6.003 Quiz 1 Spring 2020

Name: Answers

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one 8.5×11 sheet of paper (two sides).
You may NOT use any electronic devices (including calculators, phones, etc).

If you have questions, please **come to us at the front** to ask them.

Please enter all solutions in the boxes provided.

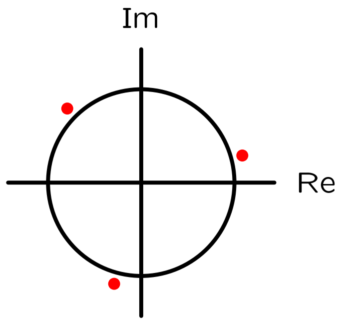
Extra work may be taken into account when assigning partial credit,
but only work on pages with QR codes will be considered.

Question 1: 20 Points
Question 2: 24 Points
Question 3: 24 Points
Question 4: 32 Points
Total: 100 Points

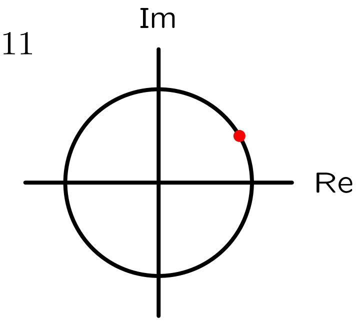
1 Plainly Complex (20 Points)

Below are eight complex-valued expressions, each paired with a depiction of the complex plane demarked by the unit circle. Evaluate each expression and mark its value on the complex plane with a dot. If the expression can represent multiple complex numbers, mark at least two of them.

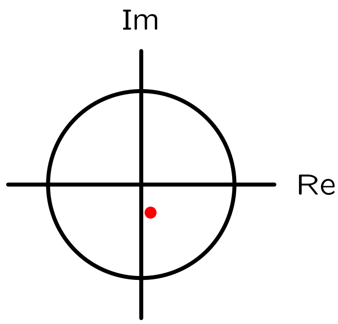
$\sqrt[3]{1+j}$



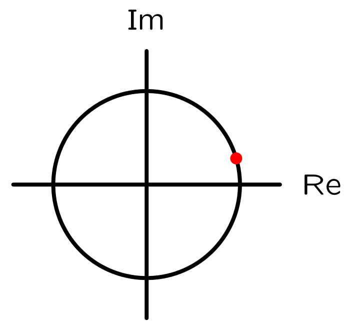
$\left(\frac{2}{\sqrt{3}+j}\right)^{11}$



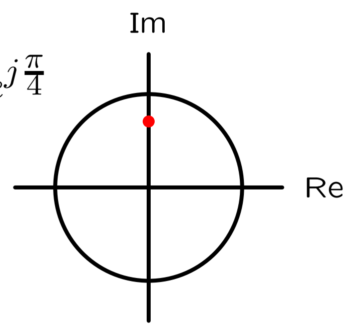
$\frac{1}{1+3j}$



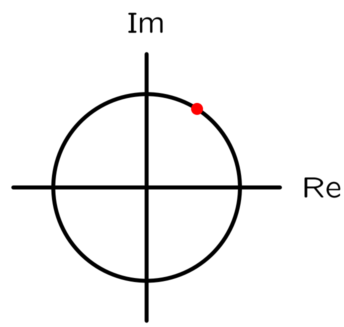
$\frac{3+4j}{4+3j}$



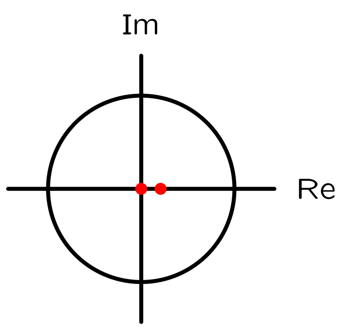
$\frac{1}{2}e^{j\frac{3\pi}{4}}+\frac{1}{2}e^{j\frac{\pi}{4}}$



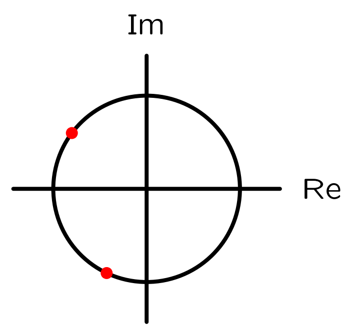
$(e)^j$



$(j)^j$



$(j)^e$



Worksheet (intentionally blank)

Part 1. $\sqrt[3]{1+j}$

Start by writing (1+j) in polar form.

$$1+j = \left(\sqrt{2}\right)\left(e^{j\pi/4}\right)$$

Now take the cube root of each part.

$$\sqrt[3]{1+j} = (2)^{1/6} \times e^{j\pi/12}$$

We should be expecting three cube roots of a number. Where are the other two?

