6.003: Signal Processing

Discrete-Time Fourier Series

Synthesis Equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k\, e^{j\frac{2\pi k}{N}n}$$

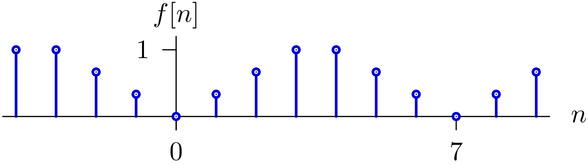
Analysis Equation

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]\, e^{-j\frac{2\pi k}{N}n}$$

September 23, 2021

Find the DT Fourier Series Coefficients

Let $f[n]$ represent a periodic DT signal with period $N = 7$:



Determine the Fourier series coefficients $F[k]$ for $f[n]$.

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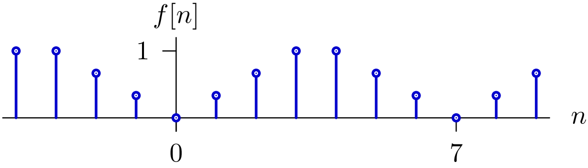


Determine the Fourier series coefficients $F[k]$ for $f[n]$.

$$F[k] = \frac{1}{7}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{7}kn}$$

$$= \frac{1}{7}\left(\frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k} + e^{-j\frac{2\pi}{7}4k} + \frac{2}{3}e^{-j\frac{2\pi}{7}5k} + \frac{1}{3}e^{-j\frac{2\pi}{7}6k}\right)$$

This is a completely well-formed answer – but we can simplify.

Find the DT Fourier Series Coefficients

Simplifying ...

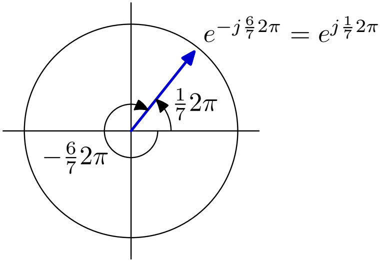
$$F[k] = \frac{1}{7}\left(\frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k} + e^{-j\frac{2\pi}{7}4k} + \frac{2}{3}e^{-j\frac{2\pi}{7}5k} + \frac{1}{3}e^{-j\frac{2\pi}{7}6k}\right)$$

The last exponential term can be rewritten with a positive exponent:

$$e^{-j\frac{2\pi}{7}6k} = e^{j\frac{2\pi}{7}7k}e^{-j\frac{2\pi}{7}6k} = e^{j\frac{2\pi}{7}k}$$

where we have used the fact that $e^{j\frac{2\pi}{7}7k} = 1$.

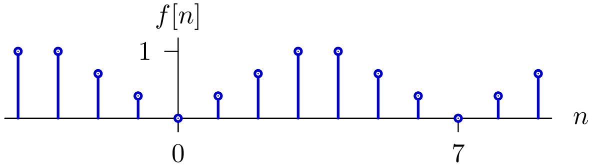
This identity is also apparent in the complex plane.



Given that $e^{-j\frac{6}{7}2\pi} = e^{j\frac{1}{7}2\pi}$ it follows that $\left(e^{-j\frac{6}{7}2\pi}\right)^k = \left(e^{j\frac{1}{7}2\pi}\right)^k$

Find the DT Fourier Series Coefficients

We could get the same answer by summing a different set of time indices.



Sum $n = -3$ to 3 instead of 0 to 6:

$$F[k] = \frac{1}{7}\sum_{n=-3}^{3} f[n]e^{-j\frac{2\pi}{7}kn}$$

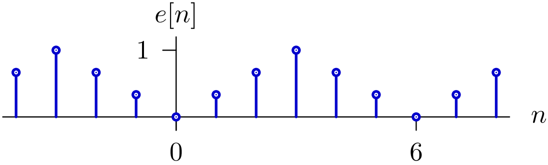
$$= \frac{1}{7}\left(e^{j\frac{2\pi}{7}3k} + \frac{2}{3}e^{j\frac{2\pi}{7}2k} + \frac{1}{3}e^{j\frac{2\pi}{7}1k} + \frac{1}{3}e^{-j\frac{2\pi}{7}k} + \frac{2}{3}e^{-j\frac{2\pi}{7}2k} + e^{-j\frac{2\pi}{7}3k}\right)$$

$$= \frac{2}{21}\cos\left(\frac{2\pi k}{7}\right) + \frac{4}{21}\cos\left(\frac{4\pi k}{7}\right) + \frac{6}{21}\cos\left(\frac{6\pi k}{7}\right)$$

Whichever way we do the math, the answer reduces to the sum of three cosine terms.

