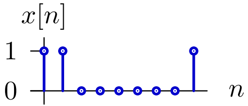
6.003: Signal Processing

Discrete Fourier Transform and Circular Convolution

October 21, 2021

Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.



Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

$$y_1[n] = (x * x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

2. using DTFTs:

$$y_2[n] = \frac{1}{2\pi}\int_{2\pi} X^2(\Omega)e^{j\Omega n}\,d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

3. using DFTs of length $N=16$:

$$y_3[n] = 16\sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16}n} \quad \text{where} \quad X[k] = \frac{1}{16}\sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16}n}$$

Convolution: Three Ways

The plots on the right show the **first ten samples** of five signals.

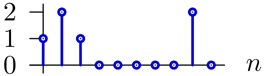
Match the signals on the left with the corresponding plots on the right.

$$y_1 = (x * x)$$

$$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$$

$$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$$

A



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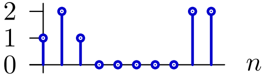

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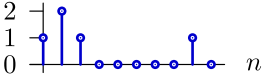

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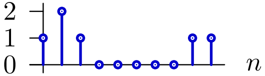

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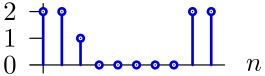
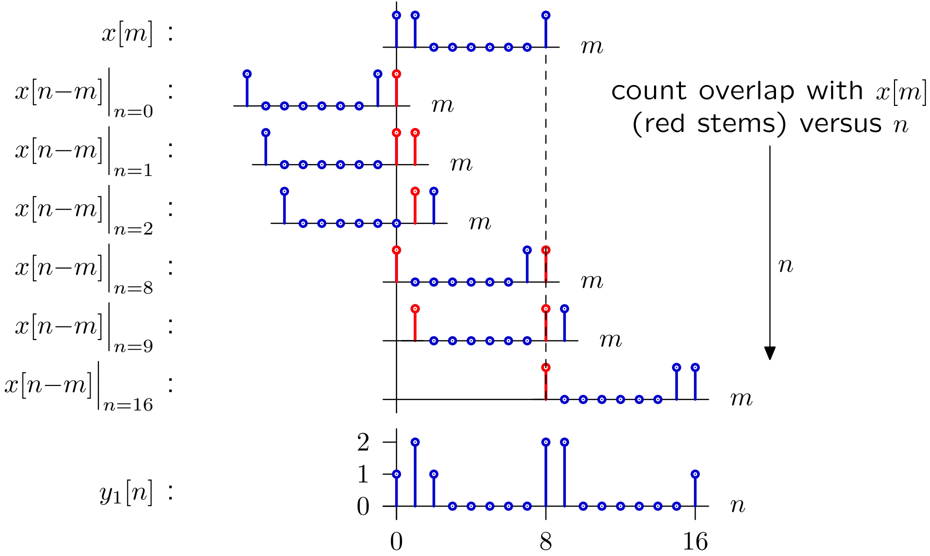


Convolution: Three Ways

Calculate $(x{*}x)[n]$ by direct convolution: flip and shift.

$$y_1[n] = (x{*}x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

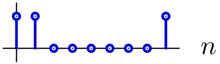


Convolution: Three Ways

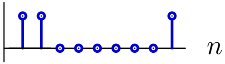
Calculate $(x*x)[n]$ by direct convolution: superposition.

$$y_1[n] = (x*x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]$$

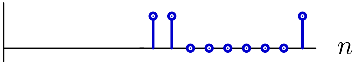
$x[0] \times x[n-0]$:



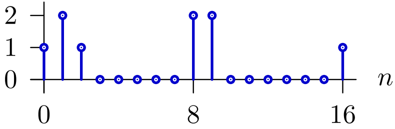
$x[1] \times x[n-1]$:



$x[8] \times x[n-8]$:



$y_1[n]$:



Note: Superposition and flip-and-shift are equivalent methods.

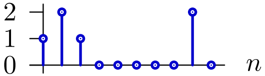
They always give the same answer.

Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals.

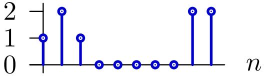
Match signals on the left with corresponding samples on the right.

A



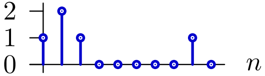
$y_1 = (x * x)$ $\boxed{\text{B}}$

B



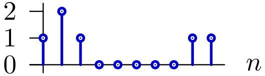
$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ $\boxed{}$

C

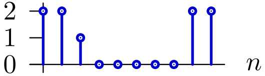


$y_3 = N{\times}\mathrm{DFT}^{-1}\big(X^2[k]\big)$ $\boxed{}$

D

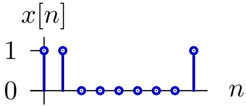


E



Convolution: Three Ways

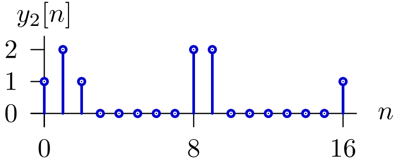
Calculate $(x*x)[n]$ using DTFTs.



