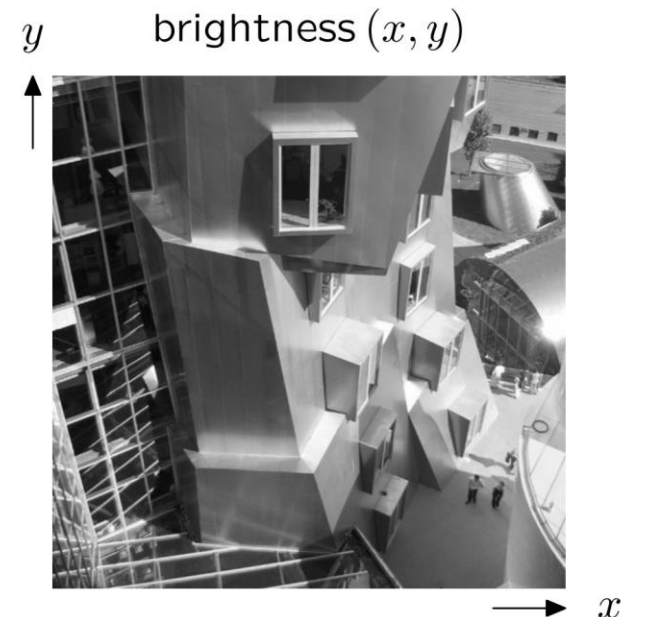
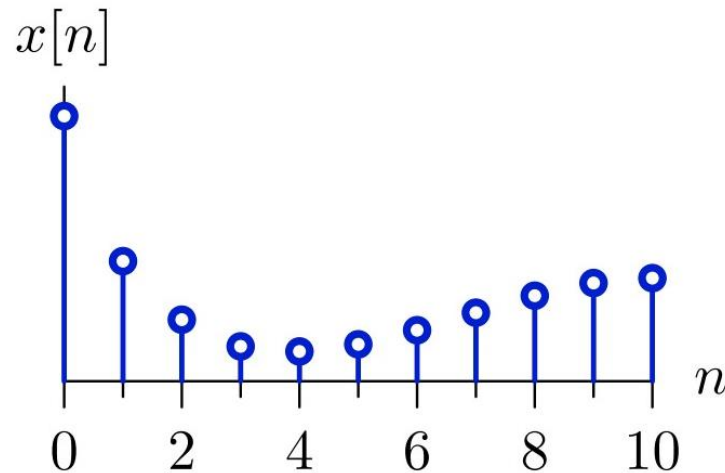
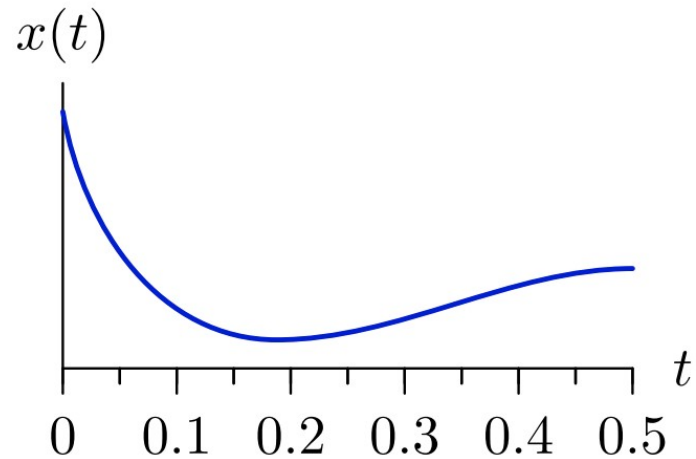
Signals: Continuous vs. Discrete

Physical signals are often of **continuous** domain:

- continuous time (in seconds)
- continuous spatial coordinates (in meters)

Computations manipulate functions of **discrete** domain:

- discrete time (in samples)
- discrete spatial coordinates (in pixels)



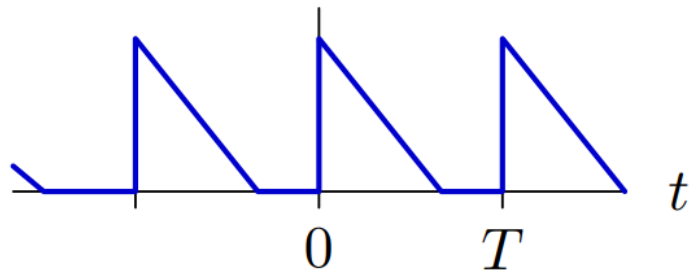
Examples?

Signals: Periodic vs Aperiodic

- Periodic signals consist of repeated cycles (periods). Important for analysis later.

