6.003 Final Exam Spring 2020

Name: Answers

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use three 8.5 × 11 sheet of paper (two sides).
You may NOT use any electronic devices (including calculators, phones, etc).

If you have questions, please **come to us at the front** to ask them.

Please enter all solutions in the boxes provided.

Extra work may be taken into account when assigning partial credit,
but only work on pages with QR codes will be considered.

Question 1:	20 Points
Question 2:	24 Points
Question 3:	24 Points
Question 4:	32 Points
Total:	100 Points

Short Answers

Part 1. Find the fundamental period T_1 of the signal $x_1(t)$ given below

$$x_1(t) = 2\cos\left(\frac{\pi}{3}t\right)\cos\left(\frac{\pi}{4}t\right)$$

and enter its value in the box provided.

$$x_1(t) = 2\cos\left(\frac{\pi}{12}t\right) + \cos\left(\frac{7\pi}{12}t\right)$$

$$T_1 = \frac{2\pi}{\pi/12} = 24$$

Part 2. A trigonometric Fourier series of a periodic DT signal has the form

$$c_0 + \sum_{k=1}^{\infty} c_k \cos(k\Omega_o n) + d_k \sin(k\Omega_o n)$$

where Ω_o is 2π divided by the period (N) of the signal. How many non-zero coefficients are needed to represent

$$x_2[n] = 3\sin(\Omega_o n) - 4\sin^3(\Omega_o n)$$

in such an expansion?

$$x_2[n] = 3\sin(\Omega_o n) - 4\sin^3(\Omega_o n) = \sin(3\Omega_o n)$$

Just one non-zero term (c_3) is required.

Part 3. Let $X_3(\omega)$ represent the Fourier transform of the following signal.

$$x_3(t) = \begin{cases} e^{-2(t-3)} & \text{if } t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

Find $X_3(0)$.

$$X_3(\omega) = \int_0^{\infty} e^{-2(t-3)} e^{-j\omega t}\, dt$$

$$X_3(0) = \int_0^{\infty} e^{-2(t-3)} e^{0}\, dt = \left.\frac{e^{-2(t-3)}}{-2}\right|_0^{\infty} = \frac{1}{2}e^6$$

Part 4. Let $X_4[k]$ represent the DFT of the signal given below

$$x_4[n] = (\delta[n])^2 + \delta[n]\delta[n-2] + 2(\delta[n-5])^2$$

when $N = 10$. Find $X_4[2]$.

$$x_4[n] = \delta[n] + 2\delta[n-5]$$

$$X_4[k] = \frac{1}{10}\sum_{n=0}^{9} x_4[n] e^{-j\frac{2\pi k}{10}n} = \frac{1}{10}\left(1 + 2e^{-j\frac{2\pi k}{10}5}\right) = \frac{1}{10}\left(1 + 2e^{-j\pi k}\right)$$

$$X_4[2] = \frac{3}{10}$$

Part 5. The discrete cosine transform (DCT) of a signal $f[n]$ is defined by the following.

$$F_C[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\cos\left(\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right)$$

The DCT of

$$x_5[n] = A\cos\left(\frac{5}{8}\pi n + \phi\right)$$

was computed with $N = 8$ and the result has a single non-zero coefficient. The value of that coefficient was 2. Find k, ϕ, and A.

$x_5[n]$ must be one of the basis functions of the DCT when $N = 8$.

$$X_C[k] = \frac{1}{8}\sum_{n=0}^{7} x_5[n]\cos\left(\frac{\pi k}{8}\left(n + \frac{1}{2}\right)\right)$$

Therefore $k = 5$, $\phi = \frac{5\pi}{16}$, and $A = 4$.

Impulsive Images

Each panel on the next page shows the magnitude of a 2D DFT $F[k_r, k_c]$, as a function of row number k_r (increasing downward) and column number k_c increasing to the right. Determine which panel corresponds to each of the following images $f_i[r, c]$ as functions of row r and column c.

Part a. Which panel (A–P) corresponds to

$$f_1[r, c] = 2\delta[r]\delta[c] + \delta[r+2]\delta[c-1] + \delta[r-2]\delta[c+1]$$

panel: G

Part b. Which panel (f_1–f_{12}) has the following 2D DFT?

