

6.003: Signal Processing

Sinusoids and Fourier Series

September 14, 2021

Fourier Series (Trigonometric Form)

If $f(t)$ is expressed as a Fourier series

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$

the Fourier coefficients are given by

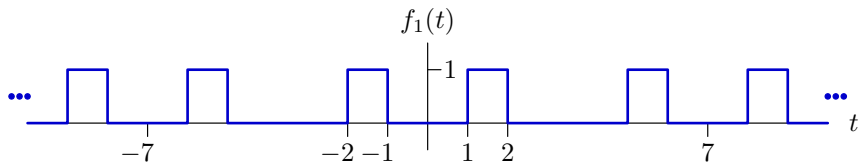
$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

Two Pulses

Let $f_1(t)$ represent the following function, which is periodic in $T = 7$:



Determine a Fourier series of the following form for $f_1(t)$.

$$f_1(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$

Two Pulses

The average value is $c_0 = \frac{1}{T} \int_T f_1(t) dt$.

There are many equivalent ways to integrate over a period:

$$\int_T dt = \int_0^7 dt = \int_{-7/2}^{7/2} dt = \dots$$

All of these are easy. The result is $c_0 = \frac{2}{7}$.

Two Pulses

For $k \geq 1$:

$$c_k = \frac{2}{T} \int_T f_1(t) \cos(k\omega_o t) dt$$

Integrating over symmetric limits simplifies the math (slightly):

$$\begin{aligned} c_k &= \frac{2}{7} \int_{-2}^2 \cos(k\omega_o t) dt - \frac{2}{7} \int_{-1}^1 \cos(k\omega_o t) dt \\ &= \frac{2}{7} \left. \frac{\sin(k\omega_o t)}{k\omega_o} \right|_{-2}^2 - \frac{2}{7} \left. \frac{\sin(k\omega_o t)}{k\omega_o} \right|_{-1}^1 \\ &= \frac{2}{\pi k} \left(\sin \frac{4k\pi}{7} - \sin \frac{2k\pi}{7} \right) \end{aligned}$$

Demonstrate other ways to do this integration, e.g.,

$$\int_{-2}^1 dt + \int_1^2 dt \quad \text{or} \quad \int_1^2 dt + \int_6^7 dt \quad \text{or} \quad 2 \int_1^2 dt$$

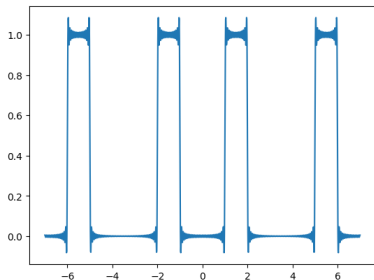
Since $f_1(t)$ is symmetric, and harmonics of the sine function are antisymmetric, the d_k coefficients are all zero.

$$d_k = \frac{2}{T} \int_T f_1(t) \sin(k\omega_o t) dt = 0$$

Checking with Python

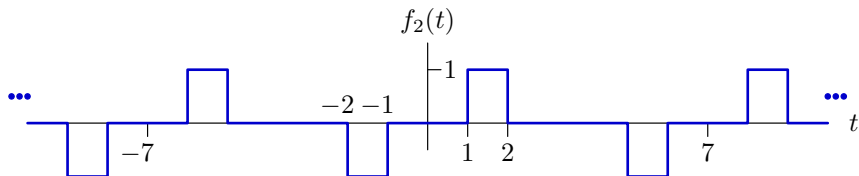
Sum the first 100 elements of series:

```
from math import sin, pi
from matplotlib.pyplot import plot, show
x = []
y = []
omega0 = 2*pi/7
t = -7
while t<7:
    x.append(t)
    y.append(2/7+sum([2*(sin(4*pi*k/7)-sin(2*pi*k/7))/pi/k*cos(k*omega0*t)
                    for k in range(1,100)]))
    t += 0.01
plot(x,y)
show()
```



Opposite Pulses

Let $f_2(t)$ represent the following function, which is periodic in $T = 7$:



Find ω_o and the Fourier series coefficients c_k and d_k so that

$$f_2(t) = \sum_{k=0}^{\infty} \left(c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \right)$$

Opposite Pulses

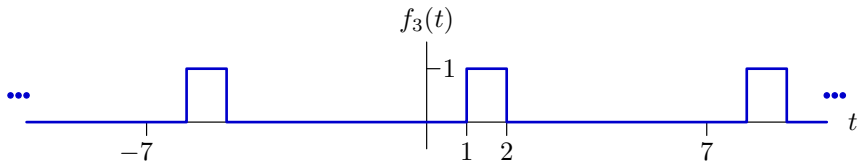
The previous problem was symmetric about $t = 0$ so there were only cosine terms.

This problem is antisymmetric about $t = 0$ so there are only sine terms.

Do the calculation using several different regions of integration.

Single Pulse

Let $f_3(t)$ represent the following function, which is periodic in $T = 7$:



Determine the Fourier series coefficients for $f_3(t)$.

Discuss the relation(s) among the Fourier series coefficients of $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Discuss the relation(s) among $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Check With Python

It's interesting that overshoots move around. They are "glued" to the discontinuities.

Trig Table

$$\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

$$\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b)$$

$$\tan(a+b) = (\tan(a)+\tan(b))/(1-\tan(a) \tan(b))$$

$$\tan(a-b) = (\tan(a)-\tan(b))/(1+\tan(a) \tan(b))$$

$$\sin(A) + \sin(B) = 2 \sin((A+B)/2) \cos((A-B)/2)$$

$$\sin(A) - \sin(B) = 2 \cos((A+B)/2) \sin((A-B)/2)$$

$$\cos(A) + \cos(B) = 2 \cos((A+B)/2) \cos((A-B)/2)$$

$$\cos(A) - \cos(B) = -2 \sin((A+B)/2) \sin((A-B)/2)$$

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

$$\sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b)$$

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b)$$

$$2 \cos(A) \cos(B) = \cos(A-B) + \cos(A+B)$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B)$$

$$2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B)$$

$$2 \cos(A) \sin(B) = \sin(A+B) - \sin(A-B)$$