6.003: Signal Processing

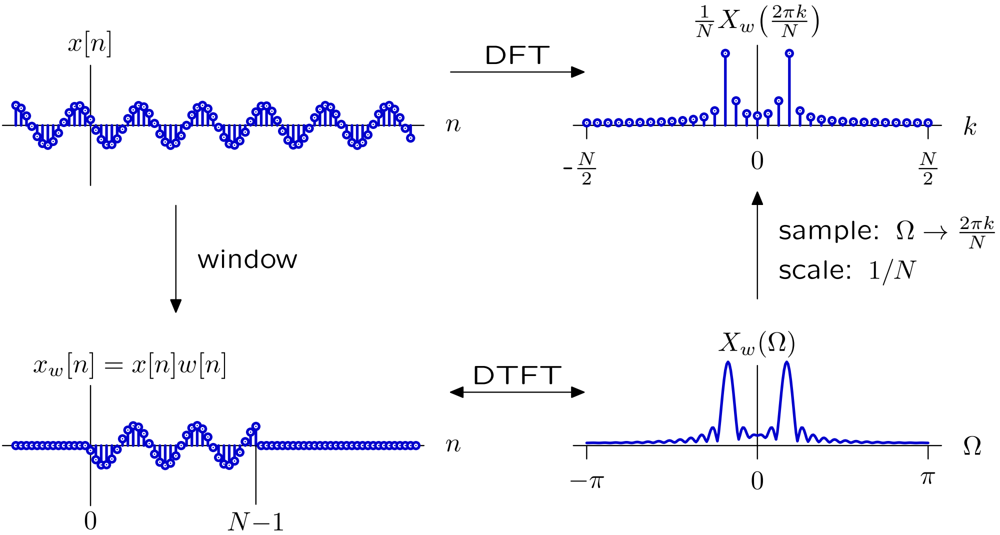
Short-Term Fourier Transform

- Window Functions
- FFT Algorithm

October 26, 2021

Window Functions

A defining feature of the DFT is its finite length N, which plays a critical role in determining both time and frequency resolution.



The finite length constraint is equivalent to multiplication by a rectangular window. What would happen if we used a different type of window?

Window Functions

Dozens of different window functions are in common use. We will look at three of them:

- rectangular window
- triangular window
- Hann window

These and other window functions have a variety of different properties. Which properties are important in which applications?

Rectangular Window

Definition:

$$w_r[n] = \begin{cases} 1 & |n| \le M \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_r[n]$ versus n.
- Determine the DT Fourier Transform $W_r(\Omega)$.
- Make a plot of $W_r(\Omega)$ versus Ω.

Trianglar Window

Definition:

$$w_t[n] = \begin{cases} 1 - \frac{|n|}{M} & |n| \leq M \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_t[n]$ versus n.
- Determine the DT Fourier Transform $W_t(\Omega)$.
- Make a plot of $W_t(\Omega)$ versus Ω.

Hann Window

Definition:

$$w_H[n] = \begin{cases} \cos^2\left(\frac{\pi n}{2M}\right) & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Make a plot of $w_H[n]$ versus n.
- Determine the DT Fourier Transform $W_H(\Omega)$.
- Make a plot of $W_H(\Omega)$ versus Ω.

Compare

Superpose the plots of $W_r(\Omega)$, $W_t(\Omega)$, and $W_H(\Omega)$.

What are the important differences?

FFT Algorithm

The FFT algorithm is both elegant and critical to modern signal processing.

Start with the definition of the DFT of an N point input $x[n]$:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

Separate even- and odd-numbered terms (assuming N is even):

$$\begin{aligned}
X[k] &= \frac{1}{N} \sum_{\substack{n=0 \\ n \ \mathit{even}}}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}} + \frac{1}{N} \sum_{\substack{n=0 \\ n \ \mathit{odd}}}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}} \\
&= \frac{1}{N} \sum_{m=0}^{N/2-1} x[2m] e^{-\frac{j2\pi k2m}{N}} + \frac{1}{N} \sum_{m=0}^{N/2-1} x[2m+1] e^{-\frac{j2\pi k(2m+1)}{N}} \\
&= \frac{1/2}{N/2} \sum_{m=0}^{N/2-1} x_e[m] e^{-\frac{j2\pi km}{N/2}} + \frac{1/2}{N/2} e^{-\frac{j2\pi k}{N}} \sum_{m=0}^{N/2-1} x_o[m] e^{-\frac{j2\pi km}{N/2}} \\
&= \frac{1}{2} X_e[k] + \frac{1}{2} e^{-\frac{j2\pi k}{N}} X_o[k]
\end{aligned}$$

where $X_e[k]$ and $X_o[k]$ represent the $N/2$ point DFTs of $x_e[n] = x[2n]$ and $x_o[n] = x[2n+1]$ respectively.

FFT Code

```
def DFT(x):
    N = len(x)
    X = []
    for k in range(N):
        X.append(sum([x[n]*e**(-2j*pi*k*n/N)/N for n in range(N)])
    return X

def FFT(x):
    N = len(x)
    if N==1:
        return x
    xe,xo = x[::2],x[1::2]
    Xe,Xo = FFT(xe),FFT(xo)
    w = e**(-2j*pi/N)
    X = [0]*N
    for k in range(N//2):
        X[k]      = (Xe[k]+w**k*Xo[k])/2
        X[k+N//2] = (Xe[k]-w**k*Xo[k])/2
    return X
```

FFT Performance

Compare the running times of DFTs and FFTs.

```
from random import random
from time import time
for N in (1024,2048,4096):
    x = [random() for n in range(N)]
    t0 = time()
    X = FFT(x)
    t1 = time()
    X = DFT(x)
    t2 = time()
    print('{0:.5f} {1:.5f}'.format(t1-t0,t2-t1))
```