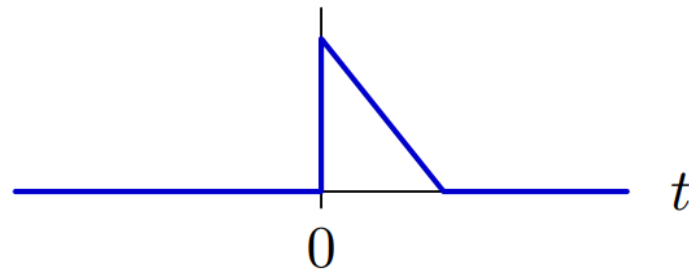
periodic

$$x(t) = x(t + T)$$

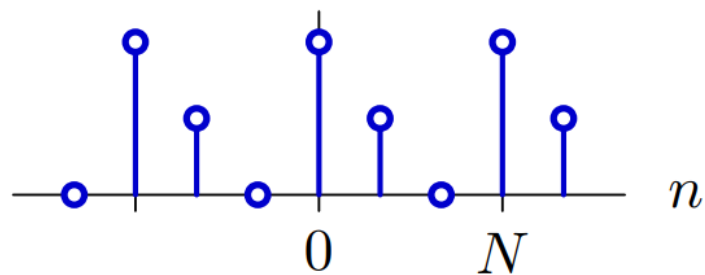


aperiodic

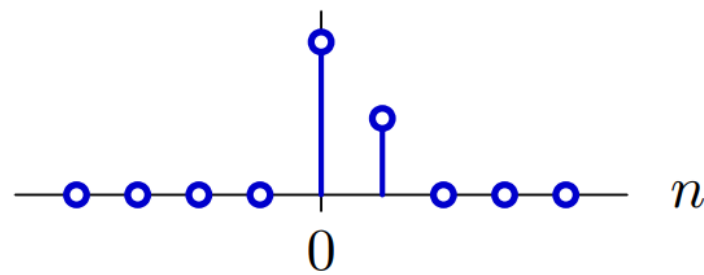
$$x(t)$$



$$x[n] = x[n + N]$$



$$x[n]$$

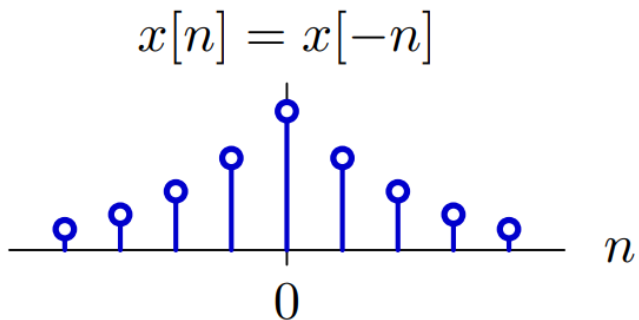
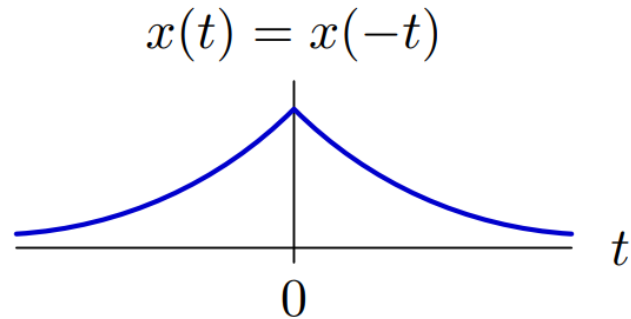


Examples?

Signals: Symmetric vs Antisymmetric

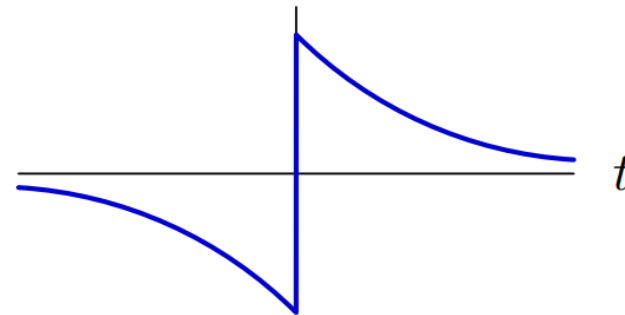
- Signals can be **symmetric** or **antisymmetric**, or neither symmetric/antisymmetric at all!

symmetric

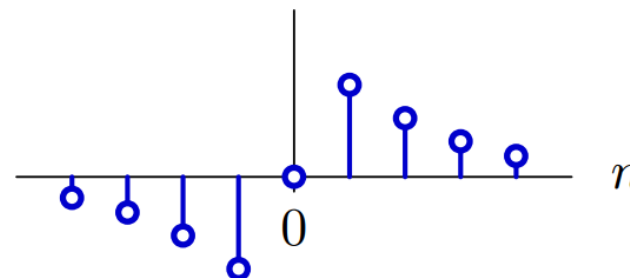


antisymmetric

$$x(t) = -x(-t)$$



$$x[n] = -x[-n]$$



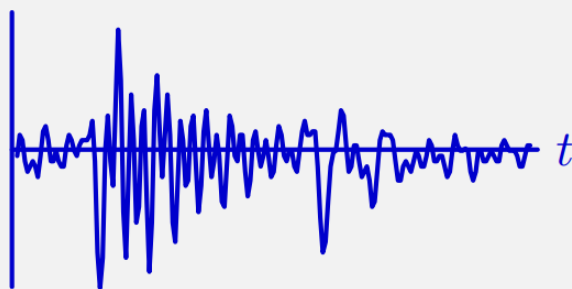
Important for analysis and intuition building later

Check yourself

- Before listening to the manipulated signals, can you think what should $f(2t)$, $-f(t)$ and $1/3f(t)$ look and sound like?

