6.003 Quiz 2 Spring 2020

Name: Answers

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one 8.5×11 sheet of paper (two sides).
You may NOT use any electronic devices (including calculators, phones, etc).

If you have questions, please **come to us at the front** to ask them.

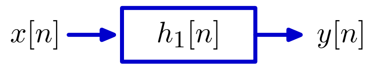
Please enter all solutions in the boxes provided.

Extra work may be taken into account when assigning partial credit,
but only work on pages with QR codes will be considered.

Question 1:	20 Points
Question 2:	24 Points
Question 3:	24 Points
Question 4:	32 Points
Total:	100 Points

1 Systems

Part 1. Let $h_1[n]$ represent the unit-sample response of a discrete-time, linear, time-invariant system.



If the input signal is

$$x[n] = \delta[n] + \delta[n-1]$$

then the output signal is

$$y[n] = \delta[n] + 2\delta[n-1] + 2\delta[n-2] + \delta[n-3]\,.$$

Find the unit-sample response $h_1[n]$, assuming that $h_1[n] = 0$ for $n < 0$.

Enter the first 5 samples of your result as a python list

$$[\ \ h_1[0],\ \ h_1[1],\ \ h_1[2],\ \ h_1[3],\ \ h_1[4]\ \ ]$$

in the box below.

```
[ 1, 1, 1, 0, 0 ]
```

One approach is to work in the time domain, using convolution.

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[0]h[n] + x[1]h[n-1] = h[n] + h[n-1]$$

Therefore

$$\begin{aligned}
y[0] &= 1 = h_1[0] + h_1[-1] &&\rightarrow\ h_1[0] = 1\\
y[1] &= 2 = h_1[1] + h_1[0] &&\rightarrow\ h_1[1] = 1\\
y[2] &= 2 = h_1[2] + h_1[1] &&\rightarrow\ h_1[2] = 1\\
y[3] &= 1 = h_1[3] + h_1[2] &&\rightarrow\ h_1[3] = 0
\end{aligned}$$

and

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]\,.$$

Another approach is to work in the frequency domain. Let

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = 1 + e^{-j\Omega n}$$

and

$$y(\Omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = 1 + 2e^{-j\Omega n} + 2e^{-j\Omega 2n} + e^{-j\Omega 3n}\,.$$

Then

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 + 2e^{-j\Omega n} + 2e^{-j\Omega 2n} + e^{-j\Omega 3n}}{1 + e^{-j\Omega n}}\,.$$

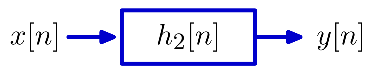
We can evaluate this ratio using long division. This leads to

$$H(\Omega) = 1 + e^{-j\Omega n} + e^{j\Omega 2n}$$

and therefore

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2]\,.$$

Part 2. Let $h_2[n]$ represent the unit-sample response of a discrete-time, linear, time-invariant system.



The input signal $x[n]$ and output signal $y[n]$ are related by the following difference equation for all n.

$$y[n+1] = \frac{1}{2}\big(x[n] + y[n]\big)$$

Part 2a. Determine the unit-sample response $h_2[n]$ of this system, assuming that $h_2[n] = 0$ for $n < 0$.

Enter the first 5 samples of your result as a python list

$[\ h_2[0],\ \ h_2[1],\ \ h_2[2],\ \ h_2[3],\ \ h_2[4]\ ]$

in the box below.

```
[ 0, 1/2, 1/4, 1/8, 1/16 ]
```

Let $x[n] = \delta[n]$. Then $y[n] = h_2[n]$. Step through the difference equation for $n = 0$ then $n = 1$ and so forth.

$$\begin{aligned}
y[0] &= (x[-1] + y[-1])/2 = (0+0)/2 = 0 \\
y[1] &= (x[0] + y[0])/2 = (1+0)/2 = 1/2 \\
y[2] &= (x[1] + y[1])/2 = (0+1/2)/2 = 1/4 \\
y[3] &= (x[2] + y[2])/2 = (0+1/4)/2 = 1/8 \\
y[4] &= (x[3] + y[3])/2 = (0+1/8)/2 = 1/16
\end{aligned}$$

Part 2b. Determine the frequency response of this system.