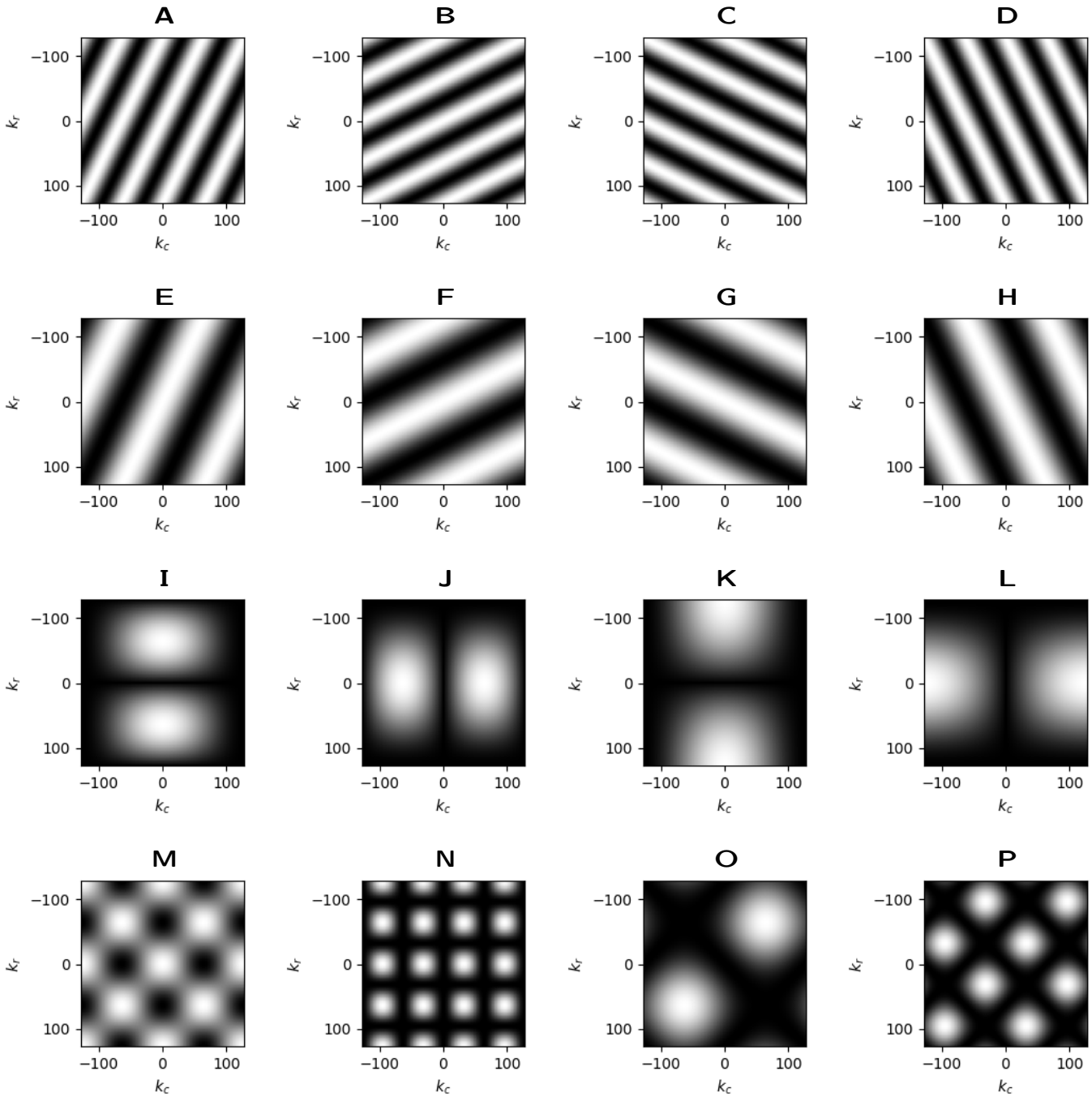
$$f_2[r, c] = \Big(\delta[r+1] + 2\delta[r] + \delta[r-1]\Big)\Big(\delta[c-1] - \delta[c+1]\Big)$$

panel: J

Part c. Which panel (f_1–f_{12}) has the following 2D DFT?