Find the DT Fourier Series Coefficients

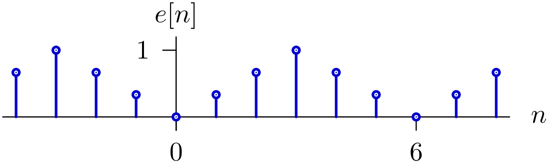
How would the answer change if the period were $N = 6$?



Determine the Fourier series coefficients $E[k]$ for $e[n]$.

Find the DT Fourier Series Coefficients

How would the answer change if the period were $N = 6$?



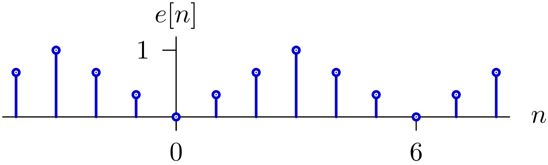
Determine the Fourier series coefficients $E[k]$ for $e[n]$.

$$E[k] = \frac{1}{6}\sum_{n=0}^{5} e[n]e^{-j\frac{2\pi}{6}kn}$$
$$= \frac{1}{6}\left(\frac{1}{3}e^{-j\frac{2\pi}{6}k} + \frac{2}{3}e^{-j\frac{2\pi}{6}2k} + \frac{3}{3}e^{-j\frac{2\pi}{6}3k} + \frac{2}{3}e^{-j\frac{2\pi}{6}4k} + \frac{1}{3}e^{-j\frac{2\pi}{6}5k}\right)$$

Can we simplify the answer by summing over indices centered on 0?

Find the DT Fourier Series Coefficients

How would the answer change if the period were $N = 6$?



Can we simplify the answer by summing over indices centered on 0?

Yes. But we must be careful at the edges.

Include $n = -3$ or $n = 3$ but not both.

$$E[k] = \frac{1}{6} \sum_{n=-3}^{2} e[n] e^{-j\frac{2\pi}{6}kn}$$

$$= \frac{1}{6}\left(e^{j\frac{2\pi}{6}3k} + \frac{1}{3}e^{j\frac{2\pi}{6}2k} + \frac{1}{3}e^{j\frac{2\pi}{6}k} + \frac{1}{3}e^{-j\frac{2\pi}{6}k} + \frac{2}{3}e^{-j\frac{2\pi}{6}2k} \right)$$

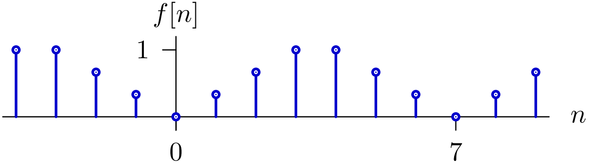
Notice that the $n = -3$ and $n = 3$ terms are equal.

$$e^{j\frac{2\pi}{6}3k} = e^{-j\frac{2\pi}{6}3k} = (e^{\pm j\pi})^k = (-1)^k$$

Find the DT Fourier Series Coefficients

Consider a new signal $g[n]$ derived from $f[n]$ as follows:

$$g[n] = 9 - 3f[n-1]$$



Find the DTFS coefficients of $g[n]$.

Find the DT Fourier Series Coefficients

Consider a new signal $g[n]$ derived from $f[n]$ as follows:

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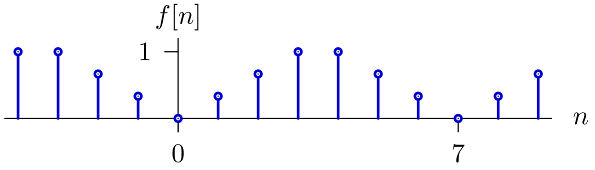


Find the DTFS coefficients of $g[n]$.

The straightforward approach is to calculate $g[n]$ for all n.

