

6.003: Signal Processing

Superposition and Convolution

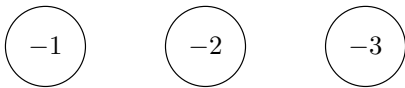
$$y(t) = (h * x)(t) = \int h(\tau)x(t - \tau) d\tau$$

$$y[n] = (h * x)[n] = \sum_m h[m]x[n - m]$$

Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

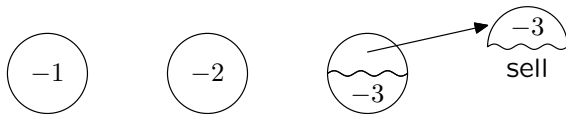
Start with 3 barrels of wine: newest at left, oldest at right.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

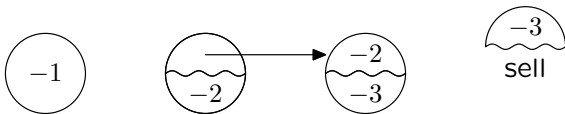
Sell half of the oldest stock.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

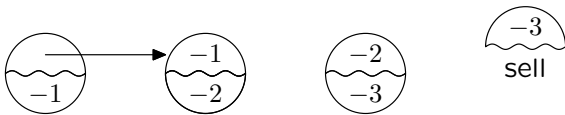
Refill oldest barrel from next-to-oldest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

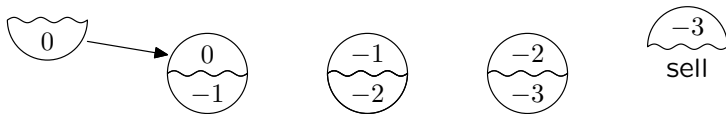
Refill next-to-oldest barrel from youngest barrel.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Refill youngest barrel with this year's harvest.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

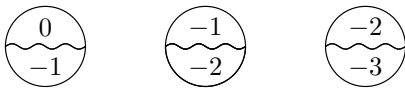
Old and new contents mix; ready for next year.



Discrete-Time Example: Solera Process

Aging and blending wines from different crops.

Old and new contents mix; ready for next year.



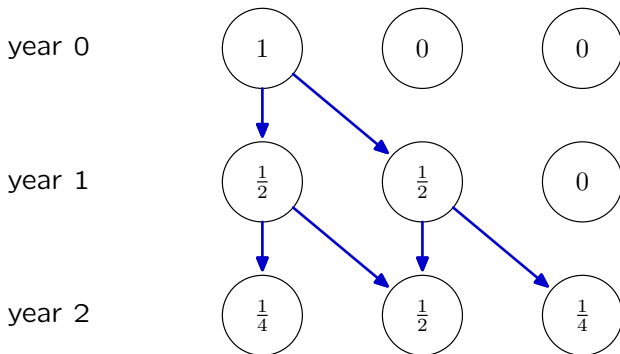
Properties of solera process:

- Mixing produces a more uniform product.
- Mitigates worst-case results of one bad year.
- Blends wines from MANY previous years.

Solera Analysis

We can analyze these effects with a tracer experiment.

Add 1 unit of tracer to new crop; track tracer through the system.

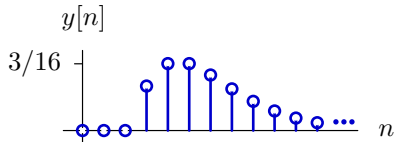
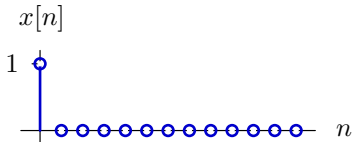


How much tracer will be in each barrel at the end of year 3?

Solera Analysis

Add 1 unit of tracer to new crop; track tracer through the system.

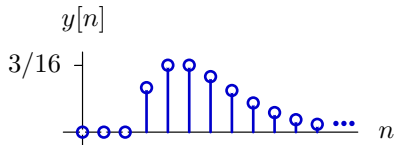
Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



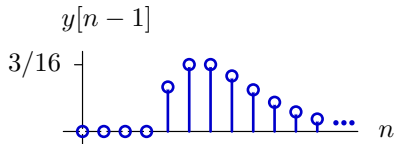
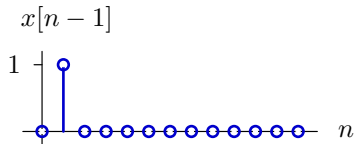
Solera Analysis

How would results change if tracer were added in year 1 (not 0)?