Computer generated speech (by Robert Donovan)





$f(t)$



Listen to the following four manipulated signals:

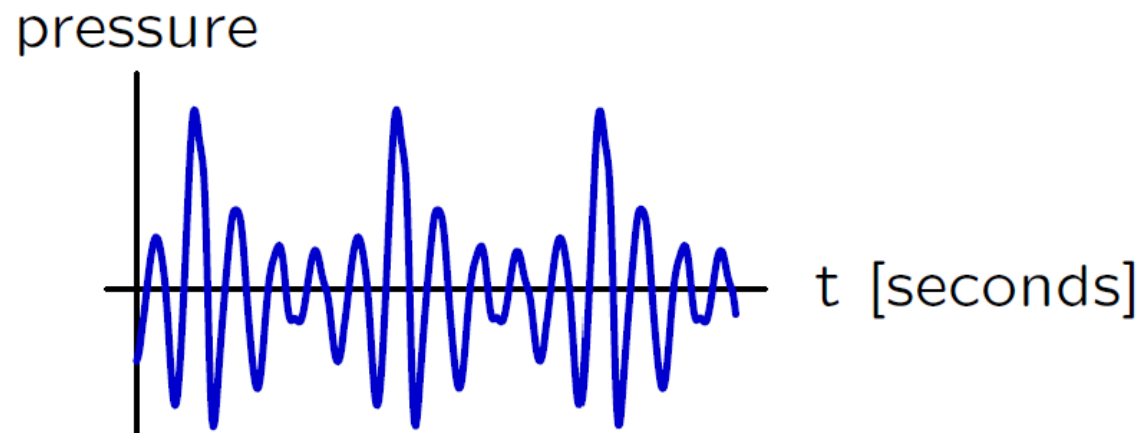
$f_1(t)$, $f_2(t)$, $f_3(t)$, $f_4(t)$.

How many of the following relations are true?

- $f_1(t) = f(2t)$ 
- $f_2(t) = -f(t)$ 
- $f_3(t) = f(2t)$ 
- $f_4(t) = \frac{1}{3}f(t)$ 

Music sounds as signals

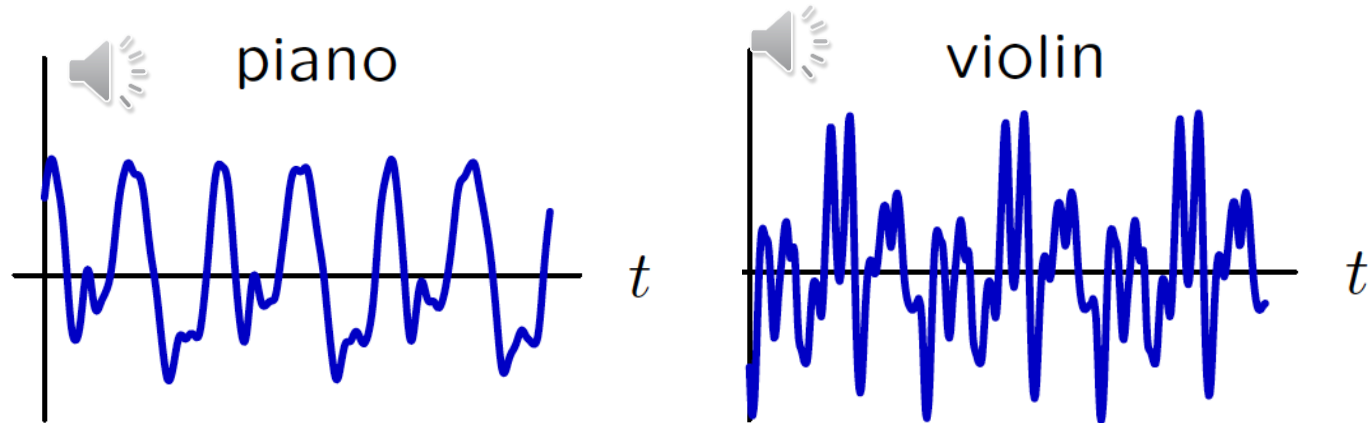
- Signals are functions that contain and convey information
- Example: a musical sound can be represented as a function of time.



- Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Music sounds as signals of time

- Even though these sounds have the same pitch, they sound different.

