

6.003: Signal Processing

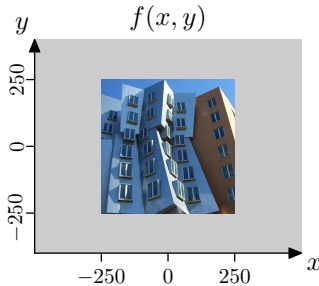
Fourier Series (Trigonometric Form)

Representing Signals as Fourier Series

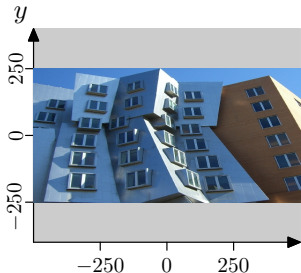
- Synthesis: making a signal from components
- Analysis: finding the components

September 9, 2021

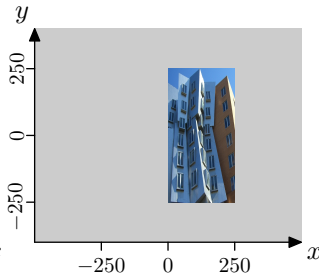
Start With Some Basic Transformations



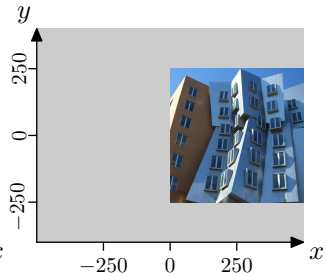
How many images match the expressions beneath them?



$$f_1(x, y) = f(2x, y) ?$$

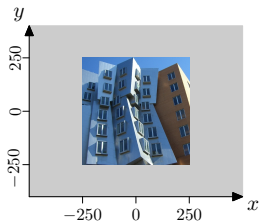


$$f_2(x, y) = f(2x - 250, y) ?$$

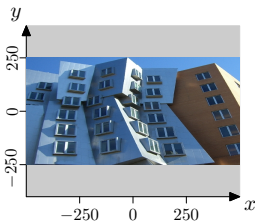


$$f_3(x, y) = f(-x - 250, y) ?$$

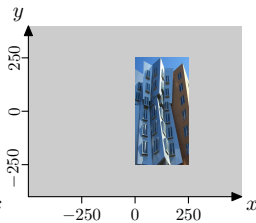
Start With Some Basic Transformations



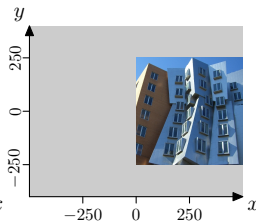
$f(x, y)$



$f_1(x, y) = f(2x, y) ?$



$f_2(x, y) = f(2x - 250, y) ?$



$f_3(x, y) = f(-x - 250, y) ?$

$x = 0 \rightarrow f_1(0, y) = f(0, y)$ ✓

$x = 250 \rightarrow f_1(250, y) = f(500, y)$ ✗

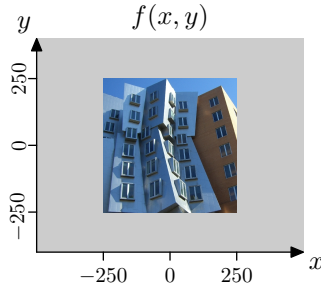
$x = 0 \rightarrow f_2(0, y) = f(-250, y)$ ✓

$x = 250 \rightarrow f_2(250, y) = f(250, y)$ ✓

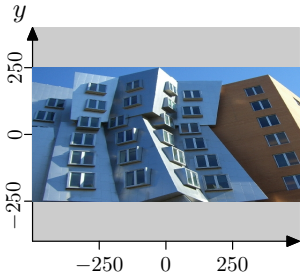
$x = 0 \rightarrow f_3(0, y) = f(-250, y)$ ✗