An easier approach is to use properties of the Fourier series.
We can use linearity to break the problem into two easier pieces:

$$g[n] = g_1[n] - g_2[n]$$

where $g_1[n] = 9$ and $g_2[n] = 3f[n-1]$.

Find the DT Fourier Series Coefficients

We can use linearity to break the problem into two easier pieces.

$$g[n] = g_1[n] - g_2[n]$$

where $g_1[n] = 9$ and $g_2[n] = 3f[n-1]$.

$$G_1[k] = \frac{1}{7}\sum_{n=0}^{6} 9e^{-j\frac{2\pi}{7}kn} = 9\delta[k]$$

Notice that we must use the same period $N = 7$ for $G_1[k]$, $G_2[k]$, and $G[k]$ in order to (later) apply linearity.

$g_2[n]$ combines a delay of 1 sample with multiplying by a scale factor 3. The delay of 1 simply multiplies the Fourier coefficients (of $f[n]$) by $e^{-j\frac{2\pi}{7}k}$. Scaling by 3 similarly multiplies the Fourier coefficients (of $f[n-1]$) by 3. The net result is

$$G_2[k] = 3e^{-j\frac{2\pi}{7}k}F[k]$$

and

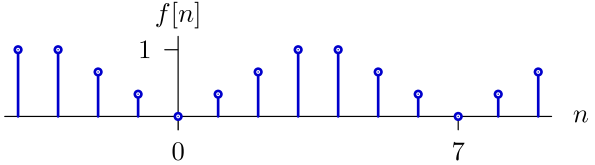
$$G[k] = 9\delta[k] - 3e^{-j\frac{2\pi}{7}k}F[k]$$

Find the DT Fourier Series Coefficients

Consider another new signal

$$h[n] = (-1)^n f[n]$$

where



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Find the DT Fourier Series Coefficients

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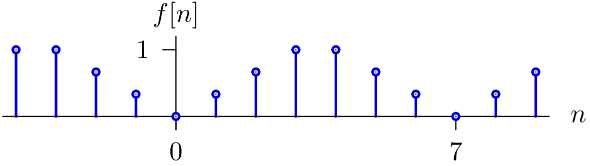


Find the DTFS coefficients of $h[n]$.

What's the effect of multiplying by $(-1)^k$?

Let $f_1[n] = (-1)^n f[n]$.
Notice that $f_1[n]$ is not periodic in $N = 7$.
We will have to analyze $f_1[n]$ with $N = 14$!

Find the DT Fourier Series Coefficients

How does changing $N = 7$ to $N = 14$ affect the Fourier series coefficients?

If the period is $N = 7$ then

$$F_7[k] = \frac{1}{7}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{7}kn}$$

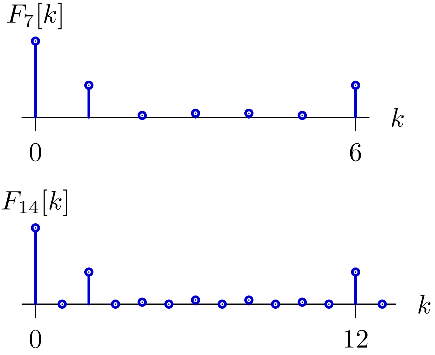
If the period is $N = 14$ then

$$\begin{aligned}
F_{14}[k] &= \frac{1}{14}\sum_{n=0}^{13} f[n]e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{14}kn} + \frac{1}{14}\sum_{n=7}^{13} f[n]e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]e^{-j\frac{2\pi}{14}kn} + \frac{1}{14}\sum_{m=0}^{6} \underbrace{f[m+7]}_{f[m]}\ \underbrace{e^{-j\frac{2\pi}{14}k(m+7)}}_{e^{-j\frac{2\pi}{14}km}e^{-j\frac{2\pi}{14}7k}} \\
&= \frac{1}{14}\sum_{n=0}^{6} f[n]\left(1+(-1)^k\right)e^{-j\frac{2\pi}{14}kn} = \begin{cases} F_7[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Find the DT Fourier Series Coefficients

How does changing $N = 7$ to $N = 14$ affect the Fourier series coefficients?

$$F_{14}[k] = \begin{cases} F_7[k/2] & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$



The components of F_7 are **stretched** in F_{14}.

There is no fundamental in F_{14}, the harmonics are 0, 2, 4, ... 12.

