6.003: Signal Processing

Synthetic Aperture Optics

- Fourier Optics
- Synthetic Aperture Microscopy

Announcements

- End-of-Term Subject Evaluations due Monday, December 13 at 9am.
- Regular office hours end this Friday, December 10 at 4pm.
- Solutions to the Practice Final Exam are posted.
- **Final Exam**: Thursday, December 16: 1:30-4:30pm, Dupont Gym.

*December 7, 2021*
Why Focus on Fourier?

What’s so special about sines and cosines?

Sinusoidal functions have interesting mathematical properties. → harmonically related sinusoids are orthogonal to each other over $[0, T]$.

Sines and cosines also play important roles in physics – especially the physics of waves.
Physical Example: Vibrating String

A taut string supports wave motion.

The speed of the wave depends on the tension on and mass of the string.
Physical Example: Vibrating String

The wave will reflect off a rigid boundary.

The amplitude of the reflected wave is opposite that of the incident wave.
Physical Example: Vibrating String

Reflections can interfere with excitations.

The interference can be constructive or destructive depending on the frequency of the excitation.
**Physical Example: Vibrating String**

We get constructive interference if round-trip travel time equals the period.

\[
x = 0 \quad x = L
\]

Round-trip travel time \( T \) is given by

\[
\text{Round-trip travel time} = \frac{2L}{v} = T
\]

\[
\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}
\]
Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

Round-trip travel time $= \frac{2L}{v} = 2T$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$
Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is $kT$.

\[
\text{Round-trip travel time } = \frac{2L}{v} = kT
\]

\[
\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o
\]

Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist. This is the basis of stringed instruments.
Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is “plucked” at $x = l$. 
Physical Example: Vibrating String

Differences in harmonic structure generate differences in timbre.
Optical Imaging

Images from even the best microscopes are blurred. Blurring is a fundamental property of lenses.
Optical Imaging

A perfect lens transforms a spherical wave of light from a target into a spherical wave that converges to an image of the target.
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Blurring is inversely related to the diameter of the lens.
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![Diagram of optical imaging](image)

Blurring is inversely related to the diameter of the lens.
Optical Imaging

A perfect lens transforms a spherical wave of light from a target into a spherical wave that converges to an image of the target.

Blurring is inversely related to the diameter of the lens.
Optical Imaging

Today’s lecture is on how the size of a lens affects image resolution, and how Fourier representations can be used to understand (and even overcome some of) these limitations.
If a target is located in the focal plane of a lens, light from a point on the target forms a plane wave as it passes through the lens.

If the target point lies on the axis of the lens, then the plane wave is perpendicular to the imaging plane.
Fourier Optics

If a target is located in the focal plane of a lens, light from a point on the target forms a plane wave as it passes through the lens.

If the target point lies on the axis of the lens, then the plane wave is perpendicular to the imaging plane.

If the target point lies off the axis of the lens, then the plane wave is no longer perpendicular to the image plane. The light striking the image plane has linearly increasing phase delay with distance.
Fourier Optics

Light from the point $x=0$ generates a plane wave, that is everywhere in phase at the imaging plane.

Light from $x=x_o$ generates a plane wave with linearly increasing phase lag.
Fourier Optics

The target can be described as a collection of point sources of light

\[ f(x) = \int f(x_o) \delta(x - x_o) \, dx_o \]

and the result in the image plane is a superposition of plane waves, one for each point in the target.

\[ g(\omega) = \int f(x) e^{-j\omega x} \, dx = F(\omega) \]

Notice that \( g(\omega) = F(\omega) \) is the Fourier transform of \( f(x) \).

**Fourier Optics:** \( f(x) \overset{\text{CTFT}}{\leftrightarrow} F(\omega) \)
Fourier Optics

If an object is more than one focal distance from the lens, then the light converges to create an image of the object in the image plane.

