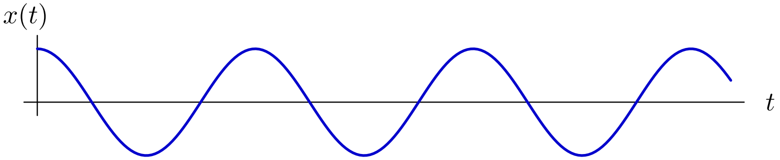
6.003: Signal Processing

Sampling and Aliasing

September 21, 2021

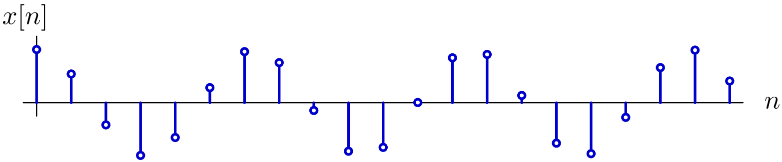
Tones and Sinusoids

A "tone" is a pressure that changes sinusoidally with time.



In 6.003, we will think of this as a "continuous-time" (CT) signal.

In contrast, a "discrete-time" (DT) signal is a sequence of numbers.



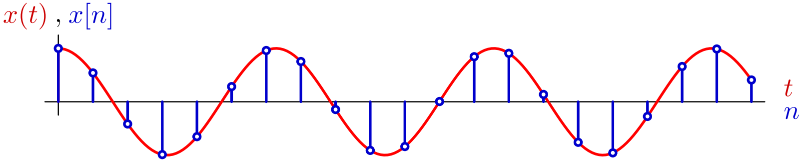
Mathematically:

$$x(t) = A\cos(\omega t)$$

$$x[n] = A\cos(\Omega n)$$

CT and DT Representations

Assume that $x[n]$ represents "samples" of $x(t)$:



$$x(t) = A\cos(\omega t)$$

$$x[n] = A\cos(\Omega n)$$

- What are the units of ω, t, Ω, and n?

Let f represent the "frequency" of the tone in cycles/second.

- Determine ω in terms of f.
- Determine Ω in terms of ω.
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CT and DT Representations

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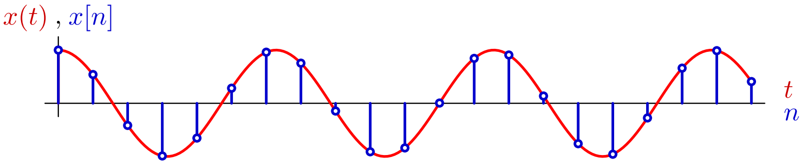

$$x(t) = A\cos(\omega t) \qquad x[n] = A\cos(\Omega n)$$

- What are the units of ω, t, Ω, and n?

The product ωt is measured in units of **radians** (dimensionless ratio).

Time t is measured in units of **seconds**.

Therefore ω is measured in units of **radians/second**.

The product Ωn is measured in units of **radians** (domain of $\cos(\cdot)$).

Discrete time n is a **dimensionless** integer.

Therefore Ω is measured in units of **radians**.

For convenience, we often think of n as measured in **number of samples** and Ω in **radians/sample**.

CT and DT Representations

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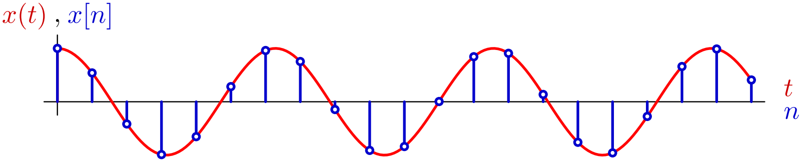


$x(t) = A\cos(\omega t)$ $\qquad$ $x[n] = A\cos(\Omega n)$

Let f represent the "frequency" of the tone in cycles/second.

- Determine ω in terms of f.
- Determine Ω in terms of ω. $[\rightarrow f_s]$
- Determine Ω in terms of f.

$$\omega[\text{rad/sec}] = 2\pi[\text{rad/cycle}]\,f\,[\text{cycles/sec}]$$

$$\Omega[\text{rad/sample}] = \frac{\omega[\text{rad/sec}]}{f_s[\text{samples/sec}]} \quad \text{where } f_s = \text{sample frequency}$$

$$\Omega[\text{rad/sample}] = \frac{2\pi[\text{rad/cycle}]\,f\,[\text{cycles/sec}]}{f_s[\text{samples/sec}]}$$

Check Yourself

Compare two signals:

$$x_1[n] = \cos\frac{3\pi n}{4}$$

$$x_2[n] = \cos\frac{5\pi n}{4}$$

How many of the following statements are true?

