6.3000: Signal Processing

Sinusoids and Series

- Relations between time and frequency.
- Fourier series for discontinuous functions.
- Fourier analysis of a vibrating string.

<table>
<thead>
<tr>
<th>Section</th>
<th>Lecture</th>
<th>Recitation</th>
<th>Common-Room Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>TR2 (3-270)</td>
<td>TR3 (36-153)</td>
<td>TR4 (4-149)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>M7-9pm (26-322)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>R7-9pm (26-322)</td>
</tr>
<tr>
<td>2</td>
<td>TR2 (3-270)</td>
<td>TR4 (36-153)</td>
<td>TR3 (4-149)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>F2-5pm (26-210)</td>
</tr>
</tbody>
</table>

September 13, 2022
Signals are functions that are used to convey information. Example: a musical sound can be represented as a function of time.

Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.
Last Time

Time functions do a poor job of conveying consonance and dissonance.

- Octave (D + D')
- Fifth (D + A)
- D + E♭

Harmonic structure conveys consonance and dissonance better.
Last Time

**Fourier series** are sums of harmonically related sinusoids.

\[ f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t)) \]

where \( \omega_o = 2\pi / T \) represents the fundamental frequency.

Basis functions:
How do we find the coefficients $c_k$ and $d_k$?

Key idea: Sift out the component of interest by
- multiplying by the corresponding basis function, and then
- integrating over a period.

This results in the following expressions for the Fourier series coefficients:

$$c_0 = \frac{1}{T} \int_T f(t) \, dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_o t) \, dt; \quad k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_o t) \, dt; \quad k = 1, 2, 3, \ldots$$
Example of Analysis

Find the Fourier series coefficients for the following triangle wave:

\[ f(t) = \begin{cases} 1 & -1 < t < 1 \\ -1 & -2 < t < -1 \\ 0 & t = 0 \\ f(t+2) & \text{otherwise} \end{cases} \]

\[ f(t) = f(t+2) \]

\[ T = 2 \]

\[ \omega_o = \frac{2\pi}{T} = \pi \]

\[ c_0 = \frac{1}{T} \int_0^T f(t) \, dt = \frac{1}{2} \int_0^2 f(t) \, dt = \frac{1}{2} \]

\[ c_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \left( \frac{2\pi kt}{T} \right) \, dt = 2 \int_0^1 t \cos(\pi kt) \, dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \ldots \end{cases} \]

\[ d_k = 0 \quad \text{(by symmetry)} \]
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k \pi t)$$

$$f(t) = \frac{1}{2} - \sum_{k=1}^{99} \frac{4}{\pi^2 k^2} \cos(k \pi t)$$

Synthesized function approaches original as number of terms increases.
Fourier Synthesis

The previous example seems to show that the sum of an infinite number of sinusoids can generate a piecewise linear function with discontinuous slope!

Fourier defended the idea that such a series is meaningful.
Lagrange ridiculed the idea that discontinuities could be generated from a sum of continuous signals.

What if the function of interest has **discontinuous values**?
Fourier Analysis of a Square Wave

Find the Fourier series coefficients for the following square wave:

\[ f(t) = f(t+2) \]

\[ T = 2 \]

\[ \omega_o = \frac{2\pi}{T} = \pi \]

\[ c_0 = \frac{1}{T} \int_0^T f(t) \, dt = \frac{1}{2} \int_0^2 f(t) \, dt = \frac{1}{2} \]

\[ c_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega_0 t) \, dt = \int_0^1 \cos(k\pi t) \, dt = \frac{\sin(k\pi t)}{k\pi} \bigg|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots \]

\[ d_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega_0 t) \, dt = \int_0^1 \sin(k\pi t) \, dt = -\frac{\cos(k\pi t)}{k\pi} \bigg|_0^1 = \left\{ \begin{array}{ll} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{array} \right. \]
Fourier Synthesis of a Square Wave

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} + \sum_{k = 1, \text{odd}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{k = 1, \text{odd}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$

The synthesized function approaches original as number of terms increases.
Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{k = 1}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.
Convergence of Fourier Series

If there is a **step discontinuity** in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as $t$ approaches $t_0$ from the left and from the right.

Let $f_K(t)$ represent the **partial sum** of the Fourier series using just $N$ terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K} \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$

As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number $\epsilon$ shrinks to zero.