$x = 250 \rightarrow f_3(250, y) = f(-500, y)$ ✗

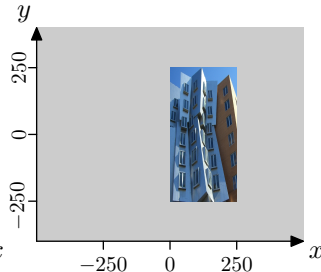
Start With Some Basic Transformations



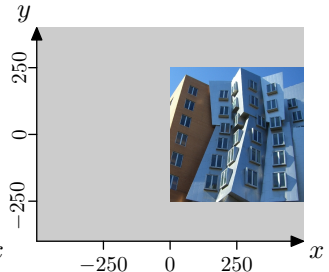
How many images match the expressions beneath them? 1



~~$f_1(x, y) = f(2x, y) ?$~~



$f_2(x, y) = f(2x - 250, y) ?$



~~$f_3(x, y) = f(x - 250, y) ?$~~

Fourier Series

Fourier representations are a major theme of this subject.

The basic ideas were described in lecture:

Synthesis Equation (making a signal from components):

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

Analysis Equations (finding the components):

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi kt}{T}\right) dt ; \quad k \geq 1$$

$$d_k = \frac{2}{T} \int_T f(t) \sin\left(\frac{2\pi kt}{T}\right) dt ; \quad k \geq 1$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

Warm Up

Find the Fourier series coefficients (c_k and d_k) for

$$f(t) = \cos(t)$$

We can find c_k and d_k directly from the synthesis equation:

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} d_k \sin\left(\frac{2\pi kt}{T}\right)$$

The function $f(t)$ is periodic in time with period

$$T = 2\pi.$$

The coefficients can be found by matching the expression on the left with that on the right:

$$c_k = \begin{cases} 1 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$d_k = 0$$

There is a single non-zero Fourier coefficient: $c_1 = 1$.

Warm Up

Alternatively, we can calculate c_k and d_k from the analysis equations:

$$f(t) = \cos(t)$$

$$c_0 = \frac{1}{2\pi} \int_T f(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t) dt = \frac{1}{2\pi} \sin(t) \Big|_{-\pi}^{\pi} = 0$$

For $k > 0$:

$$c_k = \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi kt}{T}\right) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \cos(kt) dt$$

$$c_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2t)\right) dt = \frac{1}{\pi} \left(\frac{t}{2} + \frac{1}{4} \sin(2t)\right) \Big|_{-\pi}^{\pi} = 1$$

For $k > 1$:

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos((k+1)t) + \cos((k-1)t) \right) dt \\ &= \frac{1}{2\pi} \left[\frac{\sin((k+1)t)}{k+1} + \frac{\sin((k-1)t)}{k-1} \right]_{-\pi}^{\pi} = 0 \end{aligned}$$

$$d_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(t) \sin(kt) dt = 0 \quad (\text{integrand is anti-symmetric})$$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient?

- $f_1(t) = \cos^2 t$
- $f_2(t) = \sin t \cos t$
- $f_3(t) = 4 \cos^3 t - 3 \cos t$
- $f_4(t) = \cos(12t) \cos(4t) \cos(2t)$

Fourier Series Coefficients

How many of the following functions have **exactly one** non-zero Fourier series coefficient? **2: $f_2(t)$ and $f_3(t)$**

$$f_1(t) = \cos^2(t) = \frac{1}{2} + \frac{1}{2} \cos(2t)$$

→ 2 non-zero components: c_0 and c_1 . (could this also be c_0 and c_2 ?)

$$f_2(t) = \sin(t) \cos(t) = \frac{1}{2} \sin(2t) + \frac{1}{2} \sin(0) = \frac{1}{2} \sin(2t)$$

→ 1 non-zero component: d_1 .

$$\begin{aligned} f_3(t) &= 4 \cos^3(t) - 3 \cos(t) = \cos(t) (4 \cos^2(t) - 3) \\ &= \cos(t) (2 \cos(2t) - 1) = \cos(t) + \cos(3t) - \cos(t) = \cos(3t) \end{aligned}$$

