

6.003: Signal Processing

Continuous-Time Fourier Transform

Synthesis Equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Analysis Equation

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Continuous-Time Fourier Transform

Find the Fourier transforms of the following continuous-time signals.

- $x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$
- $x_2(t) = x_1(t - t_0)$
- $x_3(t) = \text{Symmetric}\{x_1(t)\}$
- $x_4(t) = \text{Antisymmetric}\{x_1(t)\}$
- $x_5(t) = \frac{d}{dt} \text{Symmetric}\{x_1(t)\}$

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

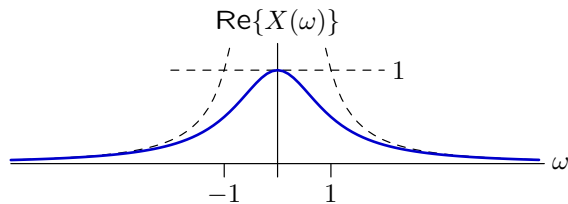
$$\begin{aligned} X_1(\omega) &= \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-t} e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(1+j\omega)t} dt \\ &= -\frac{1}{1+j\omega} e^{-(1+j\omega)t} \Big|_0^{\infty} \\ &= \frac{1}{1+j\omega} \end{aligned}$$

Continuous-Time Fourier Transform

Find the real and imaginary parts of $X(\omega)$.

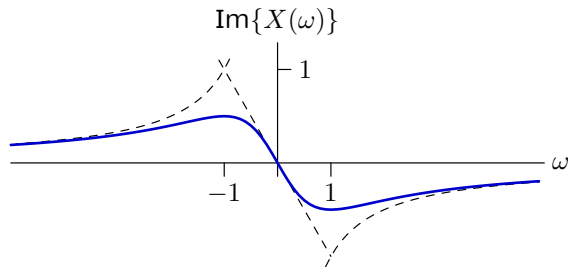
$$X(\omega) = \frac{1}{1+j\omega} \times \frac{1-j\omega}{1-j\omega} = \frac{1-j\omega}{1+\omega^2}$$

Plot these functions.



$$|\omega| \ll 1 : \frac{1}{1+\omega^2} \rightarrow 1$$

$$|\omega| \gg 1 : \frac{1}{1+\omega^2} \rightarrow \frac{1}{\omega^2}$$



$$|\omega| \ll 1 : -\frac{\omega}{1+\omega^2} \rightarrow -\omega$$

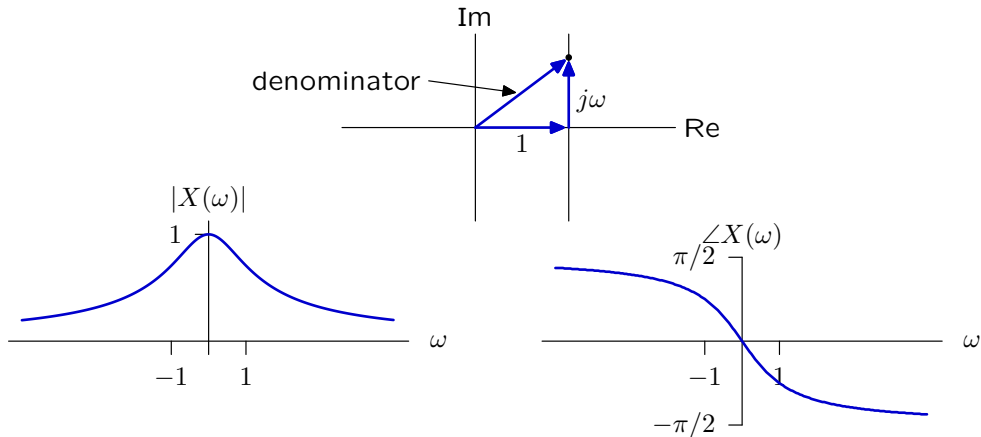
$$|\omega| \gg 1 : -\frac{\omega}{1+\omega^2} \rightarrow -\frac{1}{\omega}$$

Continuous-Time Fourier Transform

Alternative graphical method.

$$X(\omega) = \frac{1}{1 + j\omega}$$

The denominator is the sum of two vectors.