Enter an expression for $H_2(\Omega)$ in the box below.

$$\frac{1}{2e^{j\Omega} - 1}$$

We can find a relation between $X(\Omega)$ and $Y(\Omega)$ by taking the Fourier transform of the difference equation.

$$e^{j\Omega}Y(\Omega) = (X(\Omega) + Y(\Omega))/2$$

The frequency response is then

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{2e^{j\Omega} - 1}$$

Part 3. Let

$$x[n] = \begin{cases} 1 & 0 \leqslant n \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$$

and let $X[k]$ represent the DFT of $x[n]$ with analysis period $N = 8$.

Part 3a. Find $X[0]$ and $X[1]$ and enter them (as numerical expressions) in the boxes below.

$$X[0] = \frac{5}{8}$$

$$X[1] = -\frac{j}{8}\left(1+\sqrt{2}\right)$$

$$X[k] = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-j\frac{2\pi k}{8}n} = \frac{1}{8}\sum_{n=0}^{4} e^{-j\frac{2\pi k}{8}n}$$

$$X[0] = \frac{5}{8}$$

$$X[1] = \frac{1}{8}\left(1 + e^{-j\frac{2\pi}{8}1} + e^{-j\frac{2\pi}{8}2} + e^{-j\frac{2\pi}{8}3} + e^{-j\frac{2\pi}{8}4}\right)$$

$$= \frac{1}{8}\left(1 + e^{-j\pi/4} + e^{-j2\pi/4} + e^{-j3\pi/4} + e^{-j4\pi/4}\right)$$

$$= \frac{1}{8}\left(1 + \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} - j - \frac{\sqrt{2}}{2} - j\frac{\sqrt{2}}{2} - 1\right) = -\frac{j}{8}\left(1+\sqrt{2}\right)$$

Part 3b. Let $Y[k] = X^2[k]$. Determine $y[n]$ which represents the inverse DFT of $Y[k]$. Enter your result as a python list of the form

```
[ y[0], y[1], y[2], ... ]
```

$$\left[\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{8},\ \frac{1}{2},\ \frac{5}{8},\ \frac{1}{2},\ \frac{3}{8},\ \frac{1}{4}\right]$$

$y[n]$ is the circular convolution of $x[n]$ with itself divided by $N = 8$.

The regular convolution of $x[n]$ with itself is

$$(x * x)[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8]$$

The circular convolution aliases time $n = 8$ back to $n = 0$.

$$(x \circledast x)[n] = 2\delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7]$$

Thus

$$y[n] = \frac{1}{8}\left(2\delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 2\delta[n-7]\right)$$