One approach is to realize that we can always add integer multiples of 2π to a purely imaginary exponent of e.

$$1+j = \left(\sqrt{2}\right)\left(e^{j(\pi/4+2\pi m)}\right)$$

Thus

$$\sqrt[3]{1+j} = (2)^{1/6} \times e^{j(\pi/12+2\pi m/3)} = \begin{cases} (2)^{1/6} \times e^{j\pi/12} & \text{if } (m \bmod 3) = 0 \\ (2)^{1/6} \times e^{j9\pi/12} & \text{if } (m \bmod 3) = 1 \\ (2)^{1/6} \times e^{j17\pi/12} & \text{if } (m \bmod 3) = 2 \end{cases}$$

An alternative approach is to realize that there are three cube roots of 1: 1, $e^{j2\pi/3}$, and $e^{j4\pi/3}$. So after finding one cube root of (1+j), we can find the others by multipying the first cube root by $e^{j2\pi/3}$ and $e^{j4\pi/3}$.

The sixth root of 2 is greater than 1 and a lot less than 2. Assume that the sixth root of 2 is given by $1 + a$ where a is a small number. Then

$$2 = (1+a)^6 = 1 + 6a + 15a^2 + 20a^3 + 15a^4 + 6a^5 + a^6 = 2 \approx 1 + 6a$$

If $1 + 6a \approx 2$ then $a \approx \frac{1}{6}$ and $2^{1/6} \approx 1.1$.

Part 2. $\left(\frac{2}{\sqrt{3}+j}\right)^{11}$

Start by writing the denominator $\sqrt{3}+j$ in polar form.

$$\sqrt{3}+j = 2\tan^{-1}\frac{1}{\sqrt{3}} = 2e^{j\pi/6}$$

Then $\frac{2}{\sqrt{3}+j} = e^{-j\pi/6}$ and

$$\left(\frac{2}{\sqrt{3}+j}\right)^{11} = e^{-j11\pi/6} = e^{j\pi/6}$$

Part 3. $\frac{1}{1+3j}$

$$\frac{1}{1+3j} = \frac{1}{1+3j} \times \frac{1-3j}{1-3j} = \frac{1-3j}{10} = 0.1 - 0.3j$$

Part 4. $\dfrac{3+4j}{4+3j}$

$$\frac{3+4j}{4+3j}=\frac{5e^{j\tan^{-1}\frac{4}{3}}}{5e^{j\tan^{-1}\frac{3}{4}}}=e^{j(\tan^{-1}\frac{4}{3}-\tan^{-1}\frac{3}{4})}$$

The angle whose tangent is $\frac{4}{3}$ is a bit greater than $\pi/4$. Similarly, the angle whose tangent is $\frac{3}{4}$ is a bit less than $\pi/4$. The net angle is therefore a bit more than 0.

Part 5. $\frac{1}{2}e^{j\frac{3\pi}{4}}+\frac{1}{2}e^{j\frac{\pi}{4}}$

$$\frac{1}{2}e^{j\frac{3\pi}{4}}+\frac{1}{2}e^{j\frac{\pi}{4}}=\frac{1}{2}\cos\frac{3\pi}{4}+j\frac{1}{2}\sin\frac{3\pi}{4}+\frac{1}{2}\cos\frac{\pi}{4}+j\frac{1}{2}\sin\frac{\pi}{4}$$

The cosine of $\frac{3\pi}{4}$ is equal to -1 times the cosine of $\frac{\pi}{4}$. Therefore the cosine terms in the previous equation subtract out. The sine of $\frac{3\pi}{4}$ and the sine of $\frac{\pi}{4}$ are both equal to $j\frac{\sqrt{2}}{2}$. Therefore

$$\frac{1}{2}e^{j\frac{3\pi}{4}}+\frac{1}{2}e^{j\frac{\pi}{4}}=j\frac{\sqrt{2}}{2}$$

Part 6. $(e)^j$

$$(e)^j=(e)^{j1}$$

This is just a complex exponential where the angle is 1 radian.