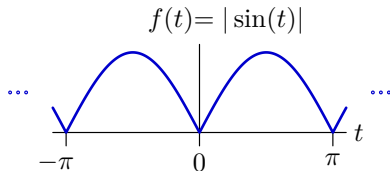
→ 1 non-zero component: c_1 .

$$\begin{aligned} f_4(t) &= \cos(12t) \cos(4t) \cos(2t) = \cos(12t) \left(\frac{1}{2} \cos(6t) + \frac{1}{2} \cos(2t) \right) \\ &= \frac{1}{4} \cos(18t) + \frac{1}{4} \cos(6t) + \frac{1}{4} \cos(14t) + \frac{1}{4} \cos(10t) \end{aligned}$$

→ 4 non-zero components: c_3 , c_5 , c_7 , and c_9 .

Rectified Sine Wave

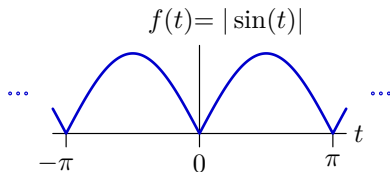
Consider a Fourier series representation of the following function.



- What is the approximate value of c_0 ?
- Which non-DC Fourier coefficient has the largest absolute value? What's the sign of that coefficient?
- Determine an expression for the Fourier coefficients of $f(t)$.
- Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Rectified Sine Wave

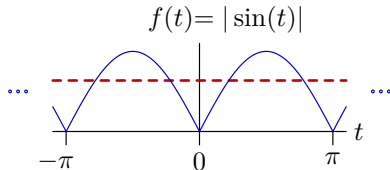
Consider a Fourier series representation of the following function.



Q: What is the approximate value of c_0 ?

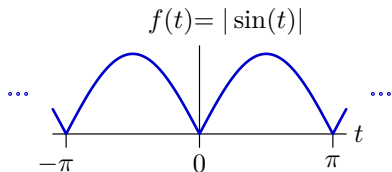
A: c_0 is the average value, which is clearly greater than $\frac{1}{2}$ but less than 1.

$$\text{More exactly, } c_0 = \frac{1}{T} \int_T f(t) dt = \frac{1}{\pi} \int_0^{\pi} \sin(t) dt = -\frac{\cos(t)}{\pi} \Big|_0^{\pi} = \frac{2}{\pi} \approx 0.64$$



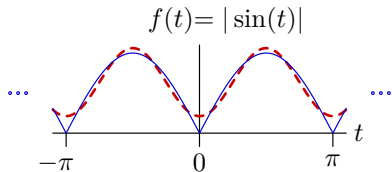
Rectified Sine Wave

Consider a Fourier series representation of the following function.



Q: Which non-DC Fourier coefficient has the largest absolute value?
What's the sign of that coefficient?

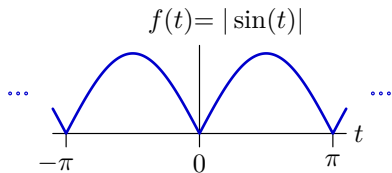
A: biggest deviations from mean at $t = 0$ and $t = \frac{\pi}{2} \rightarrow -\cos(t)$.



$$\begin{aligned}c_1 &= \frac{2}{T} \int_T f(t) \cos\left(\frac{2\pi t}{T}\right) dt = \frac{2}{\pi} \int_0^\pi \sin(t) \cos(2t) dt = \frac{1}{\pi} \int_0^\pi (\sin(3t) - \sin(t)) \\ &= \frac{1}{\pi} \left[-\frac{\cos(3t)}{3} + \cos(t) \right]_0^\pi = -\frac{4}{3\pi} \approx -0.42\end{aligned}$$

Rectified Sine Wave

Consider a Fourier series representation of the following function.



Determine an expression for the Fourier coefficients of $f(t)$.

The function is symmetric about $t = 0$, so $d_k = 0$ for all k .

