6.3000: Signal Processing

**Discrete-Time Fourier Series**
- Fourier series representations for discrete-time signals
- Comparison of Fourier series for CT and DT signals
- Properties of DT Fourier series
- Applications of Fourier analysis

HW 3 due dates are extended to accommodate tomorrow’s student holiday.
- Lab 3 check-in is due on Monday at 9pm (after Common-Room Hours).
- The rest of HW 3 is due on Friday, September 30, at noon.

To help you prepare for Quiz 1 (October 4, 2-4pm), we will distribute a practice quiz on Tuesday, September 27. The practice quiz is provided as a study aid. Your solutions will not be submitted or counted in your grade.

There will be no HW 4.

*September 22, 2022*
Continuous-Time Fourier Series

We previously explored the expansion of periodic CT functions as Fourier series using either trigonometric functions or complex exponentials

\[ f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos k\omega_o t + \sum_{k=1}^{\infty} d_k \sin k\omega_o t = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \]

where \( \omega_o = \frac{2\pi}{T} \) represents the fundamental frequency.
Continuous-Time Fourier Series

We found the Fourier series coefficients using two key insights.

1. Multiplying complex harmonics of $\omega_o$ yields a complex harmonic of $\omega_o$:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}$$

2. Integrating a complex harmonic over a period $T$ yields zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt \equiv \int_T e^{jk\omega_0 t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} = T\delta[k]$$

where $\delta[k]$ is the Kronecker delta function

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

→ Fourier components are orthogonal.
Continuous-Time Fourier Series

Use orthogonality to find the Fourier series coefficients.

\[ f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

Multiply \( f(t) \) by the complex conjugate of the basis function of interest, and then integrate over \( T \).

\[
\int_T f(t) e^{-jl\omega_0 t} dt = \int_T \left( \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) e^{-jl\omega_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-l)\omega_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = a_l T
\]

Solving for \( a_l \) and then substituting \( k \) for \( l \) yields

\[
 a_k = \frac{1}{T} \int_T f(t)e^{-jk\omega_0 t} dt
\]
Continuous-Time Fourier Series

Representing a periodic signal as a sum of harmonic sinusoids.

**Synthesis Equation**

\[ f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

**Analysis Equation**

\[ a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} \, dt \]

where \( \omega_0 = \frac{2\pi}{T} \)
Goal: Develop a Fourier representation for discrete-time signals.

While there are many similarities between series representations of CT and DT signals, there are also important differences.

Perhaps the biggest difference results from aliasing. As we saw last time the same set of samples can represent many different frequencies.

Samples (blue) of the original high-frequency signal (green) could just as easily have come from a much lower frequency signal (red).

There are other important differences between CT and DT signals.
Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

1. \( f_1[n] = \cos \left( \frac{\pi n}{12} \right) \)

2. \( f_2[n] = \cos \left( \frac{\pi n}{12} \right) + 3 \cos \left( \frac{\pi n}{15} \right) \)

3. \( f_3[n] = \cos(n) \)
Check Yourself

\[ f_1[n] = \cos \left( \frac{\pi n}{12} \right) = f_1[n+N] \]

then

\[ \frac{\pi N}{12} = 2\pi m \]

where both \( N \) and \( m \) are integers. Solving, we find

\[ N = \frac{24\pi m}{\pi} \]

which is 24 if \( m = 1 \). Therefore \( N = 24 \).
Check Yourself

Similarly if

\[ f_2[n] = \cos \left( \frac{\pi n}{12} \right) + 3 \cos \left( \frac{\pi n}{15} \right) = f_2[n+N] \]

then

\[ \frac{\pi N}{12} = 2\pi m_1 \quad \rightarrow \quad N = \frac{24\pi m_1}{\pi} \]

and

\[ \frac{\pi N}{15} = 2\pi m_2 \quad \rightarrow \quad N = \frac{30\pi m_2}{\pi} \]

for integers \( m_1 \) and \( m_2 \).