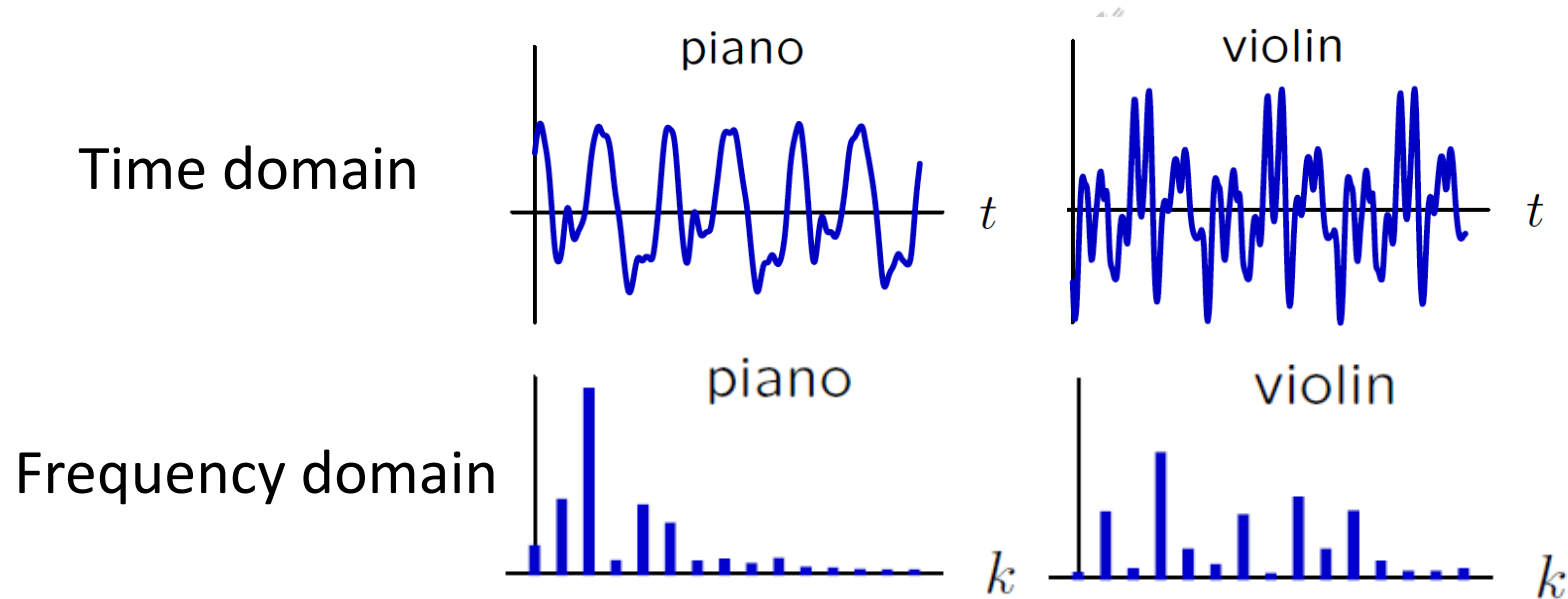


- It's not clear how the differences relate to properties of the signals. (audio clips from <http://theremin.music.uiowa.edu>)

Music sounds as signals of frequency

- Transform: reveal important properties of the signal (otherwise hidden in time domain)

Same pitch, they sound different. Why?

