6.003: Signal Processing

Fourier Series – Complex Form

Synthesis Equation (making a signal from components):

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Analysis Equation (finding the components)

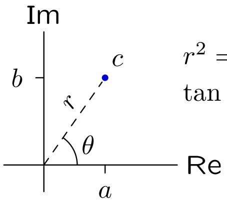
$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

where $\omega_o = \frac{2\pi}{T}$

September 16, 2021

Representations of Complex Numbers

Let c represent a complex number.



$$r^2 = a^2 + b^2$$
$$\tan\theta = \frac{b}{a}$$

rectangular form: $c = a + jb$

polar (phasor) form: $r\angle\theta$

Euler form: $r\,e^{j\theta}$

Find

$$\angle(jc) - \angle(c)$$

which can also be written as

$$\arg(jc) - \arg(c)$$

Representations of Complex Numbers

Find $\angle(jc) - \angle(c)$.

Rectangular coordinates:

$$\angle(c) = \angle(a + jb) = \mathrm{atan2}(b, a)$$

$$\angle(jc) = \angle(ja - b) = \mathrm{atan2}(a, -b)$$

$$\rightarrow \angle(jc) - \angle(c) = \mathrm{atan2}(a, -b) - \mathrm{atan2}(b, a)$$

If you are better at trig than I am, ...

$$\mathrm{atan2}(y_1, x_1) \pm \mathrm{atan2}(\mathrm{y}_2, \mathrm{x}_2) = \mathrm{atan2}(\mathrm{y}_1\mathrm{x}_2 \pm \mathrm{y}_2\mathrm{x}_1, \mathrm{x}_1\mathrm{x}_2 \mp \mathrm{y}_1\mathrm{y}_2)$$

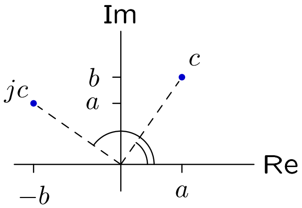
$$\mathrm{atan2}(a, -b) - \mathrm{atan2}(b, a) = \mathrm{atan2}(a^2 + b^2, -ab + ba) = \mathrm{atan2}(a^2 + b^2, 0) = \frac{\pi}{2}$$

Representations of Complex Numbers

Find $\angle(jc) - \angle(c)$.

Graphically:

$$c = a + jb$$
$$jc = ja - b$$



From the plot, we see that jc is a $\frac{\pi}{2}$ rotation of c.

Therefore $\angle(jc) - \angle(c) = \frac{\pi}{2}$.

Representations of Complex Numbers

Find $\angle(jc) - \angle(c)$.

Using Euler's equation:

$$c = re^{j\theta}$$

$$jc = jre^{j\theta} = e^{j\frac{\pi}{2}}e^{j\theta} = e^{j(\theta+\frac{\pi}{2})}$$

Therefore $\angle(jc) - \angle(c) = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}$.

The point of this question is that some operations on complex numbers are easy to think about in Cartesian coordinates, while others are easy to think about in polar coordinates (or equivalently with Euler's Formula).

Complex Numbers

How many of the following are true?

- $\dfrac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
- $(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta)$
- $\left|2 + j2 + e^{\frac{j\pi}{4}}\right| = |2 + j2| + \left|e^{\frac{j\pi}{4}}\right|$
- $\operatorname{Im}\left(j^j\right) > \operatorname{Re}\left(j^j\right)$
- $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{3}\right) = \tan^{-1} 1$

Complex Numbers

$$\frac{1}{\cos\theta + j\sin\theta} \stackrel{?}{=} \cos\theta - j\sin\theta$$

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\frac{1}{\cos\theta + j\sin\theta} = \frac{1}{e^{j\theta}} = e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$$

$$\frac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta \qquad \surd$$

Complex Numbers

$$(\cos\theta + j\sin\theta)^n \stackrel{?}{=} \cos(n\theta) + j\sin(n\theta)$$

$$(\cos\theta + j\sin\theta)^n = (e^{j\theta})^n = e^{jn\theta} = \cos(n\theta) + j\sin(n\theta)$$