We seek the smallest possible value of \( N \) so

\[ N = 24m_1 = 2 \times 2 \times 2 \times 3 \times m_1 = 30m_2 = 2 \times 3 \times 5 \times m_2 \]

and the smallest possible \( N \) is \( 2 \times 2 \times 2 \times 3 \times 5 = 120 \).
Check Yourself

If

\[ f_3[n] = \cos(n) = f_3[n+N] \]

then

\[ N = 2\pi m \]

where both \( N \) and \( m \) are integers.

This is not possible since \( \pi \) is not rational.

Therefore \( f_3[n] \) is not periodic in \( n \).
What is the fundamental (shortest) period of each of the following DT signals?

1. \( f_1[n] = \cos \left( \frac{\pi n}{12} \right) \) \( 24 \)

2. \( f_2[n] = \cos \left( \frac{\pi n}{12} \right) + 3 \cos \left( \frac{\pi n}{15} \right) \) \( 120 \)

3. \( f_3[n] = \cos(n) \) \( \infty \)
Check Yourself

What is the fundamental (shortest) period of each of the following DT signals?

1. \( f_1[n] = \cos \left( \frac{\pi n}{12} \right) \) \( \text{24} \)

2. \( f_2[n] = \cos \left( \frac{\pi n}{12} \right) + 3 \cos \left( \frac{\pi n}{15} \right) \) \( \text{120} \)

3. \( f_3[n] = \cos(n) \) \( \infty \)

The period of a periodic DT signal must be an integer.

Combined with aliasing, this constraint drastically reduces both the number of possible DT series and the complexity of each of those series.
Discrete-Time Sinusoids

There are (only) $N$ distinct complex exponentials with integer period $N$.

If $f[n] = e^{j\Omega n}$ is periodic in $N$ then

$$f[n] = e^{j\Omega n} = f[n+N] = e^{j\Omega(n+N)} = e^{j\Omega n}e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1. Therefore $e^{j\Omega}$ must be one of the $N^{th}$ roots of 1.

Example: $N = 8$

There are only 8 distinct complex exponentials with period $N = 8$:

$$e^{j0\pi/4}, e^{j1\pi/4}, e^{j2\pi/4}, e^{j3\pi/4}, e^{j4\pi/4}, e^{j5\pi/4}, e^{j6\pi/4}, e^{j7\pi/4}.$$ 

There are an infinite number of complex exponentials with period $T$ in CT!
Discrete-Time Sinusoids

There are $N$ distinct complex exponentials with period $N$.

Example: periodic in $N=3$

3 samples repeated in time

Example: periodic in $N=4$

4 samples repeated in time

If a DT signal is periodic with period $N$, then its Fourier series will contain just $N$ terms.
A DT Fourier Series has just $N$ harmonic frequencies $k\Omega_o$. 

$$f[n] = c_0 + \sum_{k=\langle N \rangle} c_k \cos(k\Omega_on) + \sum_{k=\langle N \rangle} d_k \sin(k\Omega_on)$$

where $\Omega_o$ represents the fundamental frequency (radians/sample). Otherwise, DT Fourier series are similar to CT Fourier series.
Discrete-Time Fourier Series

The same two key insights apply to both CT and DT analysis.

1. Multiplying complex DT harmonics of \( \Omega_o \) yields a new harmonic of \( \Omega_o \):

\[
e^{jk\Omega_on} \times e^{jl\Omega_on} = e^{j(k+l)\Omega_on}
\]

2. **Summing** a complex harmonic over a period \( N \) is zero unless the harmonic is at DC:

\[
\sum_{n=n_0}^{n_0+N-1} e^{jk\Omega_on} \equiv \sum_{n=\langle N \rangle} e^{jk\Omega_on} = \begin{cases} 
N & \text{if } k = 0 \\
0 & \text{if } k \neq 0 
\end{cases} = N\delta[k]
\]

→ DT Fourier components are **orthogonal**.
Discrete-Time Fourier Series

Using orthogonality to find the DT Fourier series coefficients.