Part 7. $(j)^j$

$$(j)^j=(e^{j\frac{\pi}{2}})^j=e^{-\frac{\pi}{2}}$$

Note that we can add multiples of 2π to the imaginary exponent to get additional values.

$$(j)^j=(e^{j(\frac{\pi}{2}+2\pi m)})^j=e^{-(\frac{\pi}{2}+2\pi m)}\approx\begin{cases}111 & \text{if } m=-1\\ 0.208 & \text{if } m=0\\ 0.0004 & \text{if } m=1\end{cases}$$

The case for $m=0$ is shown on the plot as the real-valued point to the right of the origin. The dot near the origin represents the infinite number of points for $m>0$. All of the solutions for $m<0$ are off the scale of this plot.

Part 8. $(j)^e$

$$(j)^e=(e^{j\pi/2})^e=e^{j\pi e/2}$$

This is just a complex exponential where the angle is e times $\pi/2$ radians.

Notice that we can add multiples of 2π to the exponent of e to find additional solutions.

$$(j)^e=(e^{j(\pi/2+2\pi m)})^e=e^{j(\pi/2+2\pi m)e}$$

2 Trigonometric Fourier Series (24 Points)

Part 1. We would like to represent the following continuous-time signal

$$f(t) = 2\cos\left(\frac{1}{3}\pi t\right)\sin\left(\frac{1}{4}\pi t\right) + 3$$

using a trigonometric Fourier series of the following form:

$$f(t) = \sum_{k=0}^{M} c_k \cos(2\pi kt/T) + \sum_{k=0}^{M} d_k \sin(2\pi kt/T)$$

Determine the following parameters of the trigonometric representation:

- T, which is the **fundamental period** of this signal,
- M, which represents the highest harmonic needed for this signal, and
- c_k and d_k, which are the coefficients of the trigonometric representation.

Enter T and M in the boxes below.

T: 24 M: 7

Enter the resulting coefficents c_k and d_k in the following table. Use a separate row for each value of k that is required. You need not list values of k for which both c_k and d_k are both zero. Leave unused rows empty.

k	c_k	d_k
0	3	–
1	0	-1
7	0	1

Notice that d_0 can take any value since the corresponding basis function is zero.

Worksheet (intentionally blank)

We can use trig identities or Euler's formula to reduce the expression for $f(t)$ to the standard trigonometric Fourier series form.

$$\begin{aligned}
f(t) &= 2\cos\left(\frac{1}{3}\pi t\right)\sin\left(\frac{1}{4}\pi t\right)+3 \\
&= 2\frac{1}{2}(e^{j\pi t/3}+e^{-j\pi t/3})\frac{1}{2j}(e^{j\pi t/4}-e^{-j\pi t/4})+3 \\
&= \frac{1}{2j}\left(e^{j7\pi t/12}-e^{-j7\pi t/12}-e^{j\pi t/12}+e^{-j\pi t/12}\right)+3 \\
&= \sin\left(\frac{7\pi t}{12}\right)-\sin\left(\frac{\pi t}{12}\right)+3 \\
&= \sin\left(\frac{2\pi t}{24}7\right)-\sin\left(\frac{2\pi t}{24}1\right)+3
\end{aligned}$$

The result has the desired form if $T = 24$ and $M = 7$. There are three non-zero terms:

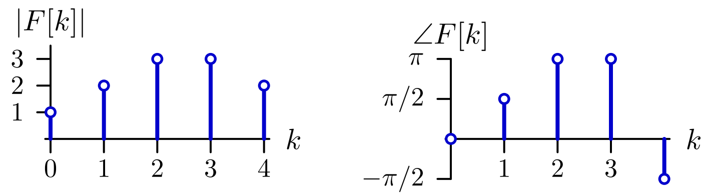
$c_0 = 3$

$d_1 = -1$

$d_7 = 1$

Note that d_0 can take any value since the corresponding basis function is zero.

Part 2. Let $f[n]$ represent a discrete-time signal whose Fourier series coefficients $F[k]$ are periodic in $N = 5$, i.e., $F[k] = F[k+5]$ for all integers k. The following plots show the magnitude and angle of $F[k]$ over one period.



We wish to find the coefficients of a trigonometric representation for $f[n]$ with the following form:

$$f[n] = \sum_{k=0}^{M} c_k \cos(2\pi kn/N) + \sum_{k=0}^{M} d_k \sin(2\pi kn/N)$$

2a. Determine the smallest value of M as well as the coefficients c_k and d_k that are needed to represent $f[n]$.

Enter that smallest value of M in the box below.

M: 2

Enter the resulting coefficents c_k and d_k in the following table. Use only as many rows as necessary, leaving unused rows blank.

k	c_k	d_k
0	1	–
1	0	−4
2	−6	0

Notice that d_0 can take any value since the corresponding basis function is zero.