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \sin(t) dt = -\frac{\cos(t)}{\pi} \Big|_0^{\pi} = \frac{2}{\pi}$$

$$\begin{aligned} c_k &= \frac{2}{\pi} \int_0^{\pi} \sin(t) \cos(2kt) dt \quad ; k \geq 1 \\ &= \frac{1}{\pi} \int_0^{\pi} \left(\sin((2k+1)t) - \sin((2k-1)t) \right) dt \\ &= \frac{1}{\pi} \left[-\frac{\cos((2k+1)t)}{2k+1} + \frac{\cos((2k-1)t)}{2k-1} \right]_0^{\pi} = \frac{-4/\pi}{4k^2 - 1} \end{aligned}$$

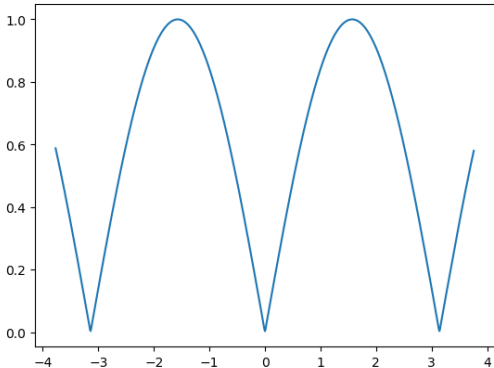
Verify Fourier Series of Rectified Sine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

Verify Fourier Series of Rectified Sine Wave Numerically

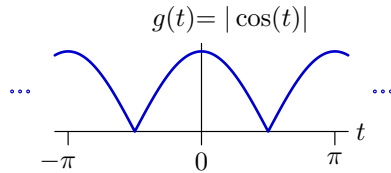
Compute the sum of the first 100 terms in the Fourier series of $f(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
ff = []
tt = []
t = -1.2*pi
while t<1.2*pi:
    ff.append(2/pi+sum([-4/pi/(4*k*k-1)*cos(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt,ff)
show()
```



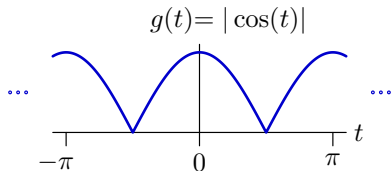
Rectified Cosine Wave

Determine the Fourier series representation of a rectified cosine.



Rectified Cosine Wave

Determine the Fourier series representation of a rectified cosine.



We could repeat the previous process, or just shift the time axis of $f(t)$:

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cos(2kt)$$

$$\begin{aligned} g(t) &= f\left(t - \frac{\pi}{2}\right) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \cos\left(2k\left(t - \frac{\pi}{2}\right)\right) \\ &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} (\cos(2kt) \cos(\pi k) + \sin(2kt) \sin(\pi k)) \\ &= \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2-1} \cos(2kt) \end{aligned}$$

The new Fourier coefficients $c'_k = (-1)^k c_k$.

Shifting half a period flips the sign of the odd numbered coefficients! Why?

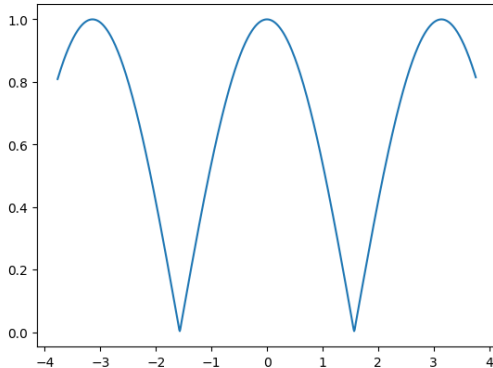
Verify Fourier Series of Rectified Cosine Wave Numerically

Compute the sum of the first 100 terms in the Fourier series of $g(t)$.

Verify Fourier Series of Rectified Cosine Wave Numerically

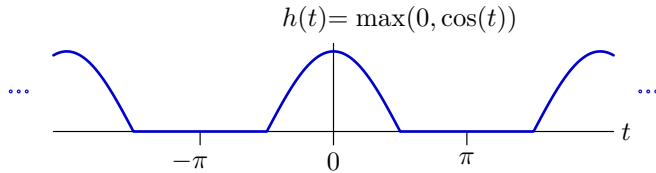
Compute the sum of the first 100 terms in the Fourier series of $g(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
gg = []
tt = []
t = -1.2*pi
while t<1.2*pi:
    gg.append(2/pi-4/pi*sum([(-1)**k/(4*k*k-1)*cos(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt,gg)
show()
```