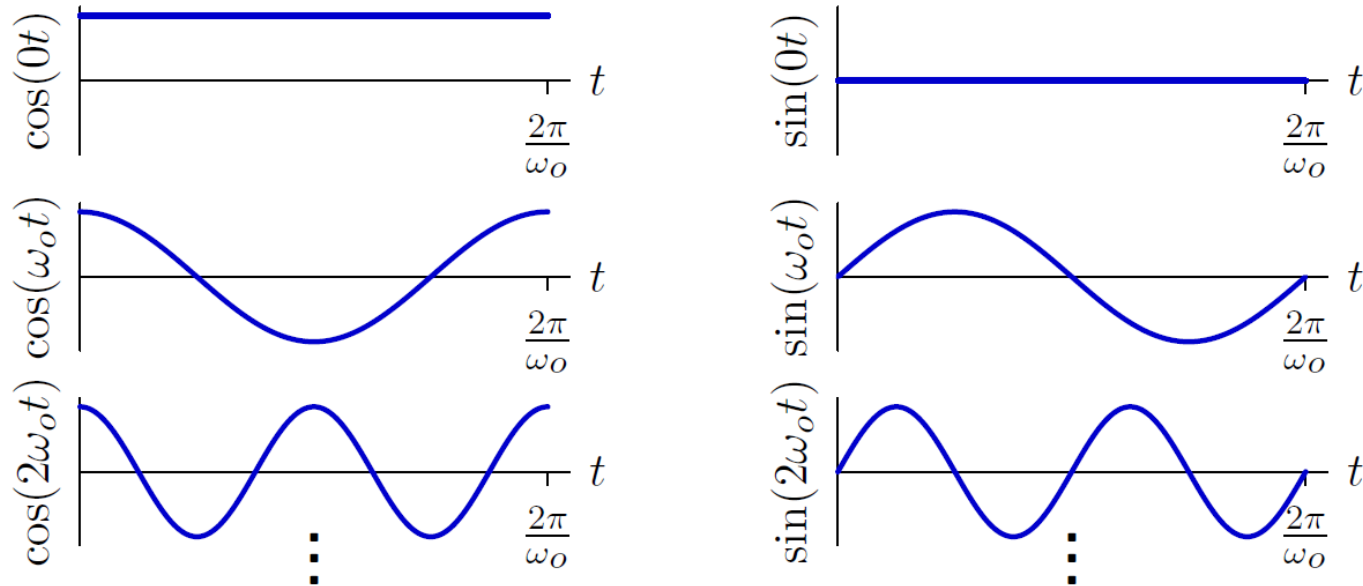


- The harmonic structures of notes from different instruments are different.

Music signals as sum of sinusoids

- How: One way to characterize differences between these signals is express them each as a sum of sinusoids

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t)$$



- Since these sounds are (nearly) periodic, the frequencies of the dominant sinusoids are (nearly) integer multiples of a **fundamental** frequency ω_0

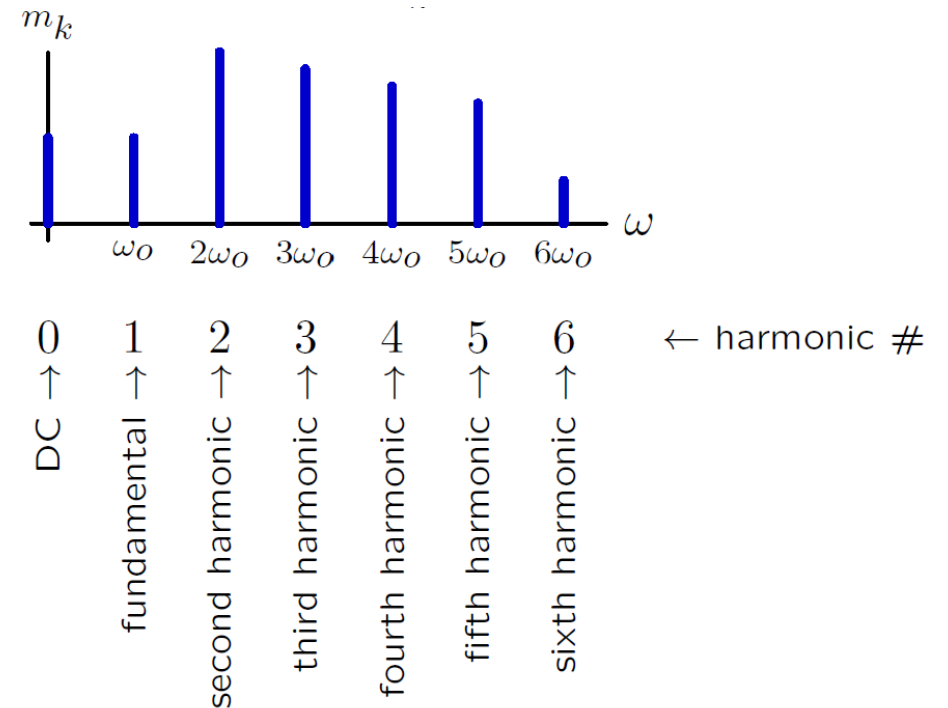
Harmonic structure

- The sum of sinusoids describes the distribution of energy across frequencies

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t) = \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k)$$

where $m_k^2 = c_k^2 + d_k^2$ and $\tan \phi_k = \frac{d_k}{c_k}$.