Original response:



Delayed input \rightarrow delayed output:

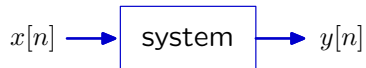


Delaying the input by a year simply delays the outputs by one year.

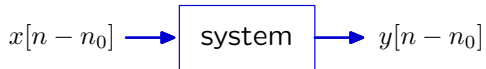
Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given



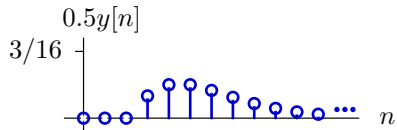
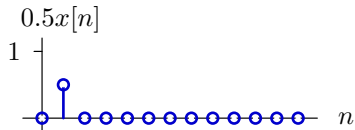
the system is **time invariant** if



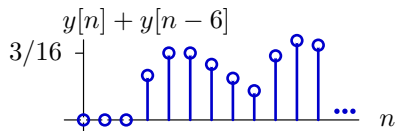
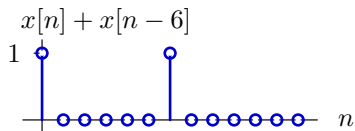
is true for all n_0 .

Solera Analysis

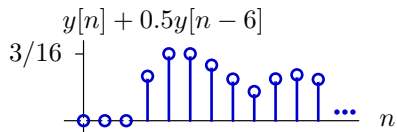
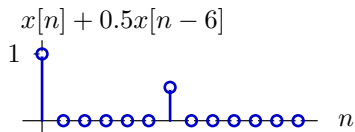
Scaling the input amplitudes:



Adding two inputs:



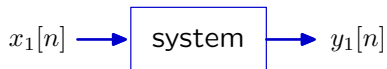
Linearly combining two inputs:



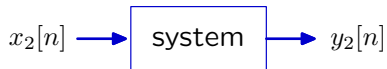
Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

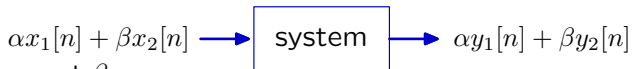
Given



and



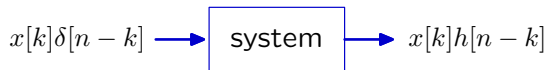
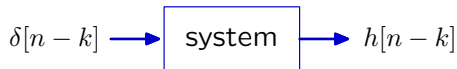
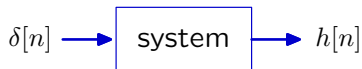
the system is linear if



is true for all α and β .

Convolution

If a system is linear and time invariant, the response to an arbitrary input is the convolution of that input with the unit-sample response.



$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \longrightarrow \text{system} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2?

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above

Check Yourself

For solera process ...

Year n	Tracer in $x[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64

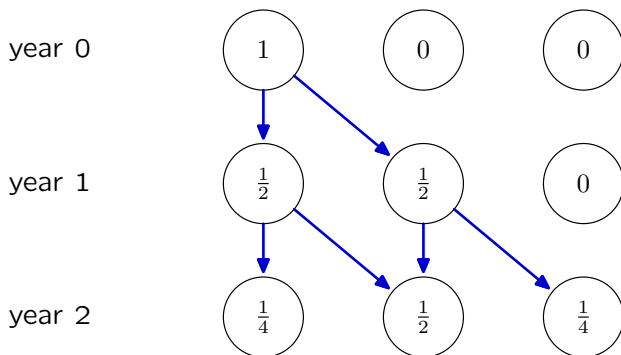
How much tracer would be found in output from year 5, if one unit of tracer is added in years 0, 1, and 2? 2

1. $\frac{21}{32}$
2. $\frac{1}{2}$
3. $\frac{3}{16}$
4. $\frac{9}{16}$
5. none of above

Divide and Conquer

The content of barrel #3 has no direct dependence on barrel #1.