Find the DT Fourier Series Coefficients

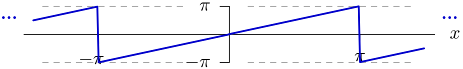
Now find the DTFS coefficients for $h[n]$:

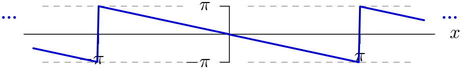
$$h[n] = (-1)^n f[n]$$

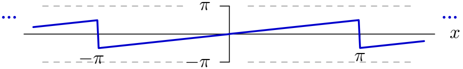
$$\begin{aligned}
H[k] &= \frac{1}{14}\sum_{n=0}^{13}(-1)^n f[n] e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{13} e^{j\pi n} f[n] e^{-j\frac{2\pi}{14}kn} \\
&= \frac{1}{14}\sum_{n=0}^{13} f[n] e^{-j\frac{2\pi}{14}(k-7)n} \\
&= F_{14}[k-7] \\
&= \begin{cases} F_7[(k-7)/2] & \text{if } k-7 \text{ is even} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} F[(k-7)/2] & \text{if } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

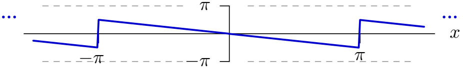
Angular Trends

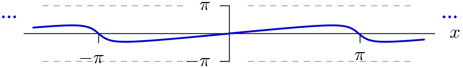
Which of the following plots shows the angle of e^{-jx}?

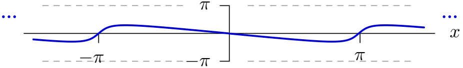
A


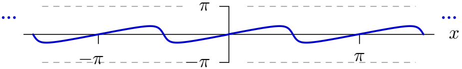
B


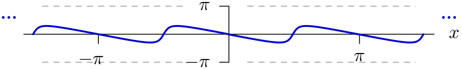
C


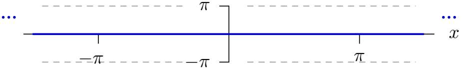
D


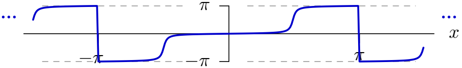
E


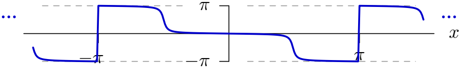
F


G


H


I


J


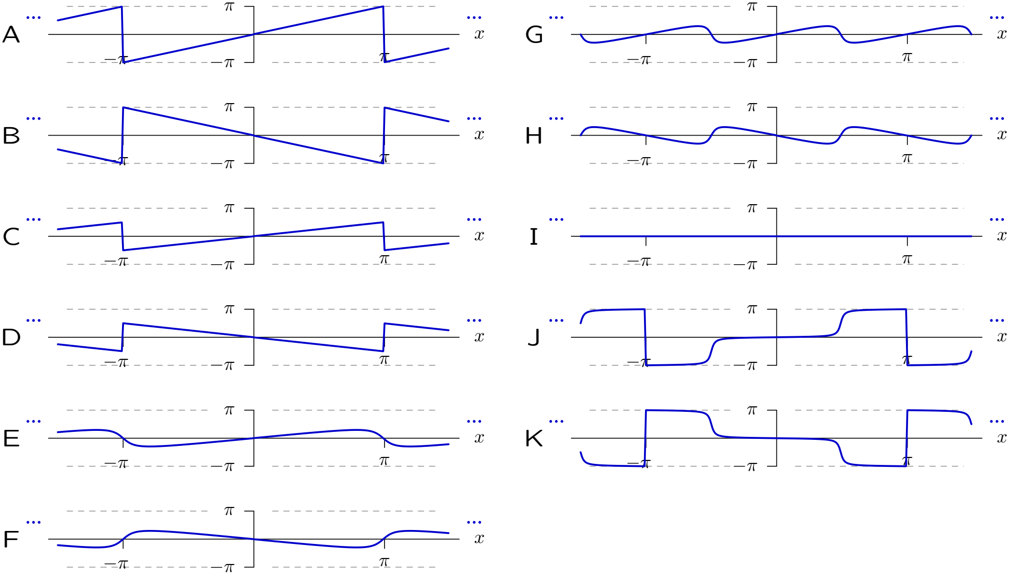
K


Angular Trends

$\angle e^{-jx}$: A complex exponential of the form $e^{j\theta}$ has magnitude 1 and angle θ. Therefore, the angle of e^{-jx} is $-x$, as shown in plot B.