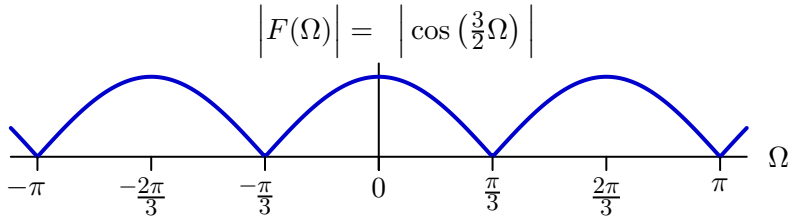
$$f_3[r, c] = 4\delta[r]\delta[c] + \delta[r]\delta[c-2] - \delta[r]\delta[c+2] + \delta[r+2]\delta[c] - \delta[r-2]\delta[c]$$

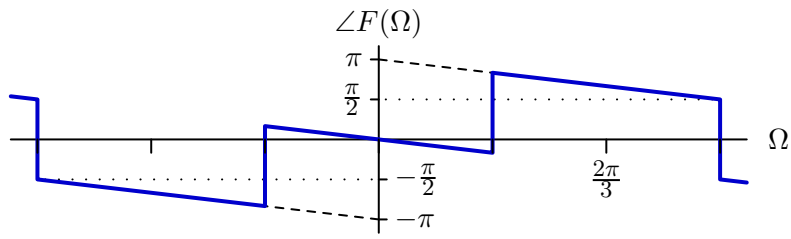
panel: P


A
B
C
D
E
F
G
H
I
J
K
L
M
N
O
P
−100
0
100
k_r
k_c

Inverse DTFT

Let $f[n]$ represent a discrete time signal whose DTFT is shown below.





Determine $f[n]$ and enter its values for $n = -3$ through 3 in the boxes below.

$f[-3]$: 0

$f[-2]$: 0

$f[-1]$: $\frac{1}{2}$

$f[0]$: 0

$f[1]$: 0

$f[2]$: $\frac{1}{2}$

$f[3]$: 0

The complex amplitude of $F(\Omega)$ is the product of its magnitude $|F(\Omega)|$ and $e^{j\angle F(\Omega)}$. Notice that the phase jumps discontinuously by π at $\Omega = \pm\frac{\pi}{3}$ and $\pm\pi$. These jumps in phase coincide with discontinuities in the magnitude function, and correspond to changes in the sign of $\cos\left(\frac{3}{2}\Omega\right)$. Therefore we can combine the magnitude and phase to derive an equivalent expression for $F(\Omega)$ as follows:

$$F(\Omega) = \cos\left(\frac{3}{2}\Omega\right)\, e^{-j\frac{1}{2}\Omega}$$

where the phase term is linear (with a slope of -1/2) after the discontinuities have been removed.

Next, we can express the cosine term as complex exponentials to get

$$F(\Omega) = \frac{1}{2}\left(e^{j\frac{3}{2}\Omega} + e^{-j\frac{3}{2}\Omega}\right)\, e^{-j\frac{1}{2}\Omega} = \frac{1}{2}e^{j\Omega} + \frac{1}{2}e^{-j2\Omega}$$

The inverse DTFT of this expression is

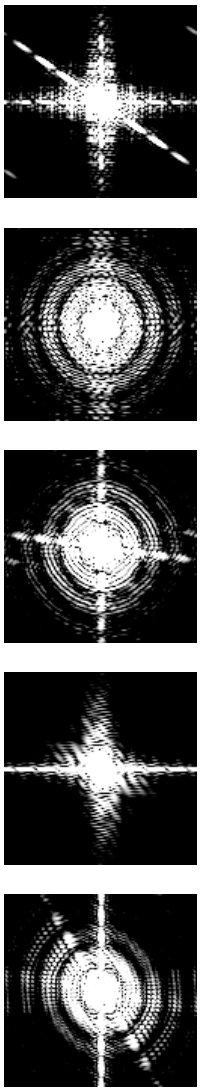
$$f[n] = \frac{1}{2}\delta[n+1] + \frac{1}{2}\delta[n-2]$$

so there are only two nonzero samples: $f[-1] = \frac{1}{2}$ and $f[2] = \frac{1}{2}$.

Number Transforms

The left panels below show images of the numbers 1 through 5. The right panels show the magnitudes of the 2D DFTs of those images, but the order is scrambled.

For each 2D DFT on the right, identify the corresponding number 1–5 on the left.



4

3

5

1

2

Inverse Transforms

Let $x[n]$ represent the following discrete-time signal

$$x[n] = \begin{cases} 0 & \text{if } n < 0 \\ \left(\frac{1}{2}\right)^{n/3} & \text{if } n \geqslant 0 \text{ and } n \bmod 3 = 0 \\ \left(\frac{1}{3}\right)^{(n-1)/3} & \text{if } n \geqslant 0 \text{ and } n \bmod 3 = 1 \\ \left(\frac{1}{4}\right)^{(n-2)/3} & \text{if } n \geqslant 0 \text{ and } n \bmod 3 = 2 \end{cases}$$

and let $X(\Omega)$ represent the DTFT of $x[n]$.