\[ f[n] = f[n+N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_{on}} \]

Multiply \( f[n] \) by the complex conjugate of the basis function of interest, and then sum over \( N \).

\[
\sum_{n=\langle N \rangle} f[n] e^{-jl\Omega_{on}} = \sum_{n=\langle N \rangle} \left( \sum_{k=\langle N \rangle} a_k e^{jk\Omega_{on}} \right) e^{-jl\Omega_{on}} \\
= \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-l)\Omega_{on}} \\
= \sum_{k=\langle N \rangle} a_k N \delta[k-l] = a_l N
\]

Solving for \( a_l \) and then substituting \( k \) for \( l \) yields

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_{on}}
\]
Discrete-Time Fourier Series

Representing a periodic DT signal as a sum of harmonic sinusoids.

**Synthesis Equation**

\[ f[n] = f[n+N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n} \]

**Analysis Equation**

\[ a_k = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_0 n} \]

where \( \Omega_0 = \frac{2\pi}{N} \)
Fourier Series Summary

CT and DT Fourier Series are similar, but DT Fourier Series require just \( N \) components while CT Fourier Series require an infinite number.

Continuous-Time Fourier Series

\[
a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} dt
\]

analysis equation

\[
f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}
\]
synthesis equation

where \( \omega_0 = \frac{2\pi}{T} \)

Discrete-Time Fourier Series

\[
a_k = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_0 n}
\]

analysis equation

\[
f[n] = f[n + N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_0 n}
\]
synthesis equation

where \( \Omega_0 = \frac{2\pi}{N} \)
Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent (but easier) operations on the series coefficients.

We will discuss four (of many) properties of Fourier series.

- linearity
- time shift
- time reversal
- conjugate symmetry
**Linearity**

The Fourier series coefficients of a linear combination of two signals is the linear combination of their Fourier series coefficients.

Let 

\[ f[n] = a f_1[n] + b f_2[n] \]

where \( f_1[n] = f_1[n+N] \) and \( f_2[n] = f_2[n+N] \)

then the Fourier series coefficients for \( f[n] \) are given by

\[
F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=\langle N \rangle} \left( a f_1[n] + b f_2[n] \right) e^{-jk \frac{2\pi}{N} n} \\
= a \frac{1}{N} \sum_{n=\langle N \rangle} f_1[n] e^{-jk \frac{2\pi}{N} n} + b \frac{1}{N} \sum_{n=\langle N \rangle} f_2[n] e^{-jk \frac{2\pi}{N} n} \\
= a F_1[k] + b F_2[k]
\]

where \( F_1[k] \) and \( F_2[k] \) are Fourier series coefficients for \( f_1[n] \) and \( f_2[n] \).
Time Shift

Shifting time changes the phases of a signal’s Fourier coefficients.

Let

\[ g[n] = f[n-n_0] \quad \text{where} \quad f[n] = f[n+N] \]

If

\[ F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk \frac{2\pi}{N} n} \]

then

\[ G[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[n-n_0] e^{-jk \frac{2\pi}{N} n} \]

\[ = \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{-jk \frac{2\pi}{N} (m+n_0)} \quad \text{where} \quad m = n-n_0 \]

\[ = e^{-jk \frac{2\pi}{N} n_0} \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{-jk \frac{2\pi}{N} m} = e^{-jk \frac{2\pi}{N} n_0} F[k] \]
Time Reversal

Reversing time reverses frequency.

Let

\[ g[n] = f[-n] \quad \text{where} \quad f[n] = f[n+N] \]

If

\[ F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk \frac{2\pi}{N} n} \]

then

\[ G[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk \frac{2\pi}{N} n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[-n] e^{-jk \frac{2\pi}{N} n} \]

\[ = \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{+jk \frac{2\pi}{N} m} \quad \text{where} \quad m = -n \]

\[ = F[-k] \]
Conjugate Symmetry

If \( f[n] \) is real-valued, then its Fourier coefficients have conjugate symmetry.