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta) \qquad \surd$$

Complex Numbers

$$|2+j2+e^{\frac{j\pi}{4}}| \stackrel{?}{=} |2+j2|+|e^{\frac{j\pi}{4}}|$$

$$\begin{aligned}
|2+j2+e^{\frac{j\pi}{4}}| &= |2\sqrt{2}\,e^{\frac{j\pi}{4}} + e^{\frac{j\pi}{4}}| \\
&= |(2\sqrt{2}+1)e^{\frac{j\pi}{4}}| \\
&= |(2\sqrt{2}+1)||e^{\frac{j\pi}{4}}| \\
&= 2\sqrt{2}+1 \\
|2+j2|+|e^{\frac{j\pi}{4}}| &= 2\sqrt{2}+1
\end{aligned}$$

$$|2+j2+e^{\frac{j\pi}{4}}| = |2+j2|+|e^{\frac{j\pi}{4}}| \qquad \checkmark$$

This is only true because the angles of $2+j2$ and $e^{\frac{j\pi}{4}}$ are equal!
$|a+b|$ is NOT generally equal to $|a|+|b|$.

Complex Numbers

$\mathrm{Im}\left(j^j\right) \overset{?}{>} \mathrm{Re}\left(j^j\right)$

$j^j = \left(e^{j\pi/2}\right)^j = e^{-\pi/2}$ which is real and > 0.

Therefore Im$\left(j^j\right) = 0$ and is always less than the real part.

Caveat: There are other ways to express j.

$$j^j = (e^{j2\pi(n+\frac{1}{4})})^j = e^{-2\pi(n+\frac{1}{4})}$$

All of these alternatives lead to real numbers that are > 0.
Therefore the original premise is always false.

$\mathrm{Im}\left(j^j\right) > \mathrm{Re}\left(j^j\right)$ ✗

Notice that j^j is multi-valued, much like the n^{th} root of 1.

Complex Numbers

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right) \stackrel{?}{=} \tan^{-1}1$$

Let $c_1 = 2+j$ and $c_2 = 3+j$ so that $c_3 = (2+j)(3+j) = 5+5j$.
The angle of a product is the sum of the angles of the constituents:

$$\angle c_1 + \angle c_2 = \angle c_3$$

This proves the premise.
More generally,
c_1 could be any complex number whose angle is $\tan^{-1}\left(\frac{1}{2}\right)$,
c_2 could be any complex number whose angle is $\tan^{-1}\left(\frac{1}{3}\right)$,
and the product c_1c_2 would have angle $\tan^{-1}(1)$,

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}1 \qquad \surd$$

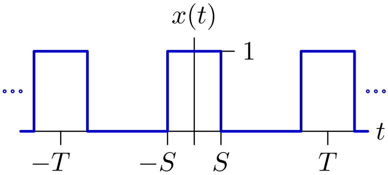
Complex Numbers

How many of the following are true?

- $$\frac{1}{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta \qquad \surd$$
- $$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta \qquad \surd$$
- $$\left|2 + j2 + e^{\frac{j\pi}{4}}\right| = |2 + j2| + \left|e^{\frac{j\pi}{4}}\right| \qquad \surd$$
- $$\mathrm{Im}\left(j^j\right) > \mathrm{Re}\left(j^j\right) \qquad \times$$
- $$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1} 1 \qquad \surd$$

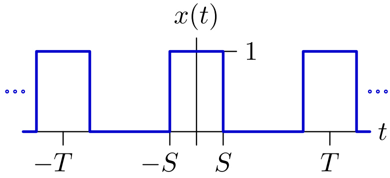
Pulse Train

Find the Fourier series coefficients a_k for $x(t)$:



Pulse Train

Find the Fourier series coefficients a_k for $x(t)$:



$$a_k = \frac{1}{T}\int_T x(t)\, e^{-j\frac{2\pi k}{T}t} dt$$

$$= \frac{1}{T}\int_{-S}^{S} e^{-j\frac{2\pi k}{T}t} dt = \frac{1}{T}\frac{e^{-j\frac{2\pi kS}{T}} - e^{j\frac{2\pi kS}{T}}}{-j\frac{2\pi k}{T}} = \frac{\sin(2\pi kS/T)}{\pi k}$$

Notice that a_k is real-valued:

$$\mathrm{Im}\,(a_k) = 0$$

and a_k is a symmetric function of k:

$$a_{-k} = a_k$$

Properties of Fourier Series

If $x(t)$ is real-valued, symmetric function of t then a_k is a real-valued, symmetric function of k.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi k}{T}t}dt$$