$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}=1+e^{-j\Omega}+e^{-j8\Omega}$$

$$X^2(\Omega)=\left(1+e^{-j\Omega}+e^{-j8\Omega}\right)^2=1+2e^{-j\Omega}+e^{-j2\Omega}+2e^{-j8\Omega}+2e^{-j9\Omega}+e^{-j16\Omega}$$

$$\begin{aligned}y_2[n]&=\frac{1}{2\pi}\int_{2\pi}X^2(\Omega)e^{j\Omega n}\,d\Omega\\&=\frac{1}{2\pi}\int_{2\pi}\left(1+2e^{-j\Omega}+e^{-j2\Omega}+2e^{-j8\Omega}+2e^{-j9\Omega}+e^{-j16\Omega}\right)e^{j\Omega n}\,d\Omega\\&=\delta[n]+2\delta[n-1]+\delta[n-2]+2\delta[n-8]+2\delta[n-9]+\delta[n-16]\end{aligned}$$



Multiplying DTFTs is always equivalent to direct convolution.

Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals.

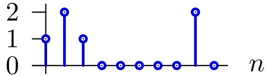
Match signals on the left with corresponding samples on the right.

$y_1 = (x * x)$ B

$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ B

$y_3 = N{\times}\mathrm{DFT}^{-1}\big(X^2[k]\big)$

A



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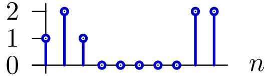

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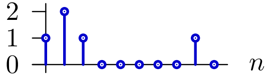

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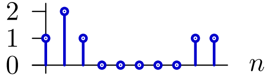

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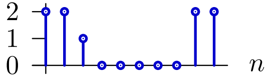
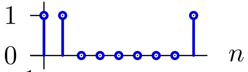


Convolution: Three Ways

Calculate $(x*x)[n]$ using DFTs ($N=16$).

$x[n]$



$$X[k]=\frac{1}{16}\sum_{n=0}^{15}x[n]e^{-j\frac{2\pi k}{16}n}=\frac{1}{16}\left(1+e^{-j\frac{2\pi k}{16}}+e^{-j\frac{2\pi 8k}{16}}\right)$$

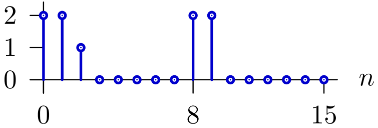
$$X^2[k]=\frac{1}{256}\left(1+2e^{-j\frac{2\pi k}{16}}+e^{-j2\frac{2\pi k}{16}}+2e^{-j8\frac{2\pi k}{16}}+2e^{-j9\frac{2\pi k}{16}}+\underbrace{e^{-j16\frac{2\pi k}{16}}}_{=1}\right)$$

$$y_3[n]=16\sum_{k=0}^{15}X^2[k]e^{j\frac{2\pi k}{16}n}$$

$$=\frac{16}{256}\sum_{k=0}^{15}\left(2+2e^{-j\frac{2\pi k}{16}}+e^{-j2\frac{2\pi k}{16}}+2e^{-j8\frac{2\pi k}{16}}+2e^{-j9\frac{2\pi k}{16}}\right)e^{j\frac{2\pi k}{16}n}$$

$$=2\delta[n]+2\delta[n-1]+\delta[n-2]+2\delta[n-8]+2\delta[n-9]$$

$y_3[n]$



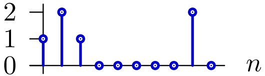
Since $N{=}16$, the sample at $n{=}16$ in direct convolution **aliases** to $n{=}0$.

Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals.

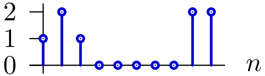
Match signals on the left with corresponding samples on the right.