Worksheet (intentionally blank)

Using the plots and the fact that $F[k]$ is periodic in k with period $N = 5$, we can see that

$F[0] = 1$
$F[1] = -F[-1] = j2$
$F[2] = F[-2] = -3$

It follows that we can write $f[n]$ as

$$f[n] = 1 - 4\sin\left(\frac{2\pi}{N}n\right) - 6\cos\left(\frac{4\pi}{N}n\right)$$

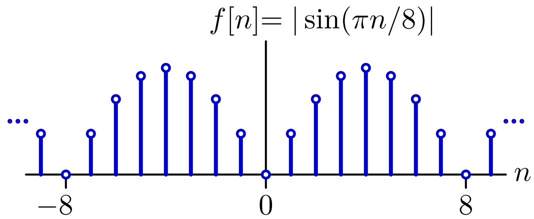
Therefore, $M = 2$ and

$c_0 = 1$
$d_1 = -4$
$c_2 = -6$

Notice that d_0 can take any value since the corresponding basis function is zero.

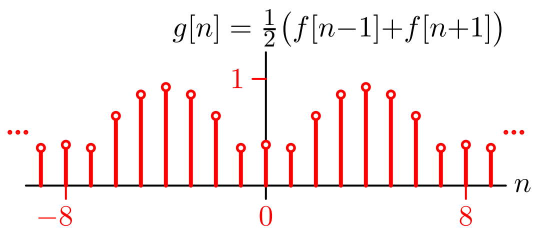
3 Rectified Sine Wave (24 Points)

Part a. Let $f[n]$ represent the following periodic, discrete-time signal:



Let $g[n] = \frac{1}{2}\big(f[n-1]+f[n+1]\big)$.

Sketch $g[n]$ on the following axes. Label the important parameters of your plot.



Let $F[k]$ represent the Fourier series coefficients for $f[n]$ computed with **period** $N = 8$:

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-j\frac{2\pi k}{N}n}$$

Let $G[k]$ represent the Fourier series coefficients for $g[n]$ computed with **same period** $N = 8$.

Determine the relation between the $G[k]$ coefficients and the $F[k]$ coefficients.

In the table below, enter an expression for each of $G[0]$ through $G[7]$ in terms of $F[0]$, $F[1]$, $F[2]$, ...

In addition to $F[0]$, $F[1]$, $F[2]$, . . ., your table entries can contain real and/or imaginary numbers and constants such as e and π. Your entries should not contain integrals or summations.

k	$G[k]$
0	$F[0]$
1	$\frac{\sqrt{2}}{2}F[1]$
2	0
3	$-\frac{\sqrt{2}}{2}F[3]$
4	$-F[4]$
5	$-\frac{\sqrt{2}}{2}F[5]$
6	0
7	$\frac{\sqrt{2}}{2}F[7]$

Worksheet (intentionally blank)

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-j\frac{2\pi k}{N}n}$$

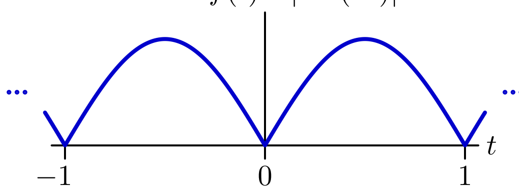
$$\begin{aligned}
G[k] &= \frac{1}{N}\sum_{n=\langle N\rangle} g[n]e^{-j\frac{2\pi k}{N}n} \\
&= \frac{1}{N}\sum_{n=\langle N\rangle} \frac{1}{2}\big(f[n-1]+f[n+1]\big)e^{-j\frac{2\pi k}{N}n} \\
&= \frac{1}{2N}\sum_{n=\langle N\rangle} f[n-1]e^{-j\frac{2\pi k}{N}n} + \frac{1}{2N}\sum_{n=\langle N\rangle} f[n+1]e^{-j\frac{2\pi k}{N}n}
\end{aligned}$$

Let $m = n-1$ in the first summation and $l = n+1$ in the second.

$$\begin{aligned}
G[k] &= \frac{1}{2N}\sum_{m=\langle N\rangle} f[m]e^{-j\frac{2\pi k}{N}(m+1)} + \frac{1}{2N}\sum_{l=\langle N\rangle} f[l]e^{-j\frac{2\pi k}{N}(l-1)} \\
&= \frac{1}{2}e^{-j\frac{2\pi k}{N}}\frac{1}{N}\sum_{m=\langle N\rangle} f[m]e^{-j\frac{2\pi k}{N}m} + \frac{1}{2}e^{j\frac{2\pi k}{N}}\frac{1}{N}\sum_{l=\langle N\rangle} f[l]e^{-j\frac{2\pi k}{N}l} \\
&= \frac{1}{2}e^{-j\frac{2\pi k}{N}}F[k] + \frac{1}{2}e^{j\frac{2\pi k}{N}}F[k] \\
&= \frac{1}{2}\left(e^{-j\frac{2\pi k}{N}} + e^{j\frac{2\pi k}{N}}\right)F[k] \\
&= \cos\left(\frac{2\pi k}{N}\right)F[k]
\end{aligned}$$