Choose symmetric region of integration and expand the exponential.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\Big(\cos(2\pi kt/T) - j\sin(2\pi kt/T)\Big)dt$$

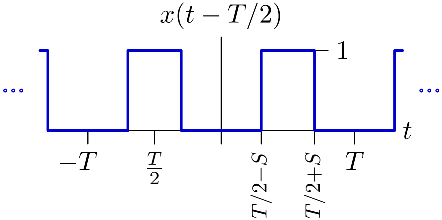
If $x(t)$ is real and symmetric, then the imaginary part integrates to zero.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cos(2\pi kt/T)\,dt$$

The result is a real-valued and symmetric function of k.

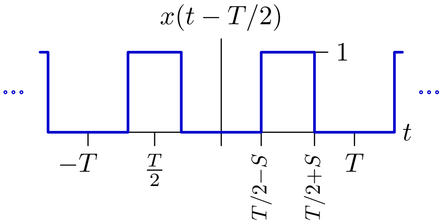
Pulse Train

What would happen to Fourier series if you delayed $x(t)$ by $T/2$?



Pulse Train

What would happen to Fourier series if you delayed $x(t)$ by $T/2$?



$$a'_k = \frac{1}{T}\int_{\frac{T}{2}-S}^{\frac{T}{2}+S} e^{-j\frac{2\pi k}{T}t}dt$$

$$= \frac{1}{T}\,\frac{e^{-j\frac{2\pi k(T/2+S)}{T}} - e^{-j\frac{2\pi k(T/2-S)}{T}}}{-j\frac{2\pi k}{T}} = e^{-j\pi k}\left(\frac{\sin\left(2\pi kS/T\right)}{\pi k}\right)$$

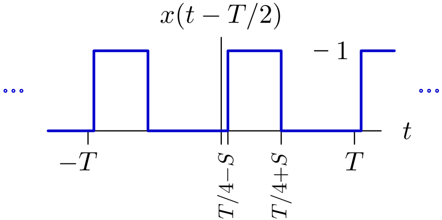
Delay by $T/2$ changes the phase but not the magnitude.

$$x(t) \quad \overset{\text{CTFS}}{\Longleftrightarrow} \quad a_k$$

$$x(t-T/2) \quad \overset{\text{CTFS}}{\Longleftrightarrow} \quad e^{-j\pi k}\,a_k$$

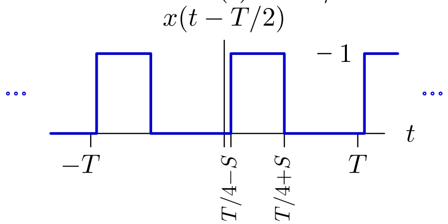
Pulse Train

What would happen if you delayed $x(t)$ by $T/4$?



Pulse Train

What would happen if you delayed $x(t)$ by $T/4$?



$$a'_k = \frac{1}{T}\int_{\frac{T}{4}-S}^{\frac{T}{4}+S} e^{-j\frac{2\pi k}{T}t}dt$$

$$= \frac{1}{T}\frac{e^{-j\frac{2\pi k(T/4+S)}{T}} - e^{-j\frac{2\pi k(T/4-S)}{T}}}{-j\frac{2\pi k}{T}} = e^{-j\pi k/2}\left(\frac{\sin(2\pi kS/T)}{\pi k}\right)$$

Delay by $T/4$ changes the phase but not the magnitude.

$$x(t) \overset{\text{CTFS}}{\Longleftrightarrow} a_k$$

$$x(t-T/4) \overset{\text{CTFS}}{\Longleftrightarrow} e^{-j\pi k/2}\,a_k$$

Delay Property of Fourier Series

Delays in time change only the phase of the Fourier series.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi k}{T}t}dt$$

$$a_k' = \frac{1}{T}\int_T x(t-t_0)e^{-j\frac{2\pi k}{T}t}dt$$

Let $\tau = t - t_0$.

$$a_k' = \frac{1}{T}\int_T x(\tau)e^{-j\frac{2\pi k}{T}(\tau+t_0)}\,d\tau$$

$$= e^{-j\frac{2\pi k}{T}t_0}\left(\frac{1}{T}\int_T x(\tau)e^{-j\frac{2\pi k}{T}\tau}d\tau\right) = e^{-j\frac{2\pi k}{T}t_0}\,a_k$$

$$x(t) \quad \overset{\text{CTFS}}{\Longleftrightarrow} \quad a_k$$

$$x(t-t_0) \quad \overset{\text{CTFS}}{\Longleftrightarrow} \quad e^{-j\frac{2\pi k}{T}t_0}\,a_k$$