A



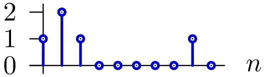
$y_1 = (x * x)$ B

B



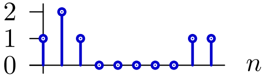
$y_2 = \mathrm{DTFT}^{-1}\big(X^2(\Omega)\big)$ B

C



$y_3 = N \times \mathrm{DFT}^{-1}\big(X^2[k]\big)$ E

D



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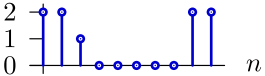


Circular Convolution

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

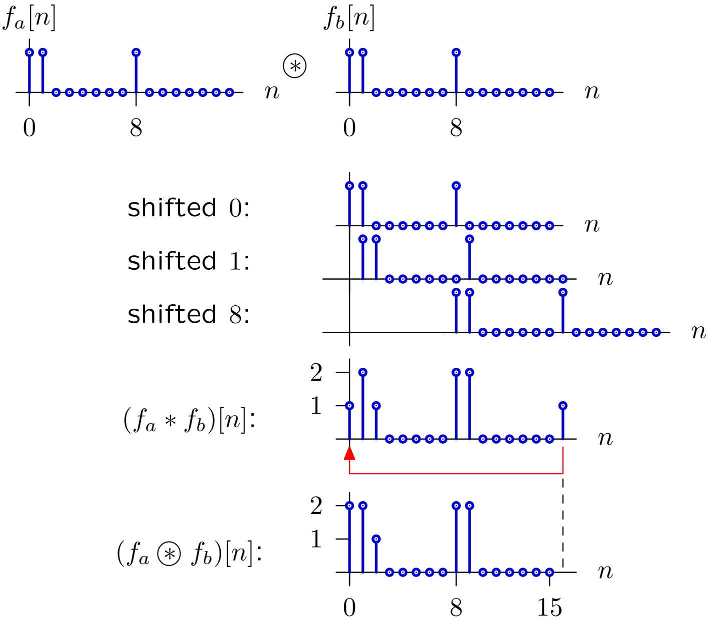
$$
\begin{aligned}
f[n] &= \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N}n} \\
&= \sum_{k=0}^{N-1} F_a[k] \left(\frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N}m} \right) e^{j\frac{2\pi k}{N}n} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N}(n-m)} \\
&= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]
\end{aligned}
$$

where $f_{ap}[n] = f_a[n \bmod N]$ is a periodically extended version of $f_a[n]$. We refer to this as **circular** or **periodic** convolution:

$$
\frac{1}{N}(f_a \circledast f_b)[n] \quad \stackrel{\text{DFT}}{\Longleftrightarrow} \quad F_a[k] \times F_b[k]
$$

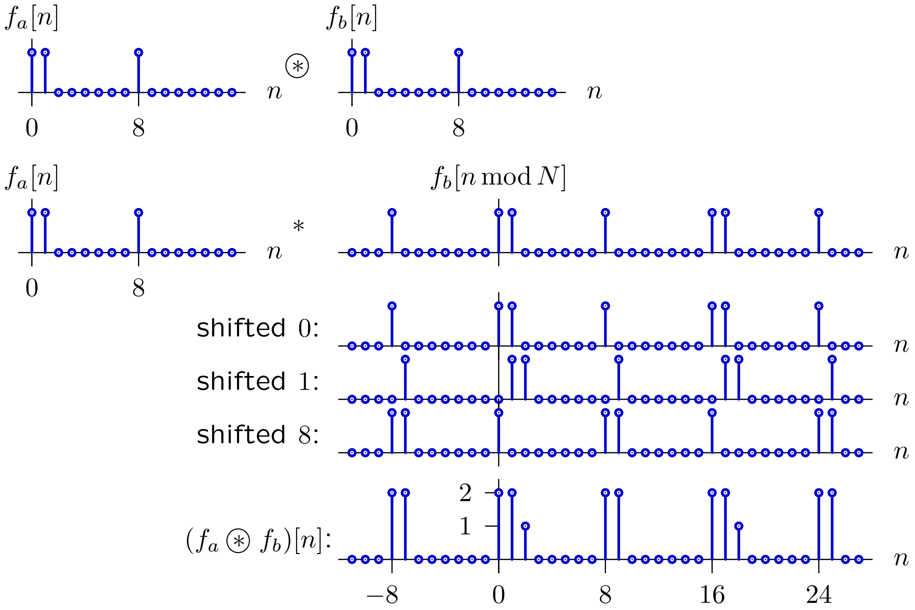
Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.



Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.



Summary

One of the most useful properties of the DTFT is its filter property: convolution in time corresponds to multiplication in frequency.

$$(f * g)[n] \quad \stackrel{\text{DTFT}}{\Longleftrightarrow} \quad F(\Omega)G(\Omega)$$

The DFT is slightly more complicated since the DFT is equivalent to the DTFS of a periodicaly extended version of $x[n]$:

$$x[n] = x[n + mN] \quad \text{for all integers } m$$

A result of this periodicity is that the convolution that results when two DFTs are multiplied is also periodic.

We refer to this type of convolution as "circular convolution."

$$\frac{1}{N}(f \circledast g)[n] \quad \stackrel{\text{DFT}}{\Longleftrightarrow} \quad F[k]G[k]$$