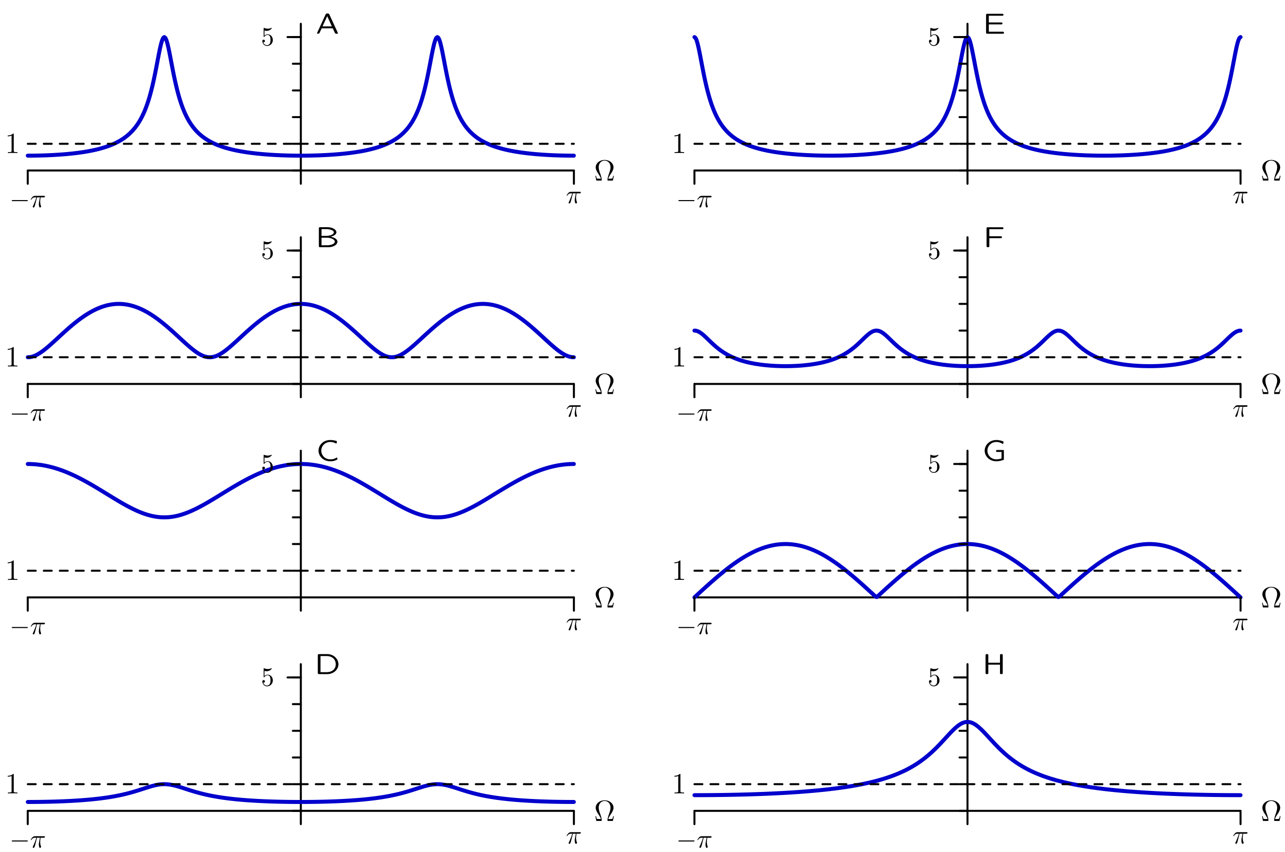
2 Peaks and Valleys

Each of the plots below shows the magnitude of the frequency response of a discrete-time system that can be described by the following difference equation, where α and m are parameters.

$$y[n] = x[n] + \alpha y[n-m]$$

Each row in the following table gives the parameters for one of the plots.
Write the letter of the corresponding plot in the right column of the table.

α	m	Frequency Response Enter A–H
0.7	1	H
−0.5	3	F
−0.8	2	A
−2	2	D
0.8	2	E



Worksheet (intentionally blank)

Find the frequency response from the difference equation.

$$Y(\Omega) = X(\Omega) + \alpha e^{-j\Omega m} Y(\Omega)$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \alpha e^{-j\Omega m}}$$

The magnitude of the frequency response is

$$\left|H(\Omega)\right| = \frac{1}{\left|1 - \alpha\cos(\Omega m) + j\alpha\sin(\Omega_m)\right|} = \frac{1}{\sqrt{(1 - \alpha\cos(\Omega m))^2 + (\alpha\sin(\Omega_m))^2}} = \frac{1}{\sqrt{1 - 2\alpha\cos(\Omega m) + \alpha^2}}$$

The maximum magnitude of H occurs when the magnitude of its denominator is at a minimum. The minimum magnitude of the denominator is $\left|1 - |\alpha|\right|$. Thus the maximum magnitude of $H(\Omega)$ is

$$max = \left|\frac{1}{1 - |\alpha|}\right|$$

The minimum magnitude of $H(\Omega)$ occurs when the magnitude of its denominator is at a maximum. The maximum magnitude of the denominator is $1 + |\alpha|$. Thus the minimum magnitude of H is

$$min = \frac{1}{1 + |\alpha|}$$

	α	max	min	matches
line 1	0.7	3.3	0.6	H
line 2	−0.5	2	0.7	F
line 3	−0.8	5	0.6	A or E
line 4	−2	1	0.3	D
line 5	0.8	5	0.6	A or E

To determine which of A or E belongs on line 3, we can compute the DC magnitudes.