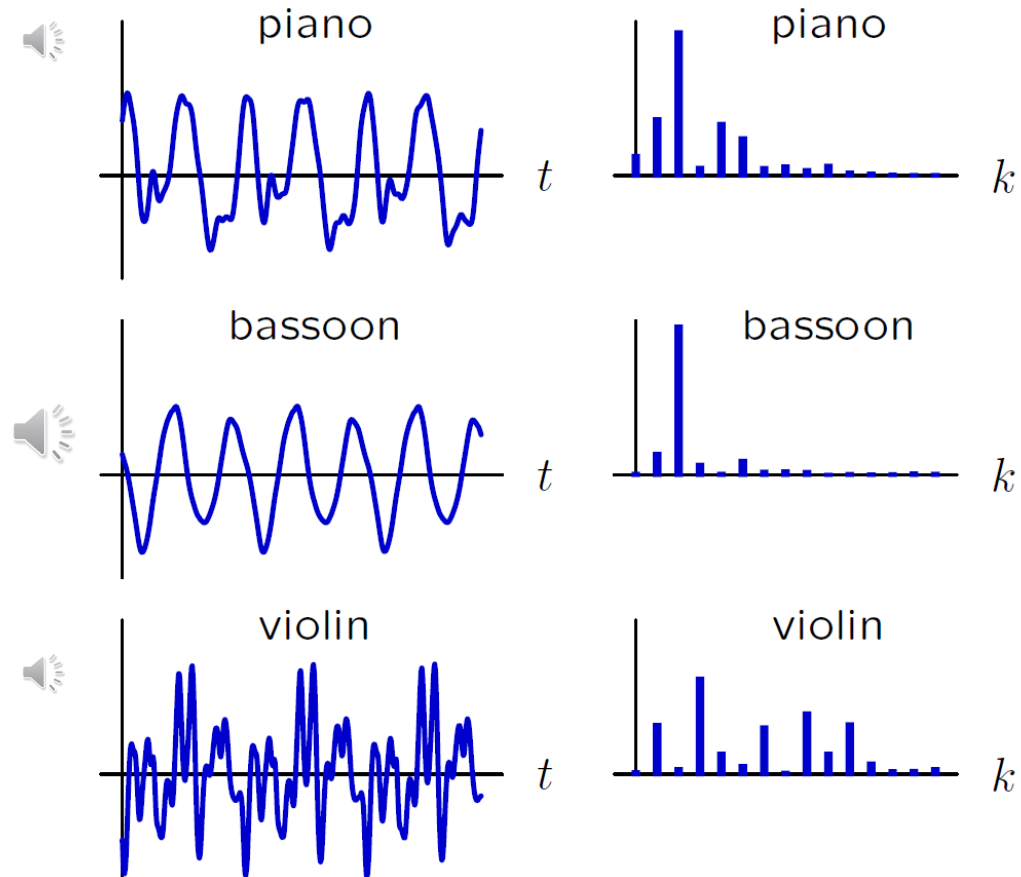
- Transform: signal of continuous time \rightarrow signal of discrete harmonic numbers



- The distribution represents the **harmonic structure** of the signal.

Harmonic structure

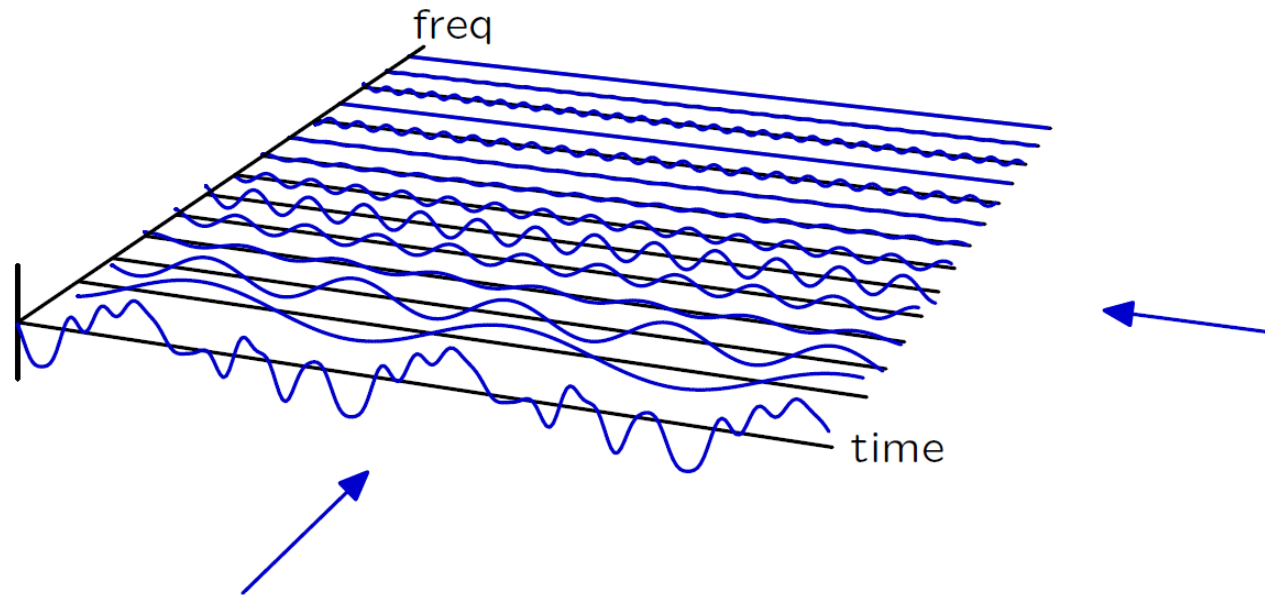
- The harmonic structures of note from different instruments are different.



