Sampling and Aliasing
**Tones and Sinusoids**

A “tone” is a pressure that changes sinusoidally with time.

In 6.003, we will think of this as a “continuous-time” (CT) signal. In contrast, a “discrete-time” (DT) signal is a sequence of numbers.

Mathematically:

\[ x(t) = A \cos(\omega t) \]

\[ x[n] = A \cos(\Omega n) \]
CT and DT Representations

Assume that $x[n]$ represents “samples” of $x(t)$:

$$x(t), x[n]$$

$$x(t) = A \cos(\omega t)$$

$$x[n] = A \cos(\Omega n)$$

• What are the units of $\omega$, $t$, $\Omega$, and $n$?

Let $f$ represent the “frequency” of the tone in cycles/second.

• Determine $\omega$ in terms of $f$.
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- What are the units of $\omega$, $t$, $\Omega$, and $n$?

The product $\omega t$ is measured in units of radians (dimensionless ratio).

Time $t$ is measured in units of seconds.

Therefore $\omega$ is measured in units of radians/second.

The product $\Omega n$ is measured in units of radians (domain of $\cos(\cdot)$).

Discrete time $n$ is a dimensionless integer.

Therefore $\Omega$ is measured in units of radians.

For convenience, we often think of $n$ as measured in number of samples and $\Omega$ in radians/sample.
CT and DT Representations

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- Determine $\omega$ in terms of $f$.
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$\omega[\text{rad/sec}] = 2\pi[\text{rad/cycle}] f[\text{cycles/sec}]$

$\Omega[\text{rad/sample}] = \frac{\omega[\text{rad/sec}]}{f_s[\text{samples/sec}]}$ where $f_s = \text{sample frequency}$

$\Omega[\text{rad/sample}] = \frac{2\pi[\text{rad/cycle}] f[\text{cycles/sec}]}{f_s[\text{samples/sec}]}$
Check Yourself

Compare two signals:

\[ x_1[n] = \cos \frac{3\pi n}{4} \]

\[ x_2[n] = \cos \frac{5\pi n}{4} \]

How many of the following statements are true?

1. \( x_1[n] \) has period \( N=8 \).
2. \( x_2[n] \) has period \( N=8 \).
3. \( x_1[n] = x_2[n] \).
Check Yourself

\[
\cos \frac{3\pi t}{4} = \cos \omega_1 t
\]

\[
\cos \frac{5\pi t}{4} = \cos \omega_2 t
\]

\(\omega_1 \neq \omega_2\)

but \(x_1[n] = x_2[n]\)!

\(\omega_1 + \omega_2 = 2\pi\)

\[
\cos \omega_1 n = \cos (2\pi - \omega_2)n = \cos (2\pi n - \omega_2 n) = \cos (-\omega_2 n) = \cos \omega_2 n
\]
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- \( x_2[n] \) has period \( N=8 \). √
- \( x_1[n] = x_2[n] \). √
Frequencies

Consider the following CT signal:

\[ f(t) = 6 \cos(42\pi t) + 4 \cos(18\pi t - 0.5\pi) \]

What is the fundamental period of this signal?
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What is the fundamental period of this signal?

We need to find the smallest time \( T \) for which both \( \cos(42\pi t) \) and \( \cos(18\pi t - 0.5\pi) \) go through an integer number of cycles.

\( \cos(42\pi t) \) goes through one cycle every \( \frac{1}{21} \) seconds, and \( \cos(18\pi t - 0.5\pi) \) goes through one cycle every \( \frac{1}{9} \) seconds. So we want the smallest integers \( m \) and \( n \) such that \( T = \frac{m}{21} = \frac{n}{9} \). Solving we find that \( m = 7 \) and \( n = 3 \), which gives us \( T = \frac{1}{3} \) seconds.
Frequencies

Now imagine that this same signal

\[ f(t) = 6 \cos(42\pi t) + 4 \cos(18\pi t - 0.5\pi) \]

is sampled with a sampling rate of \( f_s = 60 \text{ Hz} \) to obtain a discrete-time signal \( f[n] \), which is periodic in \( n \) with fundamental period \( N \). Determine the DT frequency components of \( f[n] \).
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Determine the DT frequency components of \( f[n] \).

Sampling at \( f_s = 60 \) Hz results in a periodic DT signal with fundamental period \( N = 20 \) samples:

\[ f[n] = 6 \cos(42\pi \frac{n}{60}) + 4 \cos(18\pi \frac{n}{60} - 0.5\pi) \]

Our goal is to express \( f[n] \) in the form

\[ f[n] = \sum_k e^{j \frac{2\pi k}{20} n} \]

We can use Euler’s formula to convert the cosine terms in \( f[n] \) to complex exponentials. The result has non-zero coefficients at \( k = \pm 3 \) and \( \pm 7 \).

To completely specify \( f[n] \), we must provide all of the components in one period of \( a_k \). Thus we could alternatively use \( k = 3, 7, 13, \) and 17.
Frequencies

The DT signal

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has a fundamental period of \( N = 20 \). However, this signal is also periodic in \( N = 80 \).

Which discrete frequencies are present if we reanalyze with \( N = 80 \)?
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has a fundamental period of \( N = 20 \). However, this signal is also periodic in \( N = 80 \).

Which discrete frequencies are present if we reanalyze with \( N = 80 \)?

\[ k = \pm 12 \text{ and } \pm 28 \text{ or } k = 12, 28, 52, \text{ and } 68. \]
Tones in Python

Determine EXPR1 and EXPR2 below to generate a 1000 Hz cosine tone using a sampling rate of 44,100 samples/second. The tone should last 2.5 seconds.

```python
import math
from lib6003.audio import wav_write
f = [cos(EXPR1 * n) for n in range(0, EXPR2)]
wav_write(f, 44100, 'output.wav')
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`EXPR1` is the DT frequency, which we can calculate as follows:

\[
\Omega = 2\pi \left( \frac{\text{radians}}{\text{cycle}} \right) \times f \left( \frac{\text{cycles}}{\text{second}} \right) / f_s \left( \frac{\text{sample}}{\text{second}} \right)
\]

Substituting the constants above yields

\[
EXPR1=2*\text{math.pi}*1000/44100
\]

`EXPR2` corresponds to the total number of samples needed for 2.5 seconds of audio, which is

\[
EXPR2=\text{int}(2.5*44100)
\]