For line 3, the DC magnitude is $\frac{1}{1-(-0.8)}$ which is less than 1.

For line 5, the DC magnitude is $\frac{1}{1-0.8}$ which is 5.

Thus E belongs on line 5 and A belongs on line 3.

3 Composite Systems

In this problem, we consider eight linear, time-invariant systems whose unit-sample responses are expressed in terms of $g_1[n]$ and $g_2[n]$ where

$$g_1[n] = \delta[n] - \delta[n-1] + \delta[n-2] \quad \text{and}$$

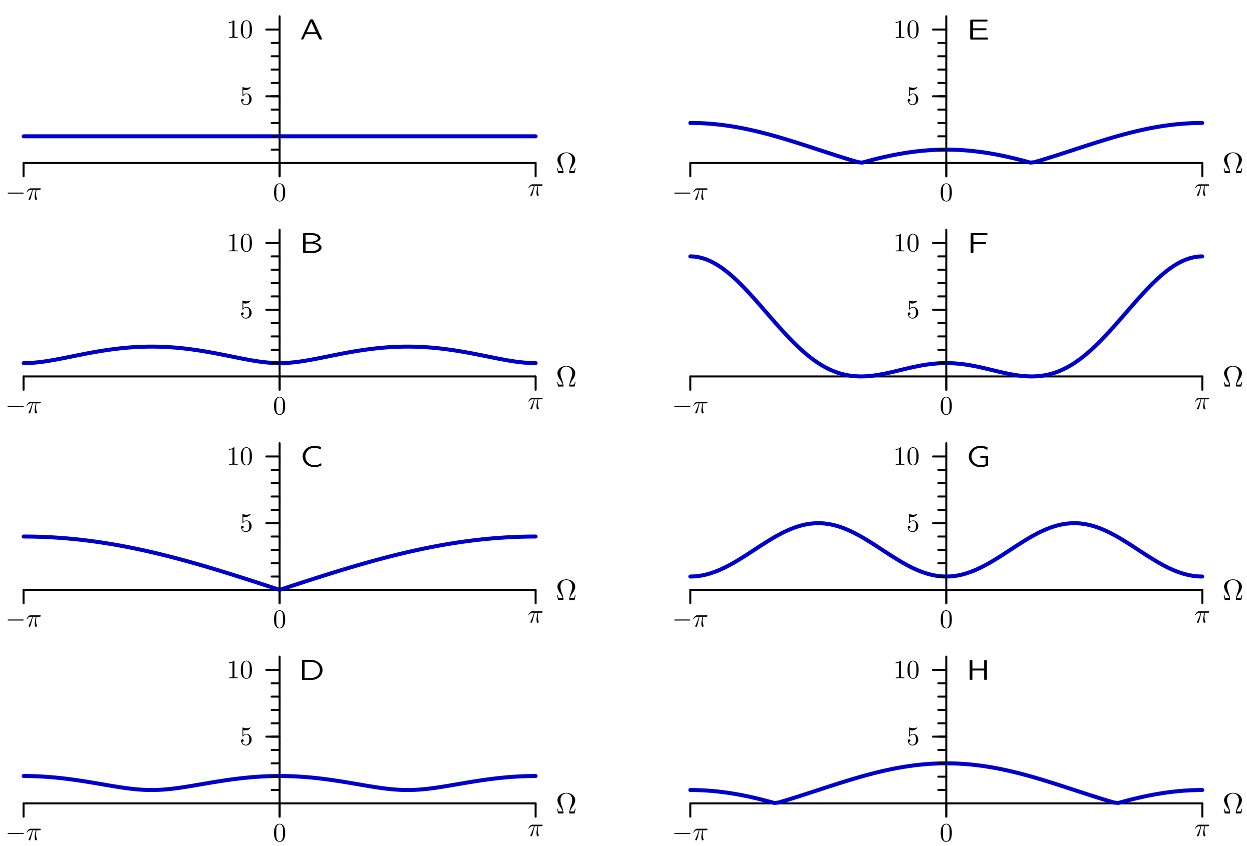
$$g_2[n] = \delta[n] + \delta[n-1] - \delta[n-2]\,.$$

The unit-sample response $h_i[n]$ of each system is given by the expression in the center column of the table below.