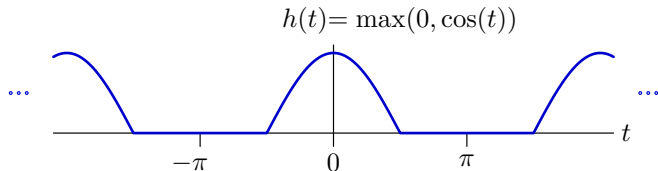
Half-Wave Rectified Cosine

Determine the Fourier series representation of a half-wave rectified cosine.



Half-Wave Rectified Cosine

Determine the Fourier series representation of a half-wave rectified cosine.



We could apply the original method, or realize that $h(t) = \frac{g(t) + \cos(t)}{2}$.

$$g(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1} \cos(2kt)$$

$$h(t) = \frac{1}{2}g(t) + \frac{1}{2}\cos(t) = \frac{1}{\pi} + \frac{1}{2}\cos(t) - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^2 - 1} \cos(2kt)$$

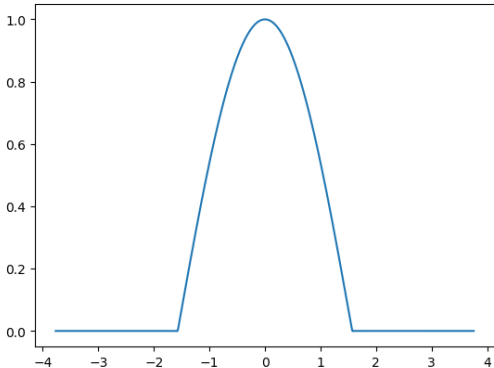
Note: the period has changed! Harmonics of $h(t)$ are stretched to larger k .

k :	0	1	2	3	4	5	6	7	8	...
c_k^J :	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$-\frac{4}{15\pi}$	$\frac{4}{35\pi}$	$-\frac{4}{63\pi}$	$\frac{4}{99\pi}$	$-\frac{4}{143\pi}$	$\frac{4}{195\pi}$	$-\frac{4}{255\pi}$...
$c_k^{J'}$:	$\frac{1}{\pi}$	$\frac{1}{2}$	$\frac{2}{3\pi}$	0	$-\frac{2}{15\pi}$	0	$\frac{2}{35\pi}$	0	$-\frac{2}{63\pi}$...