- Some musical qualities are more easily seen in time, others in frequency

Express each signal as a sum of sinusoids

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k) \\ &= m_1 \cos(\omega_0 t + \phi_1) + m_2 \cos(2\omega_0 t + \phi_2) + m_3 \cos(3\omega_0 t + \phi_3) + \dots \end{aligned}$$



- Two views: as a function of time and as a function of frequency

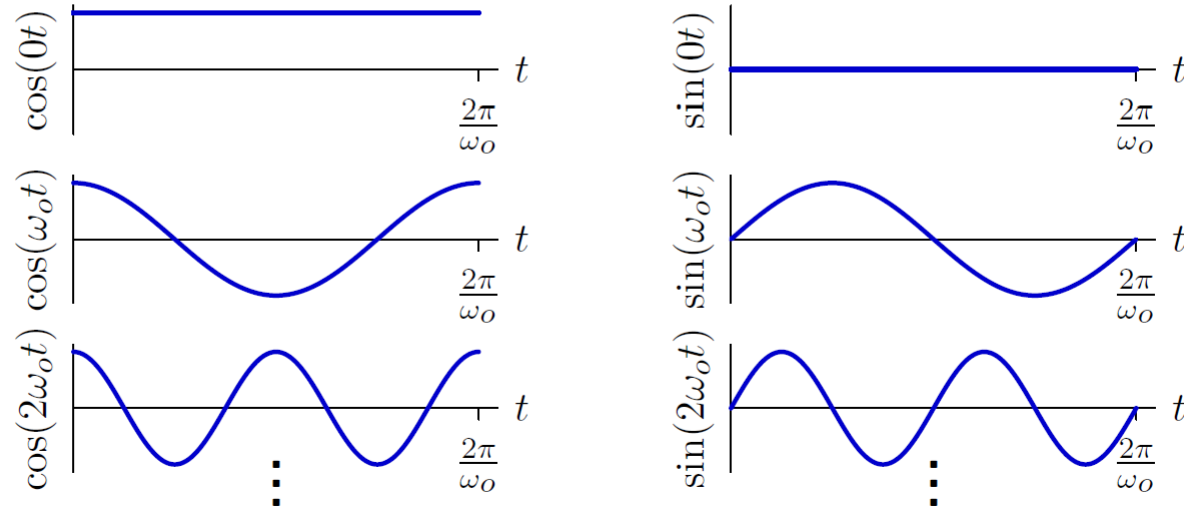
Fourier representations of signals

- Fourier series are sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

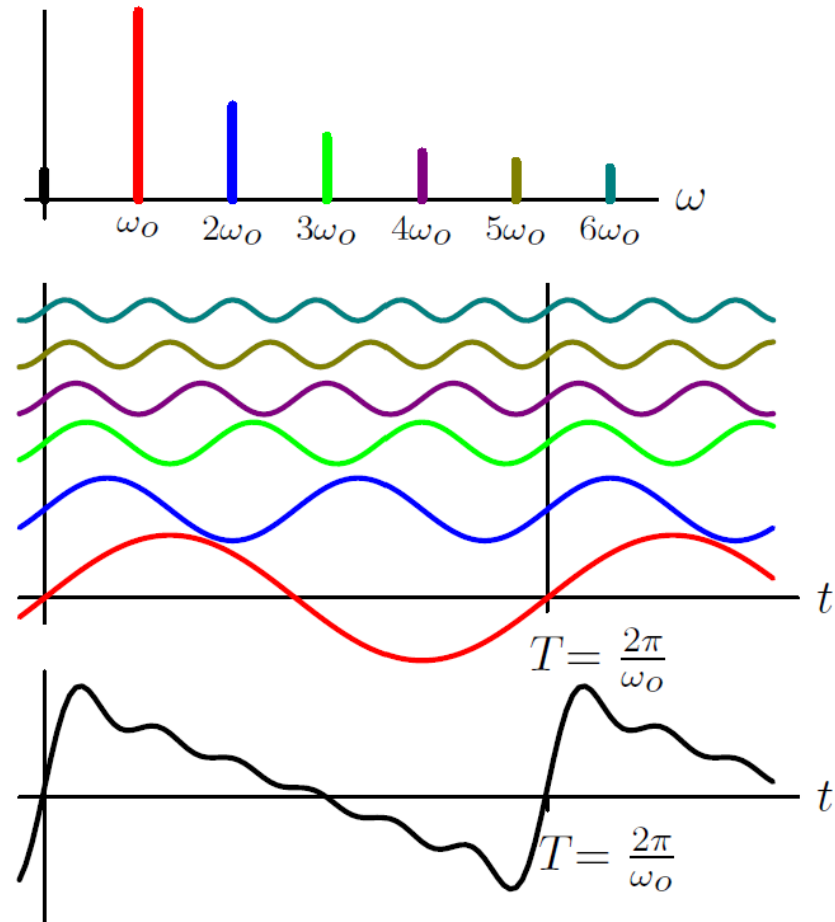
where $\omega_0 = 2\pi/T$ represents the fundamental frequency.

Basis functions:



- Q1: Under what conditions can we write $f(t)$ as a Fourier series?
- Q2: How do we find the coefficients c_k , d_k ?

Fourier series can only represent periodic signals



- All harmonics of ω_0 ($\cos(k\omega_0 t)$ or $\sin(k\omega_0 t)$) are periodic in $T = 2\pi/\omega_0$.
- all sums of such signals are periodic in $T = 2\pi/\omega_0$.
- Fourier series can only represent periodic signals.