System	Unit-sample response	Frequency Response Enter A–H
1	$h_1[n] = g_1[n]$	E
2	$h_2[n] = g_2[n]$	B
3	$h_3[n] = g_1[n] + g_2[n]$	A
4	$h_4[n] = g_1[n] - g_2[n]$	C
5	$h_5[n] = g_1[n] \times g_1[n]$	H
6	$h_6[n] = g_1[n] \times g_2[n]$	B
7	$h_7[n] = (g1 * g1)[n]$	F
8	$h_8[n] = (g2 * g2)[n]$	G

The **magnitude** of the frequency response for each of systems 1–8 is shown by one of the plots A–H below.

Determine which applies and enter the appropriate letter (A–H) in the right column. Answers may be repeated.



Worksheet (intentionally blank)

Start by determining expressions for $h_1[n]$ through $h_8[n]$.

Determine the magnitudes at $\Omega = 0$ and π.

	sample at time n					$\lvert H(0)\rvert$	$\lvert H(\pi)\rvert$	match
	0	1	2	3	4			
$h_1[n]$	1	-1	1	0	0 1	3	E	
$h_2[n]$	1	1	-1	0	0 1	1	B, G	
$h_3[n]$	2	0	0	0	0 2	2	A	
$h_4[n]$	0	-2	2	0	0 0	4	C	
$h_5[n]$	1	1	1	0	0 3	1	H	
$h_6[n]$	1	-1	-1	0	0 1	1	B, G	
$h_7[n]$	1	-2	3	-2	1 1	9	F	
$h_8[n]$	1	2	-1	-2	1 1	1	B, G	

To distinguish B and G, evaluate $H(\pi/2)$.

$H_2(\pi/2) = 2 - j$ so $|H_2(\pi/2)| = \sqrt{5}$.