Verify Fourier Series of Half-Wave Rectified Cosine

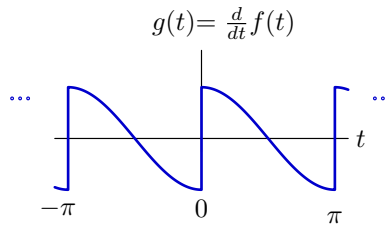
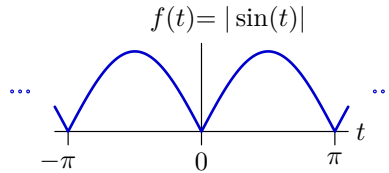
Compute the sum of the first 100 terms in the Fourier series of $h(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
hh = []
tt = []
t = -1.2*pi
while t<1.2*pi:
    hh.append(1/pi+cos(t)/2-2/pi*sum([(-1)**k/(4*k*k-1)*cos(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt, hh)
show()
```



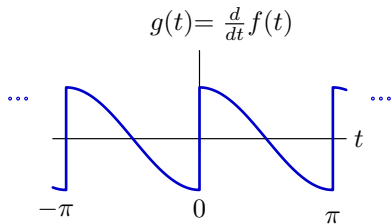
Derivative of Rectified Sine

Similarly analyze the derivative of the rectified sine with respect to time.



Derivative of Rectified Sine

Similarly analyze the derivative of the rectified sine with respect to time.



Method 1: substitute $g(t)$ for $f(t)$ in the analysis equations and integrate.

Method 2: differentiate the Fourier series of $f(t)$ term by term (easier).

$$f(t) = \frac{2}{\pi} + \sum_{k=1}^{\infty} -\frac{4/\pi}{4k^2 - 1} \cos(2kt)$$

$$g(t) = \frac{d}{dt} f(t) = \sum_{k=1}^{\infty} \frac{4/\pi}{4k^2 - 1} (2k) \sin(2kt)$$

The Fourier series of $g(t)$ has only sine terms: $d_k = \frac{8k/\pi}{4k^2 - 1}$.

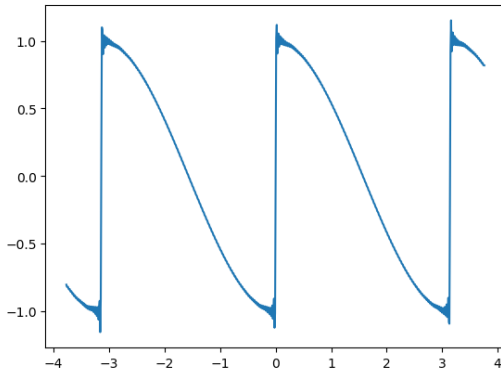
Verify Derivative of Rectified Sine Numerically

Compute the sum of the first 100 terms in the Fourier series of $g(t)$.

Derivative of Rectified Sine

Compute the sum of the first 100 terms in the Fourier series of $g(t)$.

```
from math import cos, pi
from matplotlib.pyplot import plot, show
gg = []
tt = []
t = -1.2*pi
while t < 1.2*pi:
    gg.append(sum([4/pi/(4*k*k-1)*2*k*sin(2*k*t) for k in range(1,100)]))
    tt.append(t)
    t += 0.01
plot(tt,gg)
show()
```



We will discuss the overshoots (Gibb's phenomenon) on Thursday.

Trig Table

$$\sin(a+b) = \sin(a) \cos(b) + \cos(a) \sin(b)$$

$$\sin(a-b) = \sin(a) \cos(b) - \cos(a) \sin(b)$$

$$\cos(a+b) = \cos(a) \cos(b) - \sin(a) \sin(b)$$

$$\cos(a-b) = \cos(a) \cos(b) + \sin(a) \sin(b)$$

$$\tan(a+b) = (\tan(a)+\tan(b))/(1-\tan(a) \tan(b))$$

$$\tan(a-b) = (\tan(a)-\tan(b))/(1+\tan(a) \tan(b))$$

$$\sin(A) + \sin(B) = 2 \sin((A+B)/2) \cos((A-B)/2)$$

$$\sin(A) - \sin(B) = 2 \cos((A+B)/2) \sin((A-B)/2)$$

$$\cos(A) + \cos(B) = 2 \cos((A+B)/2) \cos((A-B)/2)$$

$$\cos(A) - \cos(B) = -2 \sin((A+B)/2) \sin((A-B)/2)$$

$$\sin(a+b) + \sin(a-b) = 2 \sin(a) \cos(b)$$

$$\sin(a+b) - \sin(a-b) = 2 \cos(a) \sin(b)$$

$$\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$$

$$\cos(a+b) - \cos(a-b) = -2 \sin(a) \sin(b)$$

$$2 \cos(A) \cos(B) = \cos(A-B) + \cos(A+B)$$

$$2 \sin(A) \sin(B) = \cos(A-B) - \cos(A+B)$$

$$2 \sin(A) \cos(B) = \sin(A+B) + \sin(A-B)$$

$$2 \cos(A) \sin(B) = \sin(A+B) - \sin(A-B)$$