If \( f[n] \) is real-valued, then \( f[n] = f^*[n] \).

\[
F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}
\]

\[
F^*[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f^*[n] e^{jk\frac{2\pi}{N}n}
\]

\[
= \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{jk\frac{2\pi}{N}n}
\]

\[
= F[-k]
\]
Applications of Fourier Series

Signal processing is **widely used** in science and engineering to ... 

- **model** some aspect of the world, 
- **analyze** the model, and 
- **interpret** results to gain a new or better understanding.

We previously touched on applications in physics, including the wave equation and how it leads directly to Fourier analysis.

Applications of Fourier analysis in **hearing**.
What determines the pitch of a sound? This seemingly simple question has evoked debate (sometimes fierce) for more than 150 years.

Compare two periodic signals with the same period, each played with 4000 samples per second

Different sounds, same pitch. We would like to understand why.
**Pitch Experiments**

Early experiments were based on stringed instruments and tubes, which were known to produce not just a fundamental but also harmonic overtones.

![Diagram of harmonics]

Although different sources produced different mixtures of harmonics, it was difficult to separate effects of one harmonic from those of others.

A breakthrough occurred with the work of Seebeck who used sirens to generate more complicated sounds.

*Very clever experiment, but very controversial interpretations.*
Sirens

Seebeck used a siren to generate more complicated sounds (circa 1841) by passing a jet of compressed air through holes in a spinning disk.

The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.
Sirens

Strangely, adding a second hole per period didn’t seem to affect the pitch.

Pitch should be different if it is determined by the intervals between pulses.
There was one very interesting exception.

But hearing this exception required precise alignment of the siren’s holes.
Seebeck interpreted his results in terms of the intervals between the holes. He held that pitch results from timing with some intervals being more important than others. As the lengths of the two intervals in his experiment converged, the pitch favored what had been the second harmonic and that frequency increasingly dominated.

Georg Ohm (already known for his work on electrical conduction) interpreted Seebeck’s results using Fourier’s recently described series. He held that the pulses generated by a siren contained a fundamental and harmonics that were physically present just as much as they are in a stringed instrument.

A bitter controversy ensued.
To understand Ohm’s argument, compute the Fourier series for the siren’s sound.
Fourier Interpretation

Find the $k^{th}$ coefficient of the $i^{th}$ signal.

$$F_i[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f_i[n] e^{-j\frac{2\pi k}{N} n} = \frac{1}{10} \sum_{n=0}^{9} f_i[n] e^{-j\frac{2\pi k}{10} n} = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi k}{10} i} \right)$$

DC: the $k=0$ term is $2/10$ for all $i$

$$F_i[0] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi 0}{10} i} \right) = \frac{2}{10}$$

Fundamental: $k=1$ term

$$F_i[1] = \frac{1}{10} \left( 1 + e^{-j\frac{2\pi 1}{10} i} \right)$$

Notice that $f_5[n]$ has no fundamental component!
Notice that $f_5[n]$ has no fundamental component!
Fourier Series With and Without the Fundamental

Resynthesize each waveform without its fundamental component.

Although perception of the fundamental is weakened, it is not gone!
Summary

Seebeck designed an extremely clever experiment to test pitch perception.

Ohm analyzed an important theory (from Fourier) and argued that harmonics are present even in the pulsatile sounds generated by a siren.

Neither Seebeck nor Ohm could convincingly account for experimental results that demonstrated the dominance of the fundamental, even when it was weak or missing.

Progress in understanding the “missing fundamental” awaited Helmholtz, who demonstrated the importance of “combination tones” in the ear.
Summary

Today we focused on discrete-time Fourier analysis.

- We developed Fourier series for discrete-time signals.
- We compared Fourier series for CT and DT signals.
- We looked at four (of many) properties of DT Fourier series.
- We looked briefly at applications of Fourier analysis in hearing.

Next time: Fourier analysis of aperiodic signals (CT and DT).