# 6.003: Signal Processing

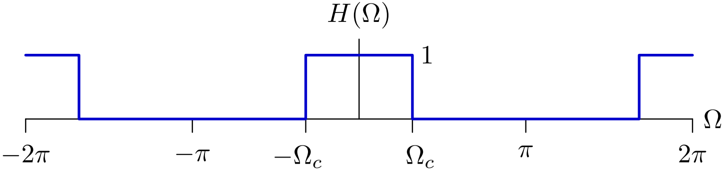
## Convolution and Filtering

| time domain | frequency domain |
|---|---|
| $y(t) = (h * x)(t) = \int h(\tau)x(t-\tau)\,d\tau$ | $Y(\omega) = H(\omega)X(\omega)$ |
| $y[n] = (h * x)[n] = \sum_m h[m]x[n-m]$ | $Y(\Omega) = H(\Omega)X(\Omega)$ |

*October 14, 2021*

# The "Ideal" Low-Pass Filter

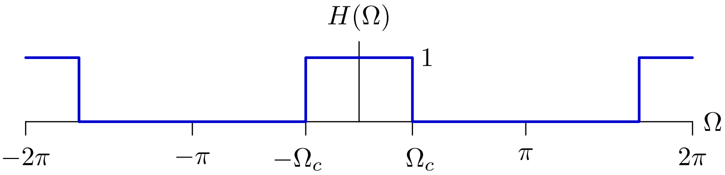
Consider a system characterized by the following purely real frequency response:




Such a system is called a **low-pass filter**, because it allows low frequencies to pass through unmodified, while attenuating high frequencies.

We could apply this filter to a signal by multiplying the DTFT of that signal by the values above. But we could also apply the filter by operating in the time domain.

# The “Ideal” Low-Pass Filter

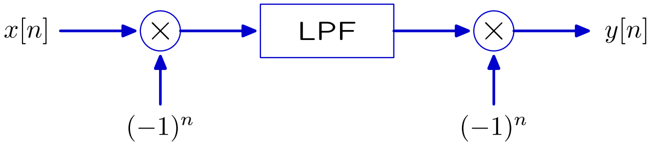



We can apply this filter to a signal by convolving with its unit sample response. What is the unit sample response of the system whose frequency response is shown above?

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_C}^{\Omega_C} e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\frac{1}{jn}e^{j\Omega n}\Big|_{\Omega=-\Omega_C}^{\Omega_C} \\
&= \frac{\sin(\Omega_c n)}{\pi n}
\end{aligned}
$$

# Cascaded System

Consider the following system, where LPF represents a lowpass filter of the form discussed on the previous slides.



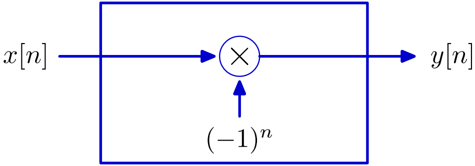

How many of the following statements are true?

- The transformation from $x[n]$ to $y[n]$ is linear.
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.

# Consider Each Part Separately

Start with the multiplier.

Is the following system linear?



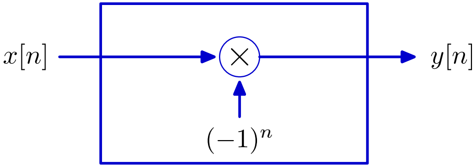

Assume the response to the input $x_1[n]$ is $y_1[n]$ and the response to the input $x_2[n]$ is $y_2[n]$. Calculate the response to $x[n] = \alpha x_1[n] + \beta x_2[n]$.

$$\begin{aligned} y[n] &= (-1)^n x[n] \\ &= (-1)^n (\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha(-1)^n x_1[n] + \beta(-1)^n x_2[n] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

$\therefore$ linear!

# Consider Each Part Separately

Is this system time-invariant?




Assume the response to the input $x_1[n]$ is $y_1[n]$.
Calculate the response to $x[n] = x_1[n - n_0]$.

$$
\begin{aligned}
y[n] &= (-1)^n x[n] \\
&= (-1)^n x_1[n - n_0] \\
&= (-1)^{n_0} (-1)^{n-n_0} x_1[n - n_0] \\
&= (-1)^{n_0} y_1[n - n_0]
\end{aligned}
$$

Shifting the input by $n_0$ samples shifts does not just shift the output by $n_0$ samples.

$\therefore$ this system is not time-invariant!

# Consider Each Part Separately

Is the LPF linear? time-invariant?


The defining property of a **filter** is that its output is a weighted sum of (possibly) shifted versions of the frequencies in the input, so that

$$Y(\Omega) = H(\Omega)X(\Omega)\,.$$

Multiplication in frequency is the same as convolution in time, so

$$y[n] = (x * h_L)[n]$$

where $h_L[\cdot]$ represents the unit-sample response of the filter.

Convolution is both linear and time invariant (as shown on the next slides).