Part b. Let $f(t)$ represent the following periodic, continuous-time signal:

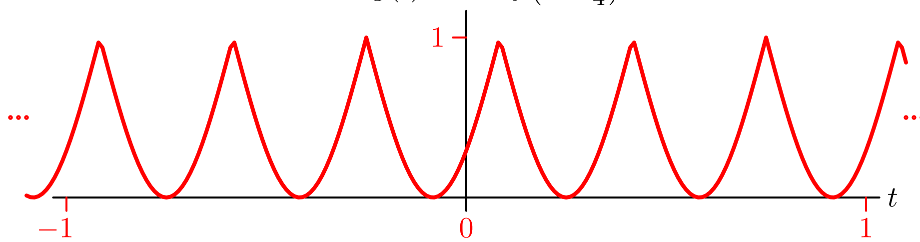
$f(t) = |\sin(\pi t)|$



Let $g(t) = 1 - f\left(3t - \frac{1}{4}\right)$.

Sketch $g(t)$ on the following axes. Label the important parameters of your plot.

$g(t) = 1 - f\left(3t - \frac{1}{4}\right)$



Let $F[k]$ represent the Fourier series coefficients for $f(t)$ computed with **period** $T = 1$:

$$F[k] = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi k}{T}t}dt$$

Let $G[k]$ represent the Fourier series coefficients for $g(t)$ computed with **same period** $T = 1$.

Determine the relation between the $G[k]$ coefficients and the $F[k]$ coefficients.

In the tables below, enter expressions for each of $G[0]$ through $G[15]$ in terms of $F[0], F[1], F[2], \ldots$.

In addition to $F[0], F[1], F[2], \ldots$, your table entries can contain real and/or imaginary numbers and constants such as e and π. Your equations should not contain integrals or summations.

k	$G[k]$
0	$1 - F[0]$
1	0
2	0
3	$j\,F[1]$
4	0
5	0
6	$F[2]$
7	0

k	$G[k]$
8	0
9	$-j\,F[3]$
10	0
11	0
12	$-F[4]$
13	0
14	0
15	$j\,F[5]$

Worksheet (intentionally blank)

$$F[k] = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi k}{T}t}dt$$
$$G[k] = \frac{1}{T}\int_T g(t)e^{-j\frac{2\pi k}{T}t}dt$$
$$= \frac{1}{T}\int_T \left(1 - f\left(3t-\frac{1}{4}\right)\right) e^{-j\frac{2\pi k}{T}t}dt$$
$$= \frac{1}{T}\int_T e^{-j\frac{2\pi k}{T}}\, dt - \frac{1}{T}\int_T f\left(3t-\frac{1}{4}\right)e^{-j\frac{2\pi k}{T}t}dt$$

Let $\tau = 3t - 1/4$. Then $d\tau = 3dt$.

$$G[k] = \delta[k] - \frac{1}{T}\int_{3T} f(\tau)e^{-j\frac{2\pi k}{T}(\frac{\tau}{3}+\frac{1}{12})}\frac{1}{3}d\tau$$
$$= \delta[k] - e^{-j\frac{2\pi k}{12T}}\frac{1}{3T}\int_{3T} f(\tau)e^{-j\frac{2\pi k}{T}(\frac{\tau}{3})}d\tau$$
$$= \delta[k] - e^{-j\frac{2\pi k}{12T}}\, F[k/3]$$

Notice that the $\delta[k]$ term contributes 1 if $k = 0$ and 0 otherwise.