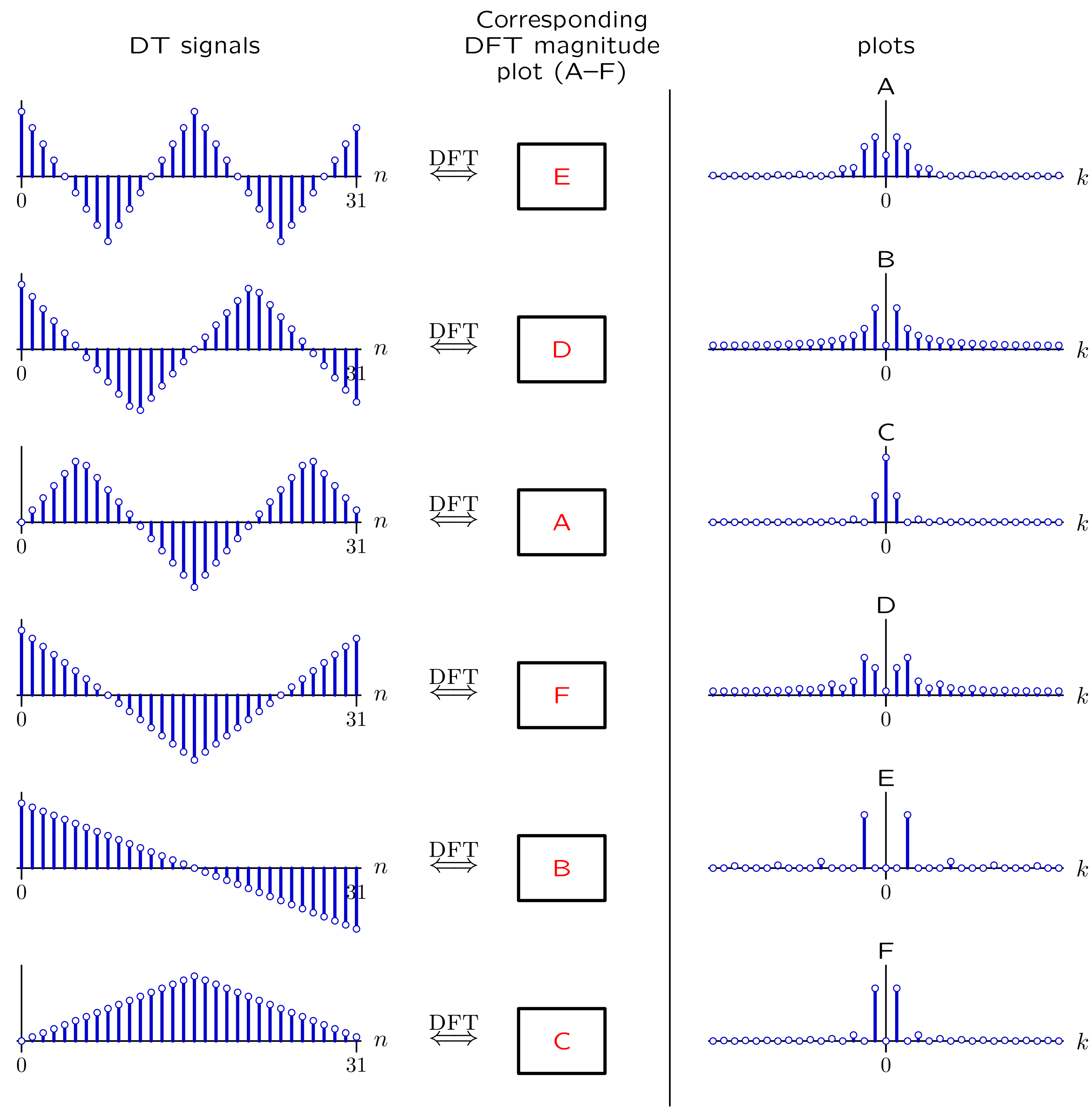
$H_6(\pi/2) = 2 + j$ so $|H_6(\pi/2)| = \sqrt{5}$.

$H_8(\pi/2) = 3 - 4j$ so $|H_8(\pi/2)| = 5$.

It follows that $H_2(\Omega)$ and $H_6(\Omega)$ correspond to B but $H_8(\Omega)$ corresponds to G.

4 Discrete Fourier Transforms

The left column below shows six discrete-time signals for $0 \leqslant n \leqslant 31$.
The right column shows plots of the magnitudes of six DFTs computed for $N = 32$.
For each discrete-time signal in the left column below, find the matching DFT magnitude (one of plots A–F) and enter its letter in the box provided.



Worksheet (intentionally blank)

The top signal shows two full cycles of a triangle wave. Therefore the fundamental frequency of the triangle wave falls at $k = 2$. There could also be harmonics of $k = 2$ (i.e., at $k = 4, 6, 8, ...$).

$\rightarrow$ plot E

The next two signals show 1.5 full cycles of a triangle wave. Therefore the magnitude will peak between $k = 1$ and $k = 2$. If the first of these is periodically extended, it will have a big discontinuity between periods. The second of these has a much smaller discontinuity. Also, the DC value of the second is much larger than the first.

$\rightarrow$ the second signal corresponds to plot D

$\rightarrow$ the third signal corresponds to plot A

The fourth signal shows 1 full cycle of a triangle wave. Therefore $k = 1$.

$\rightarrow$ plot F

When periodically extended, the fifth signal will be a sawtooth with $k = 1$. There will also be a large discontinuity at the period boundaries, so that will generate contributions at nearby k's.

$\rightarrow$ plot B

When periodically extended, the last signal will make a triangle wave at $k = 1$. Notice however that there is a large DC component.

$\rightarrow$ plot C

Worksheet (*intentionally blank*)