# Convolution is Linear

Assume the response to the input $x_1[n]$ is $y_1[n]$ and the response to the input $x_2[n]$ is $y_2[n]$. Calculate the response to $x[n] = \alpha x_1[n] + \beta x_2[n]$.

$$\begin{aligned}
y[n] &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\
&= \sum_{m=-\infty}^{\infty} \Big(\alpha x_1[m] + \beta x_2[m]\Big) h[n-m] \\
&= \alpha \sum_{m=-\infty}^{\infty} x_1[m]h[n-m] + \beta \sum_{m=-\infty}^{\infty} x_2[m]h[n-m] \\
&= \alpha y_1[n] + \beta y_2[n]
\end{aligned}$$

$\therefore$ linear!

# Convolution is Time-Invariant

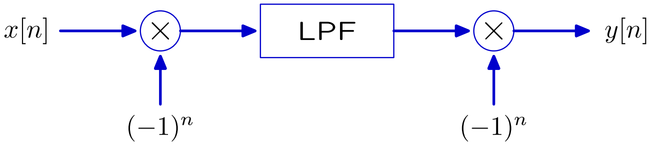
Assume the response to the input $x_1[n]$ is $y_1[n] = (x_1 * h)[n]$.
Calculate the response to $x[n] = x_1[n-n_0]$.

$$\begin{aligned} y[n] &= (x * h)[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]h[n-m] \\ &= \sum_{m=-\infty}^{\infty} x_1[m-n_0]h[n-m] \\ &= \sum_{l=-\infty}^{\infty} x_1[l]h[n-(n_0+l)] \\ &= \sum_{l=-\infty}^{\infty} x_1[l]h[(n-n_0)-l] \\ &= (x_1 * h)[n-n_0] = y_1[n-n_0] \end{aligned}$$

$\therefore$ time-invariant!

# Cascaded System

This system is the cascade of three linear subsystems, two of which are time varying.




Thus we know that the composite system is linear.

- The transformation from $x[n]$ to $y[n]$ is linear. ✓
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.

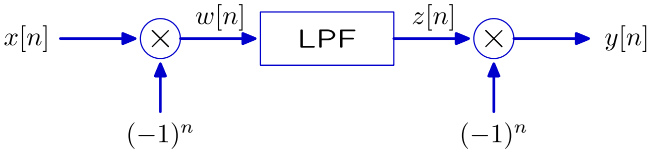
The first is true. Not sure about the others.

Could the cascade of two time-varying systems be time-invariant?

Yes. Think about the cascade of $\times(-1)^n$ with $\times(-1)^n$.

# Cascaded System

Determine an expression for $y[n]$ in terms of $x[n]$ using a time-domain approach. Assume that the unit sample response of the LPF is $h_L[n]$.




$$w[n] = (-1)^n x[n]$$

$$z[n] = (w * h_L)[n] = \sum_{m=-\infty}^{\infty} (-1)^m x[m] h_L[n-m]$$

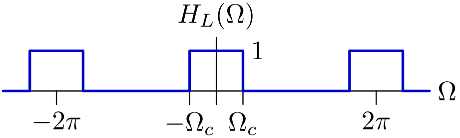
$$y[n] = (-1)^n z[n] = (-1)^n \sum_{m=-\infty}^{\infty} (-1)^m x[m] h_L[n-m]$$

$$= \sum_{m=-\infty}^{\infty} (-1)^{n+m} x[m] h_L[n-m] = \sum_{m=-\infty}^{\infty} (-1)^{n-m} x[m] h_L[n-m]$$

$$= (x * h_H)[n] \quad \text{where} \quad h_H[n] = (-1)^n h_L[n]$$

# Cascaded System

Find an expression for the unit sample response of a lowpass filter that passes frequencies in the range $-\Omega_c < \Omega < \Omega_c$.

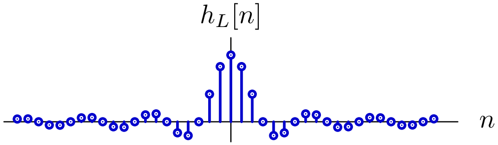


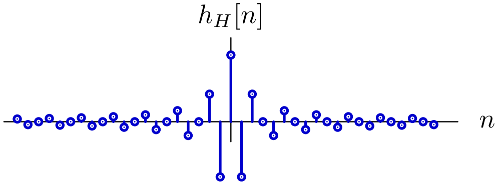

$$
\begin{aligned}
h_L[n] &= \frac{1}{2\pi}\int_{2\pi} H_L(\Omega)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\Omega_c}^{\Omega_c} \\
&= \frac{\sin(\Omega_c n)}{\pi n}
\end{aligned}
$$

$$\therefore \quad h_H[n] = (-1)^n h_L[n] = (-1)^n \frac{\sin(\Omega_c n)}{\pi n}$$

# Cascaded System

Plot $h_L[n]$ and $h_H[n]$ for $\Omega_c = \frac{\pi}{3}$.



