Discrete Cosine Transform

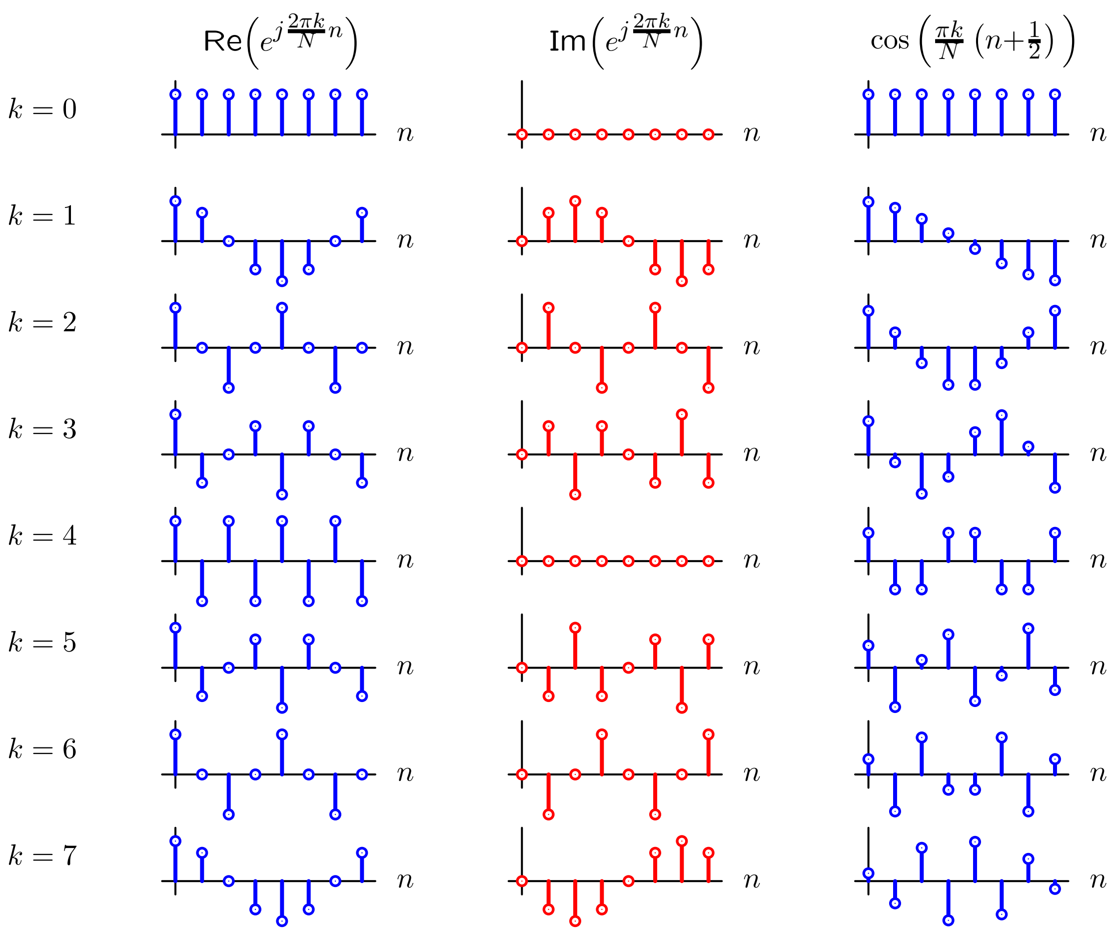
The Discrete Cosine Transform (DCT) is widely used in image coding schemes such as JPEG.

It is defined by the following analysis and synthesis equations.