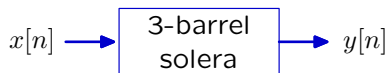
The new content of barrel #3 depends only on itself and barrel #2. All dependence on barrel #1 is through barrel #2.



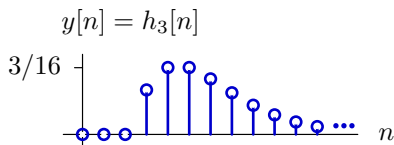
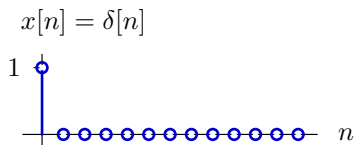
Since barrel #3 depends only on barrel #2, and barrel #2 depends only on barrel #1, the three barrel system is equivalent to the cascade of three one barrel systems!

Divide and Conquer

Making a three-barrel system by cascading three one-barrel systems.

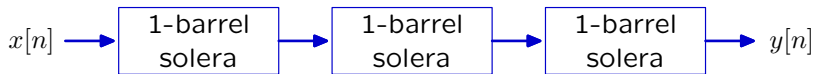


Year n	Tracer in $x[n] = \delta[n]$	Barrel #1	Barrel #2	Barrel #3	Tracer out $y[n] = h_3[n]$
0	1	1	0	0	0
1	0	1/2	1/2	0	0
2	0	1/4	2/4	1/4	0
3	0	1/8	3/8	3/8	1/8
4	0	1/16	4/16	6/16	3/16
5	0	1/32	5/32	10/32	6/32
6	0	1/64	6/64	15/64	10/64



Divide and Conquer

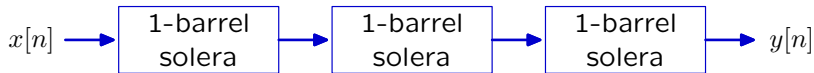
Find the unit-sample response $h_1[n]$ for a 1-barrel solera.



Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

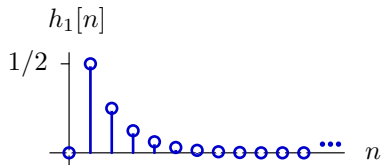
Divide and Conquer

Find the unit-sample response $h_1[n]$ for a 1-barrel solera.



Show that $h_3[n] = ((h_1 * h_1) * h_1)[n]$.

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases}$$



No tracer leaves barrel #1 on the year the tracer is added ($n = 0$).

Half leaves the following year.

Half of the remainder leaves on each subsequent year.

The sum of all that leaves (from $n = 0$ to ∞) is 1 (all of it).

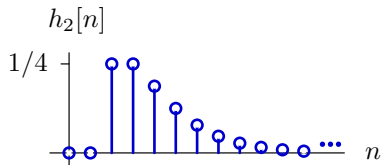
Divide and Conquer

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1]$$

$$\begin{aligned} h_2[n] &= (h_1 * h_1)[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_1[n-m] \\ &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] \left(\frac{1}{2}\right)^{n-m} u[n-m-1] \end{aligned}$$

The value being summed is zero unless $m-1 \geq 0$ and $n-m-1 \geq 0$.
Therefore $1 \leq m \leq n-1$ and $n \geq 2$:

$$h_2[n] = \sum_{m=1}^{n-1} \left(\frac{1}{2}\right)^n = (n-1) \left(\frac{1}{2}\right)^n u[n-2]$$



Divide and Conquer

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n-1] \quad \text{and} \quad h_2[n] = (n-1) \left(\frac{1}{2}\right)^n u[n-2]$$

$$\begin{aligned} h_3[n] &= (h_1 * h_2)[n] = \sum_{m=-\infty}^{\infty} h_1[m] h_2[n-m] \\ &= \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^m u[m-1] (n-m-1) \left(\frac{1}{2}\right)^{n-m} u[n-m-2] \end{aligned}$$

The value being summed is zero unless $m-1 \geq 0$ and $n-m-2 \geq 0$.
Therefore $1 \leq m \leq n-2$ and $n \geq 3$:

$$h_3[n] = \sum_{m=1}^{n-2} (n-m-1) \left(\frac{1}{2}\right)^n = \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n u[n-3]$$

