6.003: Signal Processing

Discrete Fourier Transform and Circular Convolution

October 21, 2021
Convolution: Three Ways

The signal $x[n]$, defined below, is zero outside the indicated range.

\[ x[n] \]

\[
\begin{array}{c|c c c c c c c c}
  \text{n} & 0 & 1 & & & & & & \\
\hline
  \text{x[n]} & \text{0} & \text{1} & & & & & & \\
\end{array}
\]

Consider three ways to calculate the convolution of $x[n]$ with itself.

1. direct convolution:

\[
y_1[n] = (x \ast x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]
\]

2. using DTFTs:

\[
y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega)e^{j\Omega n} d\Omega \quad \text{where} \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}
\]

3. using DFTs of length $N=16$:

\[
y_3[n] = 16 \sum_{k=0}^{15} X^2[k]e^{j\frac{2\pi k}{16} n} \quad \text{where} \quad X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n]e^{-j\frac{2\pi k}{16} n}
\]
Convolution: Three Ways

The plots on the right show the **first ten samples** of five signals. Match the signals on the left with the corresponding plots on the right.

\[
y_1 = (x * x) \quad \boxed{} \quad \text{(A)}
\]

\[
y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \quad \boxed{} \quad \text{(B)}
\]

\[
y_3 = N \times \text{DFT}^{-1}(X^2[k]) \quad \boxed{} \quad \text{(C)}
\]
Convolution: Three Ways

Calculate \((x*x)[n]\) by direct convolution: flip and shift.

\[
y_1[n] = (x*x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n-m]
\]

\[y_1[n] = \begin{array}{llll}
0 & 8 & 16 \\
0 & 1 & 2 \\
\end{array}
\]
Convolution: Three Ways

Calculate \((x \times x)[n]\) by direct convolution: superposition.

\[ y_1[n] = (x \times x)[n] = \sum_{m=-\infty}^{\infty} x[m]x[n - m] \]

\[
\begin{align*}
x[0] \times x[n - 0] : & & \quad n \\
x[1] \times x[n - 1] : & & \quad n \\
x[8] \times x[n - 8] : & & \quad n
\end{align*}
\]

\[ y_1[n] : \]

Note: Superposition and flip-and-shift are equivalent methods. They always give the same answer.
Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals. Match signals on the left with corresponding samples on the right.

\[ y_1 = (x \ast x) \]

\[ y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \]

\[ y_3 = N \times \text{DFT}^{-1}(X^2[k]) \]
Convolution: Three Ways

Calculate \((x \ast x)[n]\) using DTFTs.

\[
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = 1 + e^{-j\Omega} + e^{-j8\Omega}
\]

\[
X^2(\Omega) = \left(1 + e^{-j\Omega} + e^{-j8\Omega}\right)^2 = 1 + 2e^{-j\Omega} + e^{-j2\Omega} + 2e^{-j8\Omega} + 2e^{-j9\Omega} + e^{-j16\Omega}
\]

\[
y_2[n] = \frac{1}{2\pi} \int_{2\pi} X^2(\Omega)e^{j\Omega n} \, d\Omega
\]

\[
= \frac{1}{2\pi} \int_{2\pi} \left(1 + 2e^{-j\Omega} + e^{-j2\Omega} + 2e^{-j8\Omega} + 2e^{-j9\Omega} + e^{-j16\Omega}\right)e^{j\Omega n} \, d\Omega
\]

\[
= \delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9] + \delta[n-16]
\]

Multiplying DTFTs is always equivalent to direct convolution.
Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals. Match signals on the left with corresponding samples on the right.

\[ y_1 = (x \ast x) \]
\[ y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \]
\[ y_3 = N \times \text{DFT}^{-1}(X^2[k]) \]
Convolution: Three Ways

Calculate \((x*x)[n]\) using DFTs (\(N = 16\)).

\[
X[k] = \frac{1}{16} \sum_{n=0}^{15} x[n] e^{-j \frac{2\pi k n}{16}} = \frac{1}{16} \left(1 + e^{-j \frac{2\pi k}{16}} + e^{-j \frac{2\pi 8k}{16}}\right)
\]

\[
X^2[k] = \frac{1}{256} \left(1+2e^{-j \frac{2\pi k}{16}} + e^{-j \frac{2\pi 2k}{16}} + 2e^{-j \frac{8\pi k}{16}} + 2e^{-j \frac{9\pi k}{16}} + e^{-j \frac{16\pi k}{16}}\right)
\]

\[
y_3[n] = 16 \sum_{k=0}^{15} X^2[k] e^{j \frac{2\pi k n}{16}}
\]

\[
= \frac{16}{256} \sum_{k=0}^{15} \left(2+2e^{-j \frac{2\pi k}{16}} + e^{-j \frac{2\pi 2k}{16}} + 2e^{-j \frac{8\pi k}{16}} + 2e^{-j \frac{9\pi k}{16}}\right) e^{j \frac{2\pi k n}{16}}
\]

\[
= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + 2\delta[n-8] + 2\delta[n-9]
\]

Since \(N=16\), the sample at \(n=16\) in direct convolution aliases to \(n=0\).
Convolution: Three Ways

Plots on the left show the **first ten samples** of five signals. Match signals on the left with corresponding samples on the right.

\[ y_1 = (x \ast x) \]

\[ y_2 = \text{DTFT}^{-1}(X^2(\Omega)) \]

\[ y_3 = N \times \text{DFT}^{-1}(X^2[k]) \]
**Circular Convolution**

Multiplication of DFTs corresponds to **circular** convolution in time. Assume that $F[k]$ is the product of the DFTs of $f_a[n]$ and $f_b[n]$.

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi k}{N} n} = \sum_{k=0}^{N-1} F_a[k] F_b[k] e^{j\frac{2\pi k}{N} n}$$

$$= \sum_{k=0}^{N-1} F_a[k] \left( \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] e^{-j\frac{2\pi k}{N} m} \right) e^{j\frac{2\pi k}{N} n}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] \sum_{k=0}^{N-1} F_a[k] e^{j\frac{2\pi k}{N} (n-m)}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} f_b[m] f_{ap}[n-m]$$

where $f_{ap}[n] = f_a[n \mod N]$ is a periodically extended version of $f_a[n]$. We refer to this as **circular** or **periodic** convolution:

$$\frac{1}{N} (f_a \odot f_b)[n] \underset{\text{DFT}}{\iff} F_a[k] \times F_b[k]$$
Circular Convolution

Circular convolution is equivalent to conventional convolution followed by periodic summation of results back into base period.

\[ f_a[n] \]

\[ f_b[n] \]

\[ (f_a \ast f_b)[n] : \]

\[ (f_a \odot f_b)[n] : \]
Circular Convolution

Circular convolution of two signals is equal to conventional convolution of one signal with a periodically extended version of the other.

\[
(f_a \circledast f_b)[n] = \text{conventional convolution of } f_a[n] \text{ with } f_b[n \mod N]
\]
Summary

One of the most useful properties of the DTFT is its filter property: convolution in time corresponds to multiplication in frequency.

\[(f * g)[n] \quad \overset{\text{DTFT}}{\iff} \quad F(\Omega)G(\Omega)\]

The DFT is slightly more complicated since the DFT is equivalent to the DTFS of a periodically extended version of \(x[n]\):

\[x[n] = x[n + mN] \quad \text{for all integers } m\]

A result of this periodicity is that the convolution that results when two DFTs are multiplied is also periodic.

We refer to this type of convolution as “circular convolution.”

\[\frac{1}{N} (f \circledast g)[n] \quad \overset{\text{DFT}}{\iff} \quad F[k]G[k]\]