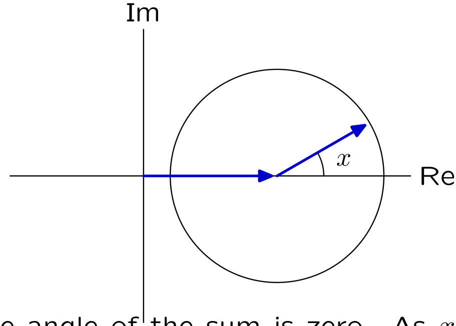
Angular Trends

Which of the following plots shows the angle of $(1 + 0.8e^{jx})$?



Angular Trends

$\angle(1+0.8e^{jx})$: The number $1+0.8e^{jx}$ is the sum of 1 with a vector of magnitude 0.8 and angle x as shown in the following plot.

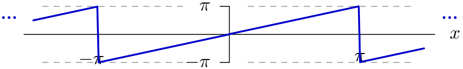


When x is small, the angle of the sum is zero. As x increases, the angle increases until x reaches about $3\pi/4$. At this point, the angle of the sum is on the order of $\pi/3$. As x increases above $3\pi/4$, the angle of the sum quickly decreases, returning to zero when $x=\pi$. From the symmetry of the figure, it follows that the angle of the sum is an odd function of x. Thus the answer is plot E.

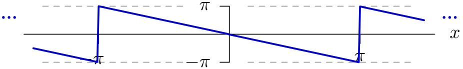
Angular Trends

Which of the following plots shows the angle of $\left(\frac{1+0.4e^{jx}}{2+0.8e^{jx}}\right)$?