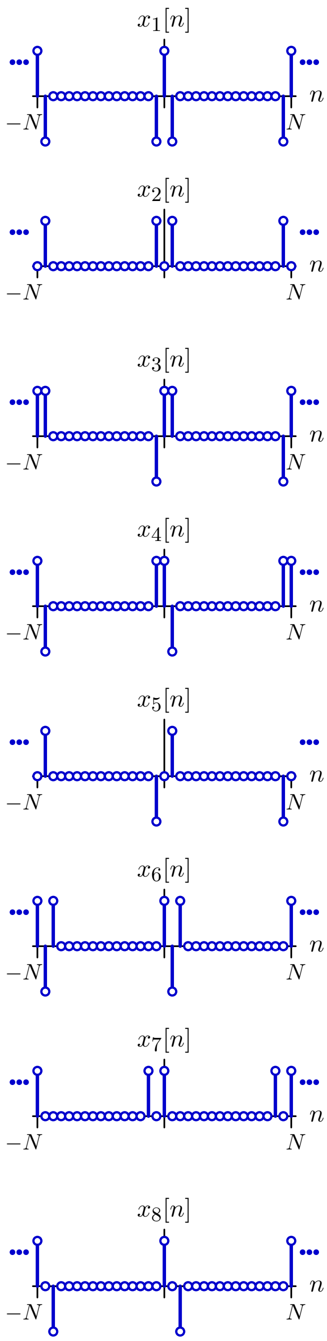
Also notice that $G[k] = 0$ unless $k \bmod 3 = 0$.

One way to think about this is that the period of $f(t)$ is 1 second, and therefore $f(t)$ can be expressed as a sum of harmonics that are integer multiples of 1 Hz. The $g(t)$ signal is a compressed version of $f(t)$, so the harmonics of $g(t)$ are spread out by a factor of three.

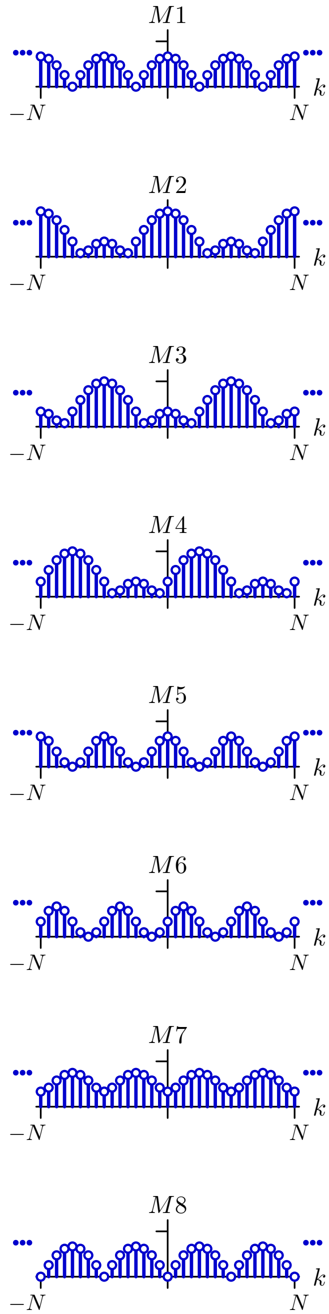
A second way to think about this is by looking at the $g(t)$ function itself. Since $g(t)$ is periodic in 1/3 second, we should be expecting that the Fourier series for $g(t)$ should only contain integer multiples of 3 Hz.

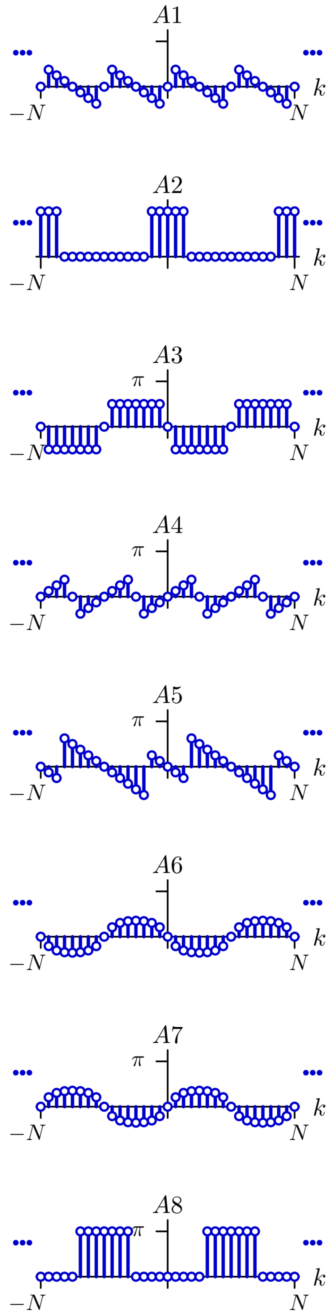
4 Fourier Series Matching (32 Points)

Each of the signals $x_i[n]$ in the left column below is periodic with period $N = 16$. Find the Fourier series coefficients $X_i[k]$ for each signal and then identify which of plots $M1 - M8$ shows the magnitude of $X_i[k]$ and which of plots $A1 - A8$ shows the angle of $X_i[k]$ as functions of k. Enter your answers in the boxes provided.