We refer to this type of overshoot as **Gibb’s Phenomenon**.

**So who was right?** Fourier or Lagrange?

Both. The series representation of a discontinuous function converges, but not uniformly.
Properties of Fourier Series

How do changes in signals affect their frequency representation?

→ investigate properties of Fourier representations
Properties of Fourier Series: Scaling Time

Find the Fourier series coefficients for the following square wave:

\[ g(t) = g(t+1) \]

We could repeat the process used to find the Fourier coefficients for \( f(t) \).

\[ f(t) = f(t+2) \]

Alternatively, we can take advantage of the relation between \( f(t) \) and \( g(t) \):

\[ g(t) = f(2t) \]
We already know the Fourier series expansion of $f(t)$:

$$f(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\pi t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\omega_0 t)$$

$$d_k = \begin{cases} 
\frac{1}{2} & k = 0 \\
\frac{1}{k\pi} & k = 1, 3, 5, \ldots \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad c_k = 0$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

Since $g(t) = f(2t)$ it follows that

$$g(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\pi 2t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(k\omega_1 t)$$

The Fourier series coefficients for $g(t)$ are thus identical to those of $f(t)$. Only the fundamental frequency has changed, from $\omega_0 = \pi$ to $\omega_1 = 2\pi$. 

The Fourier series coefficients for $g(t)$ are thus identical to those of $f(t)$.

Compressing the time axis has no effect on the $k$ axis. Only the fundamental frequency has changed.
Scaling Time

Plot the Fourier series coefficients on a frequency scale.

$f(t) = f(t+2)$

$g(t) = g(t+1)$

$\leftrightarrow$

$\omega$

$\leftrightarrow$

$\omega$

Compressing the time axis has stretched the $\omega$ axis.
Assume that $f(t)$ is periodic in time with period $T$:

$$f(t) = f(t + T).$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t - T/2).$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients $c_k$ are negated
- sine coefficients $d_k$ are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \rightarrow d_k$ and $d_k \rightarrow c_k$
Why Focus on Fourier Series?
What’s so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**. → harmonically related sinusoids are **orthogonal** to each other over \([0, T]\).

**Orthogonality:** \(f(t)\) and \(g(t)\) are orthogonal over \(0 \leq t \leq T\) if
\[
\int_T f(t)g(t) \, dt = 0
\]

Example: Calculate this integral for the \(k^{\text{th}}\) and \(l^{\text{th}}\) harmonics of \(\cos(\omega_0 t)\).
\[
\int_T \cos(k \omega_0 t) \cos(l \omega_0 t) \, dt
\]

We can use trigonometry to express the product of the two cosines as the sum of cosines at the sum and difference frequencies:
\[
\int_T \left( \frac{1}{2} \cos((k+l) \omega_0 t) + \frac{1}{2} \cos((k-l) \omega_0 t) \right) \, dt
\]

The sum and difference frequencies are also harmonics of \(\omega_0\), so their integral over \(T\) is zero (provided \(k \neq l\)).
Why Focus on Fourier Series?

What’s so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**. → harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Sines and cosines also play important roles in **physics** – especially the physics of waves.
Physical Example: Vibrating String

A taut string supports wave motion.

The speed of the wave depends on the tension on and mass of the string.
Physical Example: Vibrating String

The wave will reflect off a rigid boundary.

The amplitude of the reflected wave is opposite that of the incident wave.
Physical Example: Vibrating String

Reflections can interfere with excitations.

The interference can be constructive or destructive depending on the frequency of the excitation.
Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

Round-trip travel time \( T = \frac{2L}{v} \)

\[ \omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L} \]
Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

Round-trip travel time $= \frac{2L}{v} = 2T$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$
Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is $kT$.

$$x = 0 \quad x = L$$

Round-trip travel time $= \frac{2L}{v} = kT$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_0$$

Only certain frequencies (harmonics of $\omega_0 = \pi v/L$) persist. This is the basis of stringed instruments.
Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is “plucked” at $x = l$. 

$x = 0 \quad x = L$
Physical Example: Vibrating String

Differences in harmonic structure generate differences in timbre.
Summary

- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples → Gibb’s phenomenon.
- We investigated several properties of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are many others
- We saw how Fourier series are useful for modeling a vibrating string.