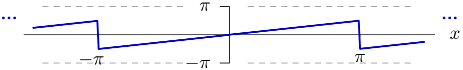
A



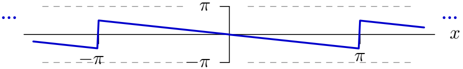
B



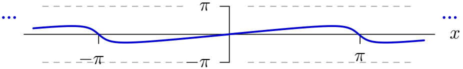
C



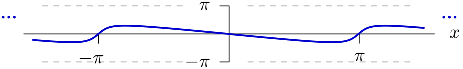
D



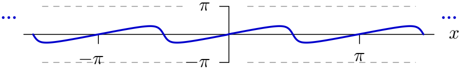
E



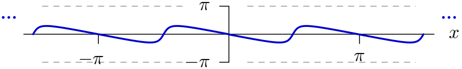
F



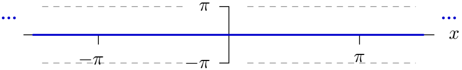
G



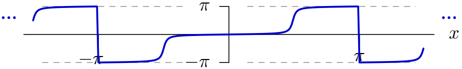
H



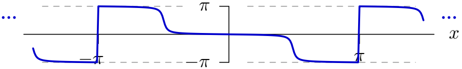
I



J



K

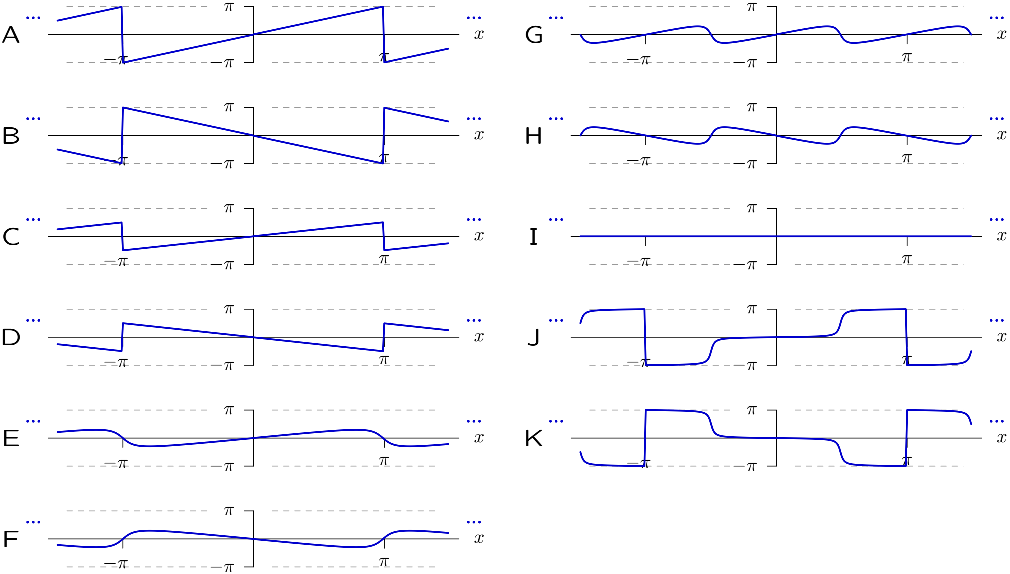


Angular Trends

$\angle\left(\frac{1+0.4e^{jx}}{2+0.8e^{jx}}\right)$: Since the denominator is twice the numerator, this is just the angle of a real number $(1/2)$, which is zero – plot I.

Angular Trends

Which of the following plots shows the angle of $(1 + e^{jx})$?



Angular Trends

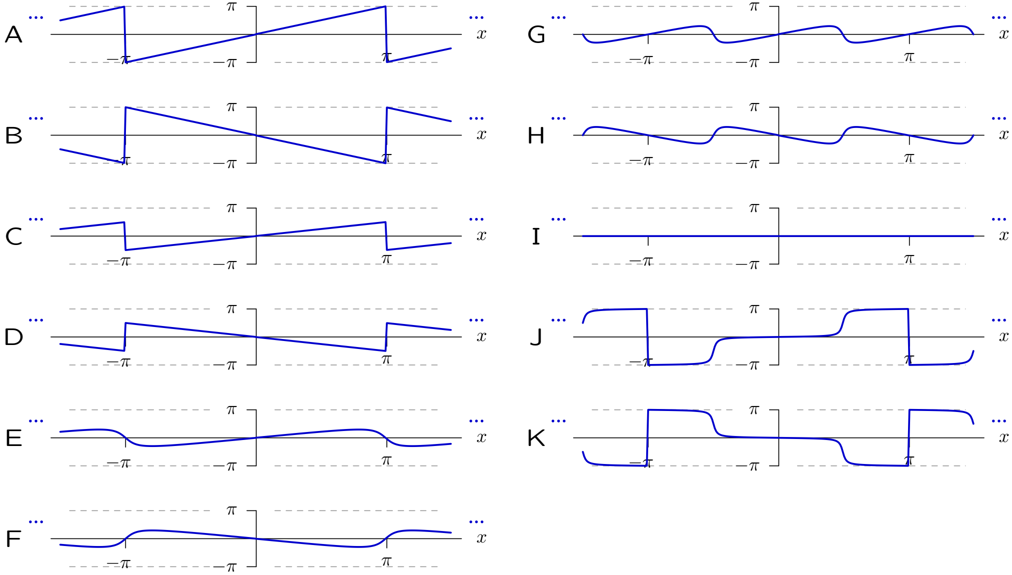
$\angle(1 + e^{jx})$:

$$1 + e^{jx} = e^{j\frac{x}{2}} \left(e^{-j\frac{x}{2}} + e^{j\frac{x}{2}} \right) = e^{j\frac{x}{2}} \, 2\cos\left(\frac{x}{2}\right)$$

Thus the angle of $1 + e^{jx}$ is $x/2$ for $-\pi < x < \pi$. At $x = \pi$ the sign of the cosine flips so that angle jumps by π. Thus the answer is plot C.

Angular Trends

Which of the following plots shows the angle of $(1 + 0.8e^{j2x})$?