Signal	M:	A:
$x_1[n]$	M3	A2
$x_2[n]$	M1	A8
$x_3[n]$	M7	A6
$x_4[n]$	M7	A7
$x_5[n]$	M8	A3
$x_6[n]$	M3	A5
$x_7[n]$	M1	A4
$x_8[n]$	M8	A1





Worksheet (intentionally blank)

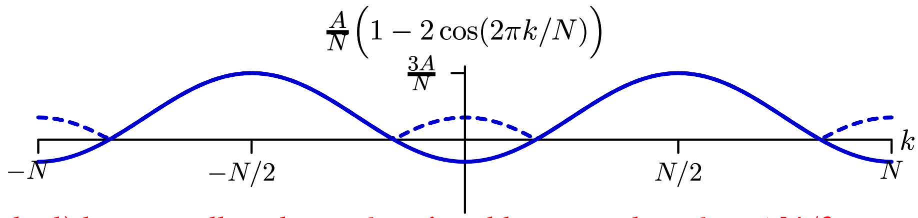
Find the Fourier series representation $X[k]$ for each of the given $x[n]$ using the analysis equation.

$$X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N}kn}$$

We will assume (arbitrarily) that the values of $x[n]$ are $-A$, 0, or A.

Since the signals are all real-valued, the corresponding magnitudes will be symmetric about $k = 0$. This eliminates M_4 and M_6.

Part 1. $X_1[k] = \frac{A}{N}(-e^{j\frac{2\pi}{N}k} + 1 - e^{-j\frac{2\pi}{N}k}) = \frac{A}{N}(1 - 2\cos(2\pi k/N))$



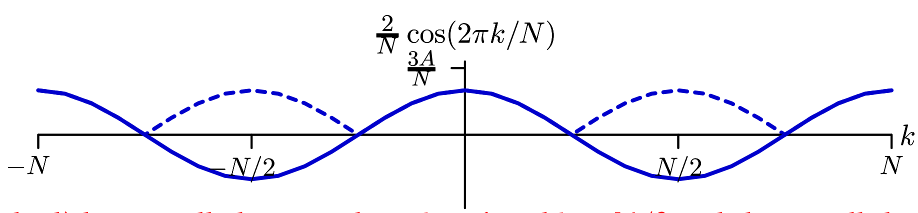
The magnitude (dashed) has a small peak near $k = 0$ and larger peaks at $k = \pm N/2$.

Answer = M3.

The angle is 0 for the range of k near $N/2$ (where there is no dashed line) and π for the range of k near 0, where the solid and dashed lines differ in sign).

Answer = A2.

Part 2. $X_2[k] = \frac{1}{N}(e^{j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k}) = \frac{2}{N}\cos(2\pi k/N)$



The magnitude (dashed) has equally large peaks at $k = 0$ and $k = N/2$ and sharp nulls between.

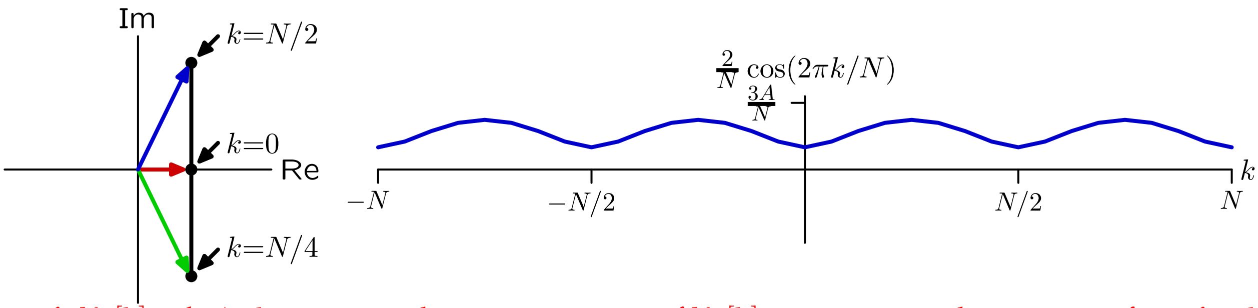
Answer = M1.

The angle is 0 for the range of k near $k = 0$ and π for the range of k near $N/2$ (where the dashed and solid curves differ in sign).

Answer = A8.