Delay Property of Fourier Series

Complex exponential form simplifies expression of delay property.

delay	complex exponential form	trig form
$T/2$	mult by $e^{-j\pi k}$	$c'_k = (-1)^k c_k$ $d'_k = (-1)^k d_k$
$T/4$	mult by $e^{-j\pi k/2}$	complicated
t_0	mult by $e^{-j\frac{2\pi k}{T}t_0}$	very complicated

Parseval's Theorem

Determine an expression for

$$\int_T (f(t))^2 dt$$

in terms of the Fourier series coefficients a_k of $f(t)$.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Parseval's Theorem

Determine an expression for

$$\int_T (f(t))^2 dt$$

in terms of the Fourier series coefficients a_k of $f(t)$.

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$\int_T (f(t))^2 dt = \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}\right)\left(\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_o t}\right) dt$$

If $a_k a_l e^{j(k+l)\omega_o t}$ is absolutely summable and absolutely integrable, then we can swap the order of summation and integration.

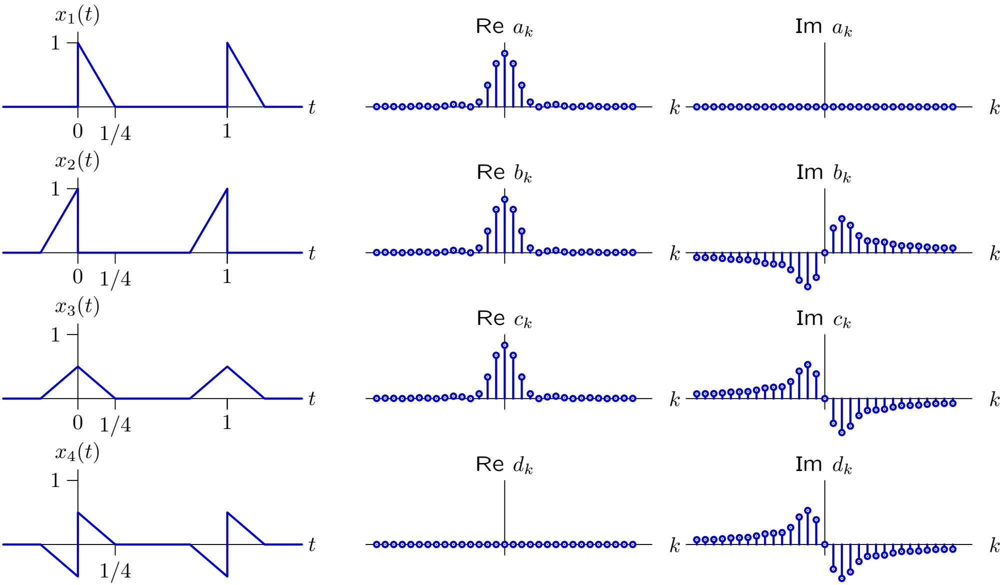
$$\int_T (f(t))^2 dt = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \int_T a_k a_l e^{j(k+l)\omega_o t} dt$$

By orthogonality, all of the exponentials integrate to zero except if $k+l=0$.

$$\int_T (f(t))^2 dt = \sum_{k=-\infty}^{\infty} T a_k a_{-k}$$

Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).



Fourier Series Matching

Match the signals (left column) to Fourier series coefficients (right).

$x_3(t)$ is a real-valued, symmetric function of time.
Therefore, its Fourier series coefficients form a real-valued, symmetric function of k.
– $x_3(t) \rightarrow a_k$

$x_4(t)$ is a real-valued, antisymmetric function of time.
Therefore, its Fourier series coefficients form a purely imaginary, antisymmetric function of k.
– $x_4(t) \rightarrow d_k$

$x_1(t) = x_3(t) + x_4(t)$, therefore its Fourier series coefficients are $a_k + d_k$.
– $x_1(t) \rightarrow c_k$

$x_2(t) = x_3(t) - x_4(t)$, therefore its Fourier series coefficients are $a_k - d_k$.
– $x_2(t) \rightarrow b_k$