This is equivalent to two lenses: one located a focal distance from the object and one located a focal distance from the image.
Fourier Optics

Now the Fourier transform relation holds for both halves of the system.

\[
F(\omega) = \int f(x) e^{-j\omega x} \, dx
\]

\[
f'(x') = \frac{1}{2\pi} \int F(\omega) e^{j\omega x'} \, d\omega
\]

Ideally, both limits of integration would be infinite. However the finite diameter of the lens limits the highest frequencies \(|\omega|\).
Fourier Optics

Light emanating from the target at large angles is not captured by the lens.

\[ F(\omega) = \int f(x) e^{-j\omega x} \, dx \]

\[ f'(x') = \frac{1}{2\pi} \int F(\omega) e^{j\omega x'} \, d\omega \]

As a result, the image at \( x' \) is a lowpass version of the target at \( x \).
Microscopy with 6.003

Dennis M. Freeman
Stanley S. Hong
Jekwan Ryu
Michael S. Mermelstein
Berthold K. P. Horn
6.003 Model of a Microscope

Microscope = low-pass filter
Phase-Modulated Microscopy

[Diagram showing the process of phase-modulated microscopy, including an image of a rose, a pattern of stripes, and the label 'microscope'.]
Phase-Modulated Microscopy

Poster: \[ \cos(\omega_c y + f(x,y)) \]
Phase-Modulated Microscopy

Poster: \[ \cos(\omega_c y + f(x,y)) \]

Projector: \[ \cos(\omega_c y) \]
Phase-Modulated Microscopy

Poster:
\[ \cos(\omega_c y + f(x,y)) \]

Projector:
\[ \cos(\omega_c y) \]

Poster with Projector:
\[ \cos(\omega_c y) \cos(\omega_c y + f(x,y)) \]

Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies.
Phase-Modulated Microscopy

$$X(\omega)$$

visible

$$\omega_c$$

low-pass

$$-2\omega_c$$

$$2\omega_c$$

Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies
Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies.
Phase-Modulated Microscopy

Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies
Phase-Modulated Microscopy

Modulated illumination enables low-pass system (eyes) to detect high spatial frequencies
Standing-wave illumination spectrum

Thanks to M. Mermelstein
Standing-wave illumination spectrum

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Thanks to M. Mermelstein
Optical transfer function

2 beams
Optical transfer function

3 beams
Optical transfer function

4 beams
Optical transfer function

5 beams
Optical transfer function

6 beams
Optical transfer function

7 beams
Aperture synthesis
Aperture synthesis
Aperture synthesis
Aperture synthesis
Aperture synthesis
Aperture synthesis

Combine multiple low-NA optics to *synthesize* high NA
M2: The first many-beam SAM
41 BEAMS IN A RING

MAKE PATTERNS LIKE THIS
PATTERNS LIKE THIS

HAVE TRANSFORMS LIKE THIS
Experimental apparatus

Stanley S. Hong
Uniform Illumination

Structured Illumination

Jekwan Ryu
Measurement of PSF

- SWI apparatus
- Isolated 200-nm-diameter fluorescent microsphere
- Objective lens

(Cross section, not to scale)
Measurement of PSF
Measurement of PSF
Measurement of PSF

Measured diameter = 290 nm
Measurement of PSF

Measured diameter = 290 nm
Predicted diameter = 250 nm
Measurement of PSF

Measured diameter = 290 nm
Predicted diameter = 250 nm
Diameter lens alone = 1,500 nm
Aperture synthesis

Combine multiple low-NA optics to \textit{synthesize} high NA
6.003 Approach to Increased Resolution

Uniform Illumination

Structured Illumination

Normalized intensity vs. Distance (µm)
Summary

Fourier transforms are important in many branches of physics, mathematics, electrical engineering, and computer science.

Today we saw how **Fourier optics** helps us to understand why optical systems blur.

We also introduced **Synthetic Aperture Optics** as a way to overcome some limitations of conventional optics.

– greatly reduced the blurring in conventional microscopy

This new method of optical imaging is directly inspired by Fourier transforms – and especially by the application of **modulation** to optics.