Part 3. $X_3[k] = \frac{1}{N}\left(-e^{j\frac{2\pi}{N}k} + 1 + e^{-j\frac{2\pi}{N}k}\right) = \frac{1}{N}\Big(1 - 2j\sin(2\pi k/N)\Big)$

This part is a bit trickier to plot since (unlike parts 1 and 2) this one has both real and imaginary parts.



When $k = 0$, $X_3[k]$ is 1. As k increases, the imaginary part of $X_3[k]$ gets increasingly negative – from 0 at $k = 0$ to -2 at $k = N/4$. Correspondingly, the magnitude increases from 1 at $k = 0$ to $\sqrt{5}$ at $k = N/4$.

As k increases from $N/4$ to $N/2$, the imaginary part of $X_3[k]$ change from -2 to 0 and the magnitude drops from $\sqrt{5}$ back to 1.

The plot of magnitude is not exactly sinusoidal, but it is smooth and does not have sharp notches.

Answer = M7.

The angle of $X_3[k]$ start at 0 for $k = 0$ and gradually decreases (going negative) for k between 0 and $N/4$. As k increases from $N/4$ to $N/2$, the angle decreases back to zero. The pattern from $N/2$ to N is similar to the pattern from 0 to $N/2$ except that the sign is now flipped to positive.

Answer = A6.

Part 4. $X_4[k] = \frac{1}{N}\left(e^{j\frac{2\pi}{N}k} + 1 - e^{-j\frac{2\pi}{N}k}\right) = 1 + 2j\sin(2\pi k/N)$

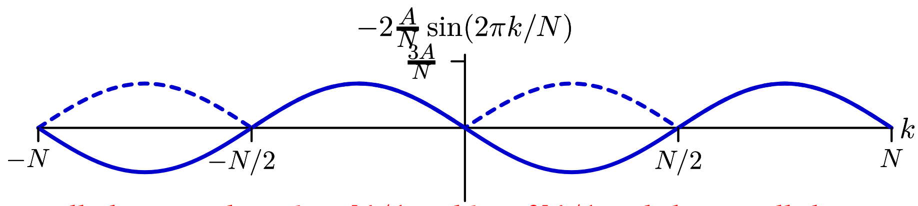
This part is similar to part 3, except that the imaginary part of $X_4[k]$ is the negative of that of $X_3[k]$.

Thus the magnitude is the same as part 3, i.e., M7.

The angle is the negative of that in part 3, i.e., A7.

Part 5. $X_5[k] = \frac{1}{N}(-e^{j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}k}) = -2j\sin(2\pi k/N)$

This part is purely imaginary.



The magnitude has equally large peaks at $k = N/4$ and $k = 3N/4$ and sharp nulls between.

Answer = M8.

The angle is $-\pi/2$ for $0 < k < N/2$ and $\pi/2$ for $N/2 < k < N$.

Answer = A3.

Part 6. $X_6[k] = \frac{1}{N}(1 - e^{-j\frac{2\pi}{N}k} + e^{-j\frac{2\pi}{N}2k}) = e^{-j2\pi k/N}(-1 + 2\cos(2\pi k/N))$

Notice that $x_6[n] = -x_1[n-1]$. Neither the delay nor the negation will affect the magnitude. Therefore the magnitude is given by M1.

To find the angle of $X_6[k]$, start with the angle of $X_1[k]$ (plot A2). Negating $x_1[n]$ adds π to all of the angles. As a result, the angle is 0 for k close to 0 and π otherwise.

Next consider the effect of the delay, which multiplies the Fourier transform by $e^{-j2\pi k/N}$. This delay adds an angle of $-2\pi k/N$ to each frequency point k. The resulting angle is shown in A5.

Part 7. $X_7[k] = \frac{1}{N}(e^{j\frac{2\pi}{N}2k} + 1) = e^{j2\pi k/N}(2\cos(2\pi k/N))$

$x_7[n]$ is a version of $x_2[n]$ that is shifted backwards in time by 1 sample. The time shift does not affect the magnitude. Therefore the magnitude is the same as part 2 – i.e., M1.

Without the delay, the angle would have been A8. The shift adds an angle of $2\pi k/N$ to each frequency point k, resulting in A4.

Part 8. $X_8[k] = \frac{1}{N}(1 - e^{-j\frac{2\pi}{N}2k}) = 2je^{-j2\pi k/N}\sin(2\pi k/N)$

This is a negated and delayed version of part 5. Neither the negation nor the delay affect the magnitude, which is therefore given by M8.

Without the delay, the angle would have been $\pi/2$ for $0 \leqslant k \leqslant N/2$ and $-\pi/2$ for $N/2 \leqslant k \leqslant N$. The delay adds a downward sloping phase, resulting in plot A1.

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