Part a. Let

$$Y[k] = X\left(\frac{2\pi}{3}k\right)$$

and let $y[n]$ represent the inverse DFT of $Y[k]$ with $N = 3$. Determine $y[0]$, $y[1]$, $y[2]$, and $y[3]$ and enter their values in the boxes below.

$y[0]$: 6

$y[1]$: 9/2

$y[2]$: 4

$y[3]$: 6

$$\begin{aligned} y[n] &= \sum_{k=0}^{2} Y[k] e^{j\frac{2\pi k}{3}n} = \sum_{k=0}^{2} X\left(\frac{2\pi}{3}k\right) e^{j\frac{2\pi k}{3}n} \\ &= \sum_{k=0}^{2} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j\frac{2\pi k}{3}m}\right) e^{j\frac{2\pi k}{3}n} = \sum_{m=-\infty}^{\infty} x[m] \sum_{k=0}^{2} e^{j\frac{2\pi k}{3}(n-m)} \\ &= \sum_{m=-\infty}^{\infty} x[m] 3\delta[(n-m) \bmod 3] = 3 \sum_{m=-\infty}^{\infty} x[n-3m] \end{aligned}$$

$$y[0] = 3 \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m = 3\left(\frac{1}{1-\frac{1}{2}}\right) = 6$$

$$y[1] = 3 \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m = 3\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{9}{2}$$

$$y[2] = 3 \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m = 3\left(\frac{1}{1-\frac{1}{4}}\right) = 4$$

$$y[3] = y[0] = 6$$

Part b. Let $w[n]$ represent the following window function

$$w[n] = \begin{cases} 1 & \text{if } 0 \leqslant n \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

which is applied to $x[n]$ (above) to obtain a new signal

$$x_w[n] = x[n]w[n]\,.$$

Let $X[k]$, $W[k]$, and $X_w[k]$ represents the DFTs of $x[n]$, $w[n]$, and $x_w[n]$, respectively, where all DFTs are computed with an analysis window length $N = 8$.

Now let $Z[k] = X_w[k]W[k]$ and find $z[n]$, which is the inverse DFT of $Z[k]$.

$z[0]$: $\frac{25}{96}$

$z[1]$: $\frac{31}{96}$

$z[2]$: $\frac{13}{32}$

$z[3]$: $\frac{7}{16}$

$z[4]$: $\frac{23}{48}$

$z[5]$: $\frac{49}{96}$

$z[6]$: $\frac{37}{96}$

$z[7]$: $\frac{25}{96}$

$$x_w[n] = \begin{cases} 1 & \text{if } n = 0, 1, 2 \\ 1/2 & \text{if } n = 3 \\ 1/3 & \text{if } n = 4 \\ 1/4 & \text{if } n = 5 \\ 0 & \text{otherwise} \end{cases}$$

Multiplication of DFTs results in circular convolution of the time functions followed by division by $N = 8$. The result can be expressed as a superposition as follows.

$$\begin{array}{cccccccc}
1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 0 \\
0 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\
0 & 0 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
\frac{1}{4} & 0 & 0 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{4} & 0 & 0 & 1 & 1 & 1 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 0 & 1 & 1 & 1 \\
\hline
\frac{25}{12} & \frac{31}{12} & \frac{13}{4} & \frac{7}{2} & \frac{23}{6} & \frac{49}{12} & \frac{37}{12} & \frac{25}{12}
\end{array}$$

The answer results after dividing by each entry by 8:

$$z[0]..z[7] = \frac{25}{96}, \frac{31}{96}, \frac{13}{32}, \frac{7}{16}, \frac{23}{48}, \frac{49}{96}, \frac{37}{96}, \frac{25}{96}$$

Continuous-Time Transforms

Consider the following three CT signals.

$$f_1(t) = \begin{cases} a & \text{if } |t| < b \\ 0 & \text{otherwise} \end{cases}$$

$$f_2(t) = f_1(t) + c\delta(t)$$

$$f_3(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} f_2(t + mT)$$

The plots on the following page show the Fourier transforms for a number of different parameter values.

Part a. Determine which plot shows the Fourier transform of $f_2(t)$ for $a = 3$, $b = 1/2$, and $c = 1$.

Enter label of plot: $\boxed{B}$

Part b. Determine which plot shows the Fourier transform of $f_2(t)$ for $a = 3/2$, $b = 1$, and $c = 1$.