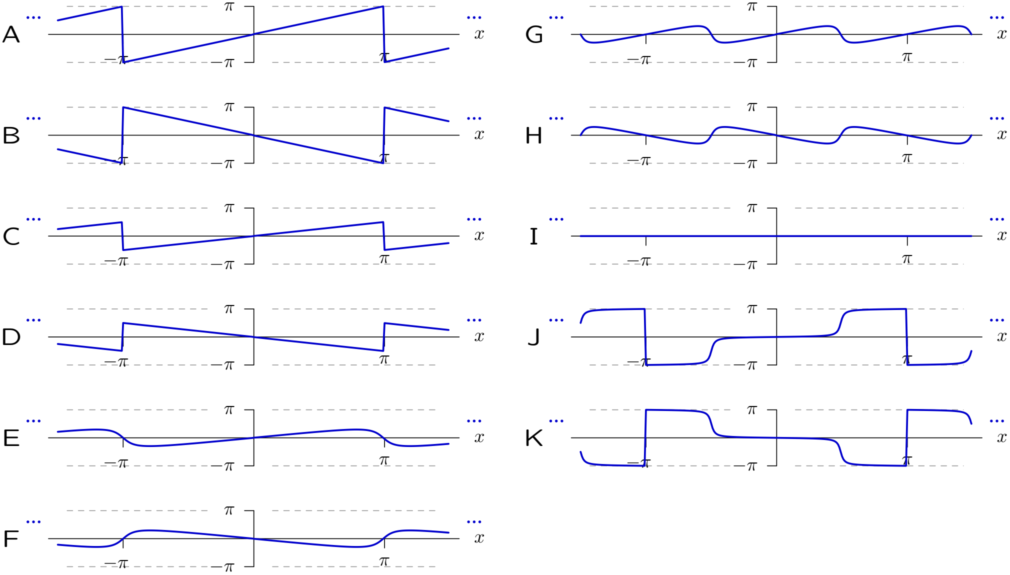


Angular Trends

$\angle(1+0.8e^{j2x})$: This expression looks like part 2 (above) except x is replaced by $2x$. Therefore the answer the same as that for part 2 except that the x-axis is compressed by a factor of 2 – generating plot G.

Angular Trends

Which of the following plots shows the angle of $(0.9e^{jx}+0.8e^{-jx})$?



Angular Trends

$\angle(0.9e^{jx}+0.8e^{-jx})$: The expression $0.9e^{jx}+0.8e^{-jx}$ can be simplified by converting to Cartesian form:

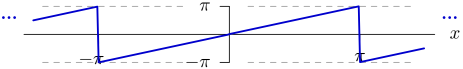
$$0.9\cos(x) + j0.9\sin(x) + 0.8\cos(x) - j0.8\sin(x) = 1.7\cos(x) + j0.1\sin(x)$$

The angle is therefore $\arctan\left(\frac{0.1\sin(x)}{1.7\cos(x)}\right) = \arctan\left(\frac{1}{17}\tan(x)\right)$ which is plot J.

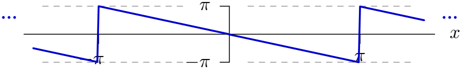
Angular Trends

Which of the following plots shows the angle of $\left(\frac{1}{1+0.8e^{jx}}\right)$?

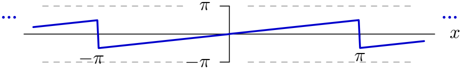
A



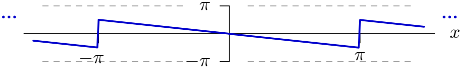
B



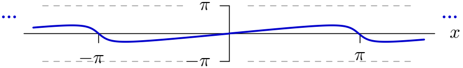
C



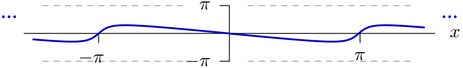
D



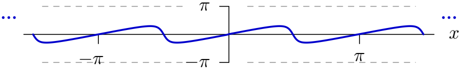
E



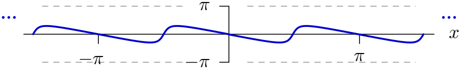
F



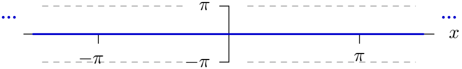
G



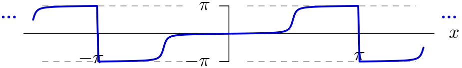
H



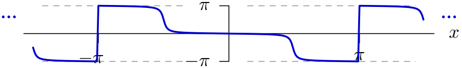
I



J



K



Angular Trends

$\angle\left(\frac{1}{1+0.8e^{jx}}\right)$: The expression $1 + 0.8e^{jx}$ was evaluated in part 2 (above). Here the expression is in the denominator, so the answer is the negative of the answer to part 2 – which yields plot F.