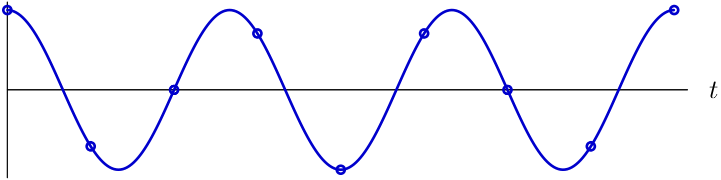
$x_1[n]$ has period N=8.

$x_2[n]$ has period N=8.

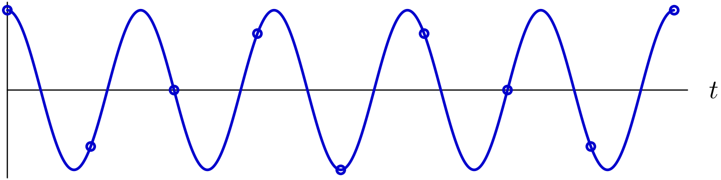
$x_1[n] = x_2[n]$.

Check Yourself

$\cos\frac{3\pi t}{4} = \cos\omega_1 t$



$\cos\frac{5\pi t}{4} = \cos\omega_2 t$



$$\omega_1 \neq \omega_2$$

but $x_1[n] = x_2[n]$!

$$\omega_1 + \omega_2 = 2\pi$$

$$\cos\omega_1 n = \cos(2\pi - \omega_2)n = \cos(2\pi n - \omega_2 n) = \cos(-\omega_2 n) = \cos\omega_2 n$$

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How many of the following statements are true? 3

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Frequencies

Consider the following CT signal:

$$f(t) = 6\cos(42\pi t) + 4\cos(18\pi t - 0.5\pi)$$

What is the fundamental period of this signal?

Frequencies

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What is the fundamental period of this signal?

We need to find the smallest time T for which both $\cos(42\pi t)$ and $\cos(18\pi t - 0.5\pi)$ go through an integer number of cycles.

$\cos(42\pi t)$ goes through one cycle every $\frac{1}{21}$ seconds, and $\cos(18\pi t - 0.5\pi)$ goes through one cycle every $\frac{1}{9}$ seconds. So we want the smallest integers m and n such that $T = \frac{m}{21} = \frac{n}{9}$. Solving we find that $m = 7$ and $n = 3$, which gives us $T = \frac{1}{3}$ seconds.

Frequencies

Now imagine that this same signal

$$f(t) = 6\cos(42\pi t) + 4\cos(18\pi t - 0.5\pi)$$

is sampled with a sampling rate of $f_s = 60\,\text{Hz}$ to obtain a discrete-time signal $f[n]$, which is periodic in n with fundamental period N.

Determine the DT frequency components of $f[n]$.

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Determine the DT frequency components of $f[n]$.

Sampling at $f_s = 60\,\text{Hz}$ results in a periodic DT signal with fundamental period $N = 20$ samples:

$$f[n] = 6\cos(42\pi\frac{n}{60}) + 4\cos(18\pi\frac{n}{60} - 0.5\pi)$$

Our goal is to express $f[n]$ in the form

$$f[n] = \sum_k e^{j\frac{2\pi k}{20}n}$$

We can use Euler's formula to convert the cosine terms in $f[n]$ to complex exponentials. The result has non-zero coefficients at $k = \pm 3$ and ± 7.

To completely specify $f[n]$, we must provide all of the components in one period of a_k. Thus we could alternatively use $k = 3, 7, 13$, and 17.

Frequencies

The DT signal

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has a fundamental period of $N = 20$. However, this signal is also periodic in $N = 80$.

Which discrete frequencies are present if we reanalyze with $N = 80$?

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Which discrete frequencies are present if we reanalyze with $N = 80$?

$k = \pm 12$ and ± 28 or $k = 12, 28, 52,$ and 68.

Tones in Python

Determine EXPR1 and EXPR2 below to generate a 1000 Hz cosine tone using a sampling rate of $44,100$ samples/second. The tone should last 2.5 seconds.

```
import math
from lib6003.audio import wav_write
f = [cos(EXPR1 * n) for n in range(0, EXPR2)]
wav_write(f, 44100, 'output.wav')
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EXPR1 is the DT frequency, which we can calculate as follows:

$$\Omega\left[\frac{\text{radians}}{\text{cycle}}\right] = 2\pi\left[\frac{\text{radians}}{\text{cycle}}\right] \times f\left[\frac{\text{cycles}}{\text{second}}\right] \;/\; f_s\left[\frac{\text{sample}}{\text{second}}\right]$$

Substituting the constants above yields

```
EXPR1=2*math.pi*1000/44100
```

EXPR2 corresponds to the total number of samples needed for 2.5 seconds of audio, which is

```
EXPR2=int(2.5*44100)
```