Enter label of plot: $\boxed{C}$

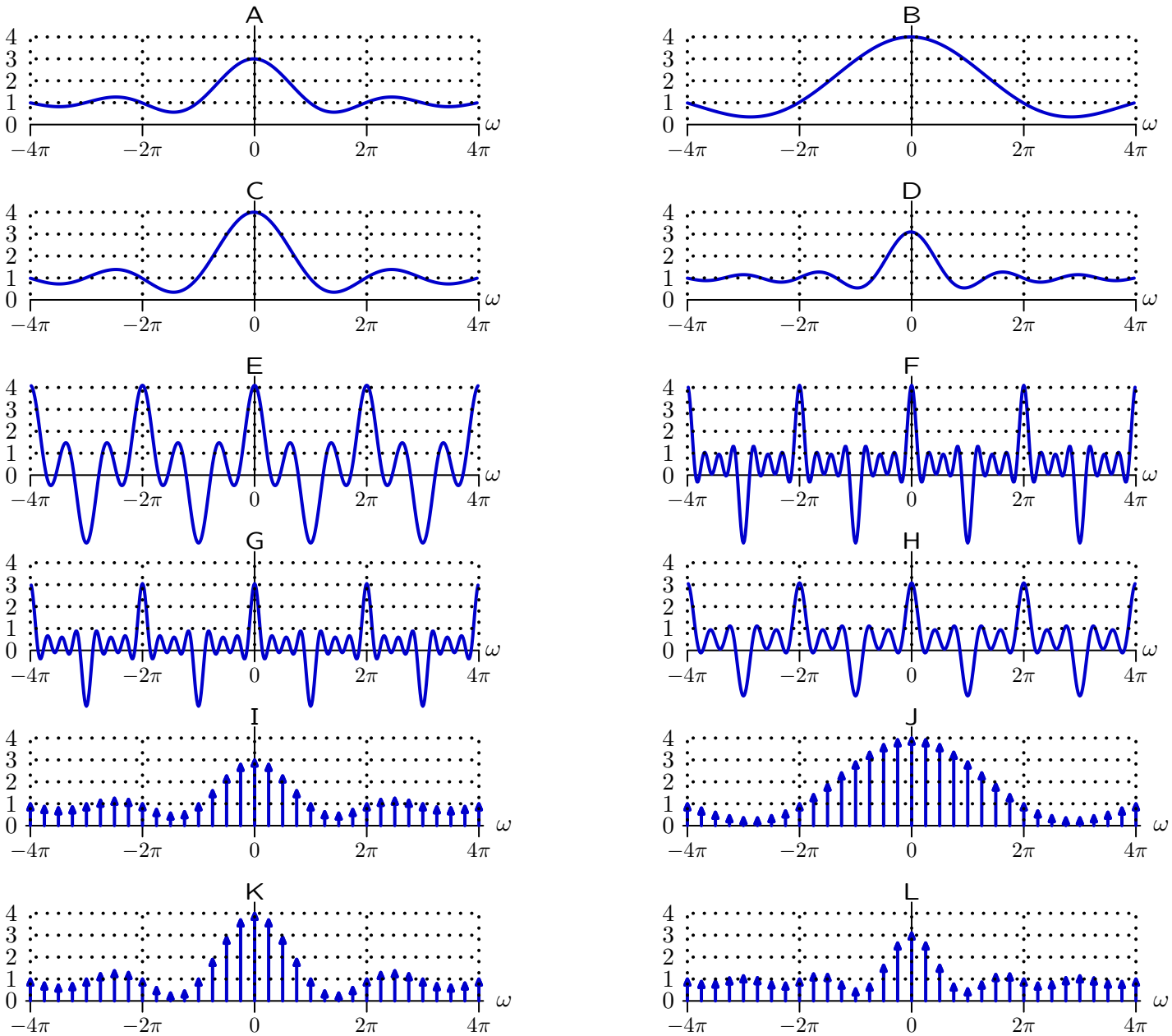
Part c. Determine which plot shows the Fourier transform of $f_3(t)$ for $a = 3$, $b = 1/2$, $c = 1$, and $T = 8$?

Enter label of plot: $\boxed{J}$

Part d. Determine which plot shows the Fourier transform of $f_3(t)$ for $a = 3/2$, $b = 1$, $c = 1$, and $T = 8$?

Enter label of plot: $\boxed{K}$

These panels show Fourier transforms. The horizontal scales are -4π to 4π for all panels. The vertical scales for panels A–H are 0 to 4 as indicated by numbers to the left of the plots. For panels I–L, the numbers to the left of the plots indicate the areas of the impulses.



Worksheet (intentionally blank)