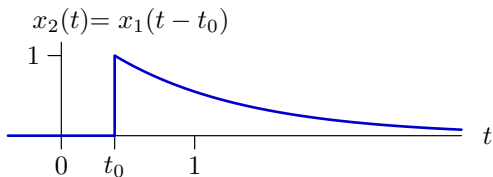
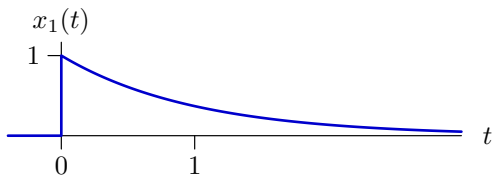


Magnitude is symmetric, angle is antisymmetric in Ω .

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$



Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$

$$\begin{aligned} X_2(\omega) &= \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x_1(t - t_0) e^{-j\omega t} dt \end{aligned}$$

Let $\tau = t - t_0$.

$$\begin{aligned} X_2(\omega) &= \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega(\tau+t_0)} d\tau \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau \\ &= e^{-j\omega t_0} X_1(\omega) \end{aligned}$$

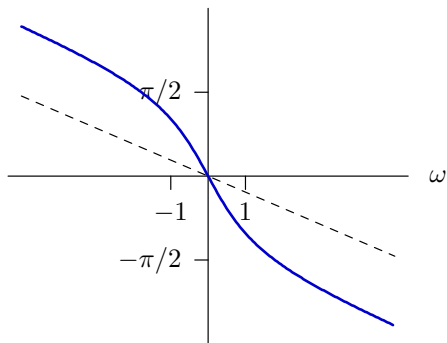
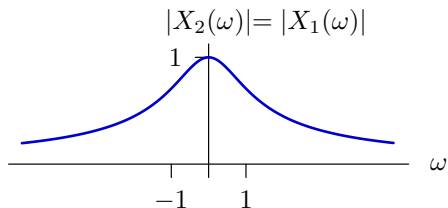
Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_2(t) = x_1(t - t_0)$$

$$X_2(\omega) = e^{-j\omega t_0} X_1(\omega)$$

$$\angle X_2(\omega) = \angle X_1(\omega) - \omega t_0$$



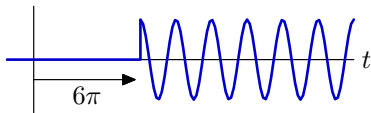
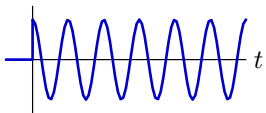
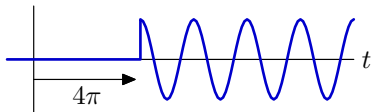
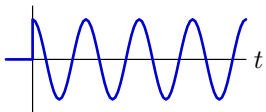
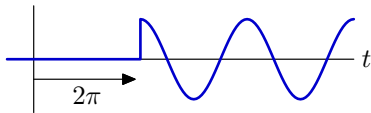
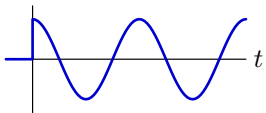
Magnitude is unchanged.

Angle offset by straight line through zero (still antisymmetric).

Why does time delay shift phase by angle proportional to frequency?

Continuous-Time Fourier Transform

Why does time delay shift phase by angle proportional to frequency?
Think about Fourier components of a signal that are each delayed by same time t_0 .



The same amount of time corresponds to different amounts of phase.