$$F_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(analysis)}$$

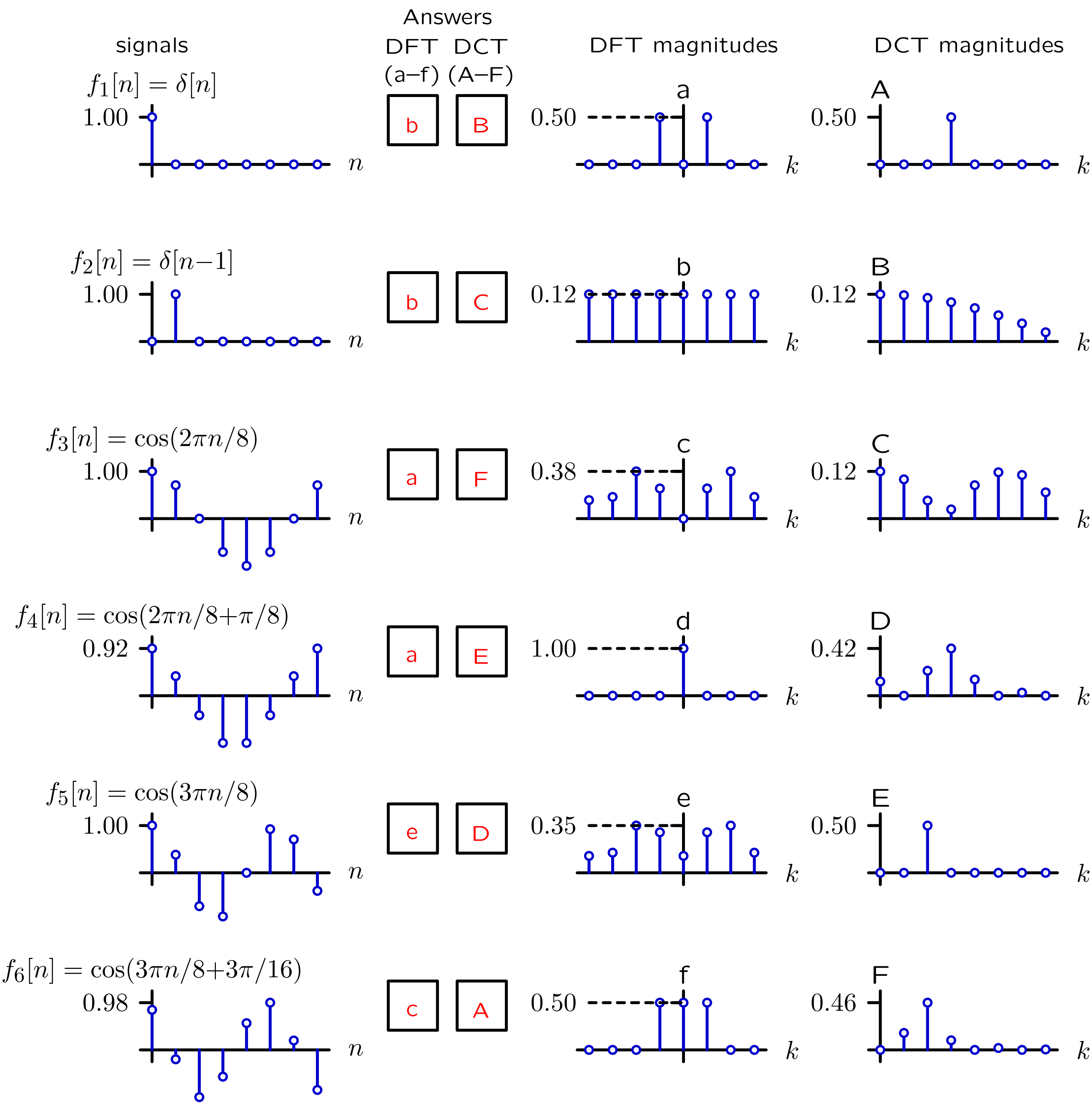
$$f[n] = F_C[0] + 2\sum_{k=1}^{N-1} F_C[k]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) \qquad \text{(synthesis)}$$

Like the DFT, the DCT expands a signal in terms of sinusoidal basis functions. The table below shows the real and imaginary parts of the DFT basis functions (left and center columns) as well as the DCT basis functions (right column) for $N = 8$.



DFT and DCT Magnitudes

For each of the DT signals shown in the left column below, find the associated DFT magnitude from the center column and DCT magnitude from the right column, and enter the appropriate labels in the answer boxes.



Part 1. The DFT of $\delta[n]$ is 1. Therefore $F_1[k] = 1 \to$ panel b.

$$F_{C1}[k] = \frac{1}{N}\sum_{n=0}^{N-1} f_1[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\sum_{n=0}^{N-1}\delta[n]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\cos\left(\frac{\pi k}{2N}\right)$$

$\to$ panel B.

The DCT of a unit-sample signal is NOT the same as the DFT of a unit-sample signal!

Part 2. The DFT of $\delta[n-1]$ is $e^{-j\frac{2\pi k}{N}}$. Therefore $|F_2[k]| = 1 \to$ panel b.

$$F_{C2}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\delta[n-1]\cos\left(\frac{\pi k}{N}\left(n+\frac{1}{2}\right)\right) = \frac{1}{N}\cos\left(\frac{\pi k}{N}\frac{3}{2}\right)$$

$\to$ panel C.