# Cascaded System

Determine an expression for $H_H(\Omega)$.

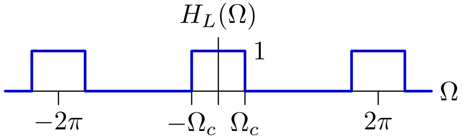
$$\begin{aligned} H_H(\Omega) &= \sum_{n=-\infty}^{\infty} h_H[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n h_L[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} e^{j\pi n} h_L[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} h_L[n] e^{-j(\Omega-\pi)n} \\ &= H_L(\Omega - \pi) \end{aligned}$$

This is an example of the frequency shifting property:

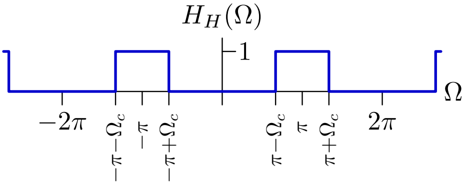
If $x[n] \overset{\text{DTFT}}{\Longleftrightarrow} X(\Omega)$ then $e^{j\Omega_o n} x[n] \overset{\text{DTFT}}{\Longleftrightarrow} X(\Omega - \Omega_o)$.

# Cascaded System

Plot $H_H(\Omega)$. Compare to $H_L(\Omega)$.



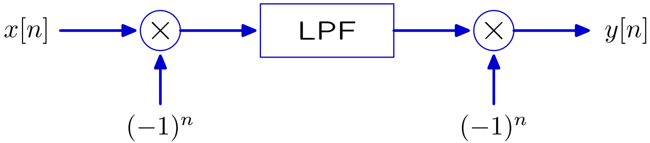

$$H_H(\Omega) = H_L(\Omega - \pi)$$




$H_H(\Omega)$ is a highpass filter!

# Cascaded System

We have just shown that the original system is a highpass system.

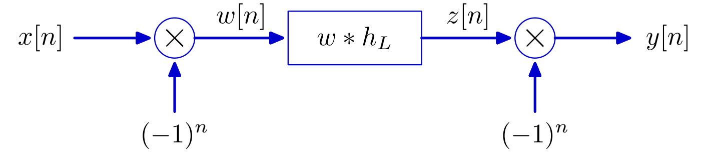


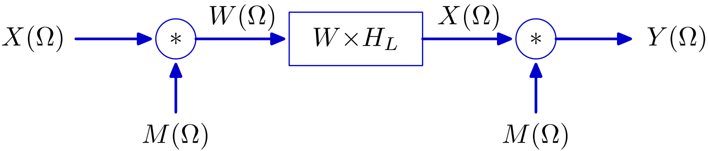

Therefore all of the following are true!

- The transformation from $x[n]$ to $y[n]$ is linear. $\surd$
- The transformation from $x[n]$ to $y[n]$ is time invariant. $\surd$
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter. $\surd$

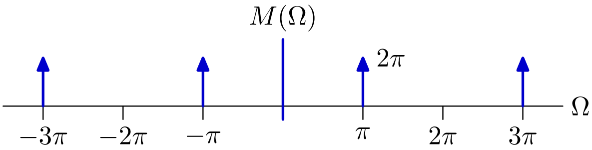
# Cascaded System

Alternatively, we could solve this problem in the frequency domain.



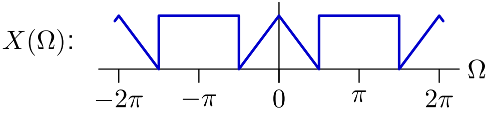




Since $(-1)^n = e^{j\pi n}$, $M(\Omega)$ is a complex sinusoid with frequency $\pi$.

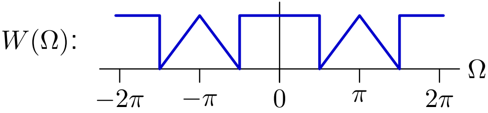


# Cascaded System

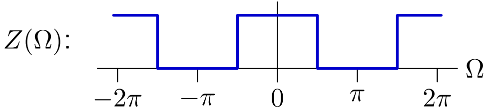
Assume $X(\Omega)$ differs at high and low frequencies, as shown below.



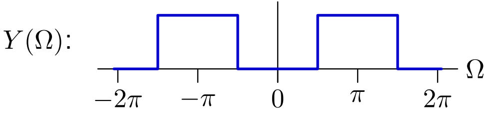

Convolving with $M(\Omega)$ shifts $X(\Omega)$ by $\Omega = \pi$:




Low pass filtering removes the high frequencies. Assume $\Omega_c = \pi/2$:




Convolving with $M(\Omega)$ shifts the result by $\Omega = \pi$:




$Y(\Omega)$ is a highpass version of $X(\Omega)$.