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_3(t) = \text{Symmetric}\{x_1(t)\}$$

$$\begin{aligned} X_3(\omega) &= \int_{-\infty}^{\infty} \text{Symmetric}\{x_1(t)\} e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (x_1(t) + x_1(-t)) e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + \frac{1}{2} \int_{-\infty}^{\infty} x_1(-t) e^{-j\omega t} dt \\ &= \frac{1}{2} X_1(\omega) + \frac{1}{2} \int_{-\infty}^{\infty} x_1(\tau) e^{j\omega\tau} d\tau \\ &= \frac{1}{2} X_1(\omega) + \frac{1}{2} X_1(-\omega) \\ &= \frac{1}{2} \left(\frac{1}{1+j\omega} + \frac{1}{1-j\omega} \right) = \frac{1}{1+\omega^2} \end{aligned}$$

real and symmetric $\stackrel{\text{CTFT}}{\iff}$ real and symmetric

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_4(t) = \text{Antisymmetric}\{x_1(t)\}$$

$$\begin{aligned} X_4(\omega) &= \int_{-\infty}^{\infty} \text{Antisymmetric}\{x_1(t)\} e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (x_1(t) - x_1(-t)) e^{-j\omega t} dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt - \frac{1}{2} \int_{-\infty}^{\infty} x_1(-t) e^{-j\omega t} dt \\ &= \frac{1}{2} X_1(\omega) - \frac{1}{2} \int_{-\infty}^{\infty} x_1(\tau) e^{j\omega \tau} d\tau \\ &= \frac{1}{2} X_1(\omega) - \frac{1}{2} X_1(-\omega) \\ &= \frac{1}{2} \left(\frac{1}{1+j\omega} - \frac{1}{1-j\omega} \right) = \frac{-j\omega}{1+\omega^2} \end{aligned}$$

real and antisymmetric $\xLeftrightarrow{\text{CTFT}}$ imaginary and antisymmetric

Continuous-Time Fourier Transform

Find the Fourier transform of the following signal.

$$x_5(t) = \frac{d}{dt} \text{Symmetric}\{x_1(t)\}$$

$$\begin{aligned} X_5(\omega) &= \int_{-\infty}^{\infty} \underbrace{\frac{dx_3(t)}{dt}}_{dv} \underbrace{e^{-j\omega t}}_u \underbrace{dt}_{dv} \\ &= \underbrace{e^{-j\omega t}}_u \underbrace{x_3(t)}_v \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \underbrace{x_3(t)}_v \underbrace{(-j\omega)e^{-j\omega t} dt}_{du} \end{aligned}$$

The first term is zero since the CTFT of $x_3(t)$ converged.

$$\begin{aligned} X_5(\omega) &= j\omega \int_{-\infty}^{\infty} x_3(t) e^{-j\omega t} dt \\ &= j\omega X_3(\omega) = \frac{j\omega}{1 + \omega^2} \end{aligned}$$

real and antisymmetric $\overset{\text{CTFT}}{\iff}$ imaginary and antisymmetric

Continuous-Time Fourier Transform

Alternative method – start with synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \frac{d}{dt} e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) (j\omega) e^{j\omega t} d\omega$$

$$Y(\omega) = j\omega X(\omega)$$

Straightforward approach (start with analysis) not always simplest.

This is a great example of how Fourier transforms simplify calculus.

Differentiation in time corresponds to multiplication by $j\omega$ in frequency.

Inverse Continuous-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

Inverse Continuous-Time Fourier Transform

Find the signal whose Fourier transform is

$$X(\omega) = e^{-|\omega|}$$

$$\begin{aligned}x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \\&= \frac{1}{2\pi} \int_{-\infty}^0 e^{\omega} e^{j\omega t} d\omega + \frac{1}{2\pi} \int_0^{\infty} e^{-\omega} e^{j\omega t} d\omega \\&= \frac{1}{2\pi} \left. \frac{e^{(jt+1)\omega}}{jt+1} \right|_{-\infty}^0 + \frac{1}{2\pi} \left. \frac{e^{(jt-1)\omega}}{jt-1} \right|_0^{\infty} \\&= \frac{1}{2\pi} \frac{1}{jt+1} - \frac{1}{2\pi} \frac{1}{jt-1} \\&= \frac{1/\pi}{1+t^2}\end{aligned}$$