Time shifts are not as simple with the DCT as they were with the DFT.

Part 3. $f_3[n]$ is the fundamental frequency for $N = 8$. The DFT has non-zero contributions only at $k = \pm 1$. Therefore $|F_3[k]| = 1 \to$ panel a.

DFT basis functions are integer multiples of the fundamental frequency. By contrast, DCT basis functions are half-integer multiples. Therefore we expect a non-zero component at $k = 2$. However, the last sample of all DCT basis functions is either equal to the first sample or equal to the negative of the first sample. Therefore $f_3[n]$ is NOT a basis function for the DCT.

$\to$ panel F.

Part 4. $f_4[n]$ is also at the fundamental frequency for $N = 8$. The DFT has non-zero contributions only at $k = \pm 1$. Therefore $|F_4[k]| = 1 \to$ panel a.

The frequency of $f_4[n]$ is the same as that of $f_3[n]$ but the phase is different. $f_4[7] = f_4[0]$. Therefore $f_4[n]$ IS a basis function.

$\to$ panel E.

Notice that both $f_3[n]$ and $f_4[n]$ have simple representations as DFTs but not as DCTs.

Part 5. The frequency of $f_5[n]$ is $\frac{3}{2}$ times the fundamental frequency. The discontinuity created by periodic extension of $f_5[n]$ generates components at all frequencies, although the peak is near $k = 3/2$. Thus the answer could be c or e. The DC component of $f_5[n]$ is positive (4 positive numbers and 3 negative).

$\to$ panel e.

Since the frequency of $f_5[n]$ is a half-integer multiple of the fundamental, it could be a DCT basis function at $k = 3$. However, the final value is not equal to the initial value or its negative.

$\to$ panel D.

Part 6. $f_6[n]$ is a phase shifted version of $f_5[n]$. Now the DC value is zero.

$\to$ panel c.

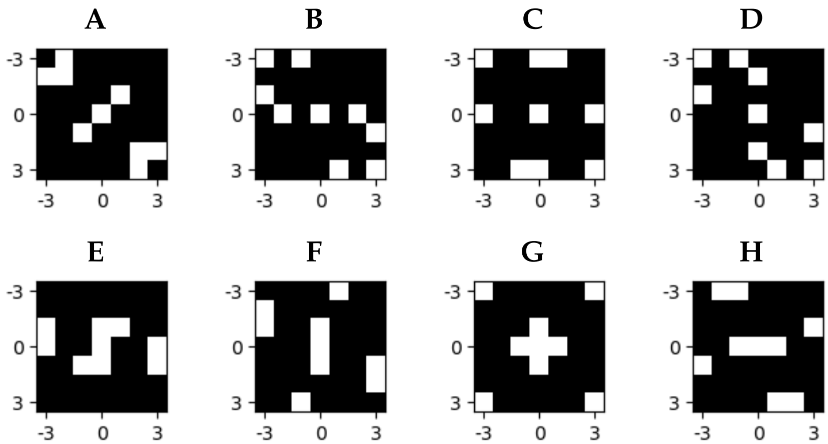
$f_6[n]$ is a basis function of the DCT.

$\to$ panel A.

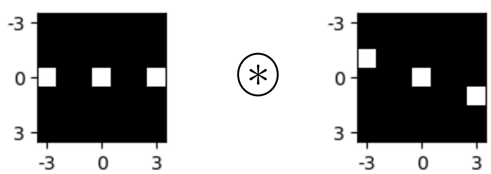
Notice the $f_6[n]$ has a simple representation as a DCT but not as a DFT.

Circular Convolution

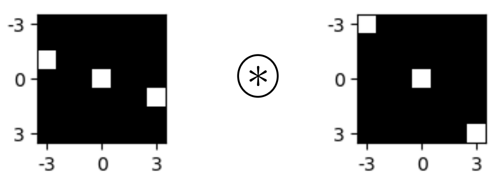
Two 7×7 images are circularly convolved by computing the 2D iDFT of the product of their 2D DFTs, where both the iDFT and DFTs are of length $N = 7$. The resulting image is one of the eight images shown below.



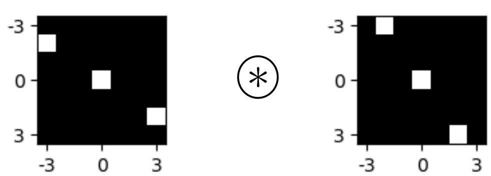
For each of the following parts, determine which of A-H results.



= E



= D



= A

= G