Fourier series can only represent periodic signals

- Definition: a signal $f(t)$ is **periodic** in T if
 - $f(t) = f(t + T)$ for all t
- Note: if a signal is periodic in T it is also periodic in $2T, 3T, \dots$
- The smallest positive number T_0 for which $f(t) = f(t + T_0)$ for all t is sometimes called the **fundamental period**. **Fundamental** frequency ω_0
- If a signal does not satisfy $f(t) = f(t + T)$ for any value of T , then the signal is **aperiodic**.

$$f(t + T) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0(t + T)) + d_k \sin(k\omega_0(t + T))) = f(t)$$

Q2: How do we find the coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos\left(\frac{2\pi k}{T}t\right) + d_k \sin\left(\frac{2\pi k}{T}t\right) \right) \quad f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

- How to sift out coefficients?

Preliminaries: Sinusoids

- Average over a period:

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) dt = 0 \quad \int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Orthogonality of the basis functions:

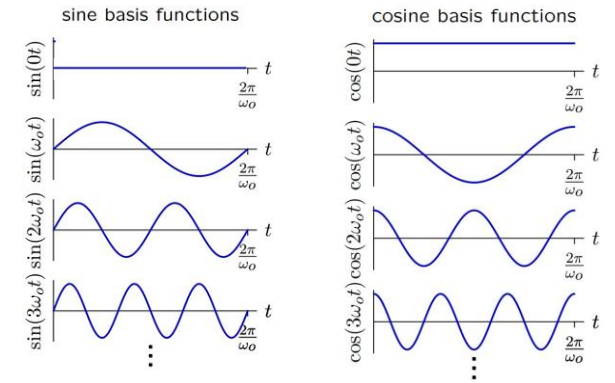
$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = 0$$

k and m are positive integers

- Orthogonality of the basis functions:

$$\int_{t_0}^{t_0+T} \cos\left(\frac{2\pi k}{T}t\right) \cos\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{t_0}^{t_0+T} \sin\left(\frac{2\pi k}{T}t\right) \sin\left(\frac{2\pi m}{T}t\right) dt = \begin{cases} T/2 & \text{if } k = m, \\ 0 & \text{otherwise} \end{cases}$$



A product of sinusoids can be expressed as sum and difference frequencies.

$$\cos(k\omega_0 t) \cos(l\omega_0 t) = \frac{1}{2} \cos((k-l)\omega_0 t) + \frac{1}{2} \cos((k+l)\omega_0 t)$$

$$\sin(k\omega_0 t) \cos(l\omega_0 t) = \frac{1}{2} \sin((k-l)\omega_0 t) + \frac{1}{2} \sin((k+l)\omega_0 t)$$

Calculating Fourier Coefficients

- Summarizing . . .

If $f(t)$ is expressed as a Fourier series

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t))$$

the Fourier coefficients are given by

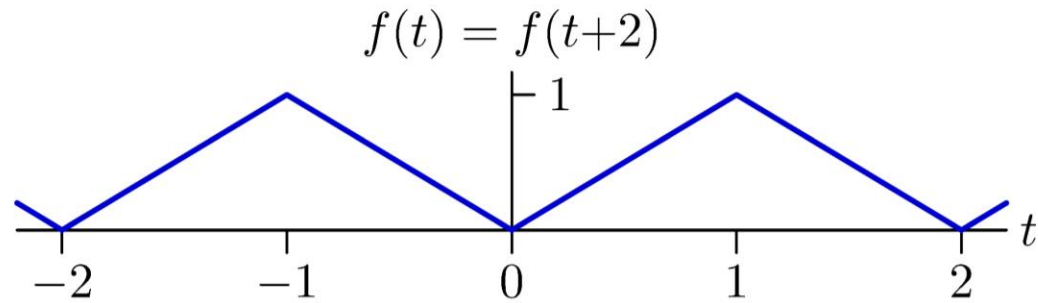
$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

Example of synthesis

Find the Fourier series coefficients for the following triangle wave:



Summary: Two views of the same signal

The harmonic expansion provides an alternative view of the signal.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \sum_{k=0}^{\infty} m_k \cos(k\omega_0 t + \phi_k)$$

We can view the musical signal as

- a function of time $f(t)$, or
- as a sum of harmonics with amplitudes m_k and phase angles ϕ_k .

Both views are useful. For example,

- the peak sound pressure is more easily seen in $f(t)$, while
- consonance is more easily analyzed by comparing harmonics.

This type of harmonic analysis is an example of **Fourier Analysis**, which is a major theme of this subject.

Recitation and common-room hours

- Live question for the lecture
 - What's your favorite type of signal? Try to express it as a function.
- We will go to 32-141 today for recitation & common hour~
- Common room hours this week
 - <https://sigproc.mit.edu/fall24/software>