6.003: Signal Processing

2D Transforms

- Introduction to 2D signal processing
- 2D Fourier Representations

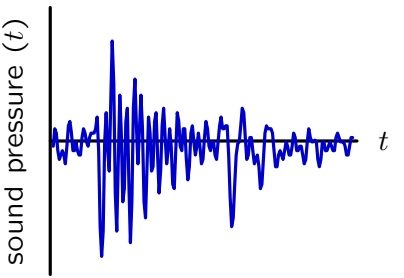
Thursday is a holiday (Veterans Day)

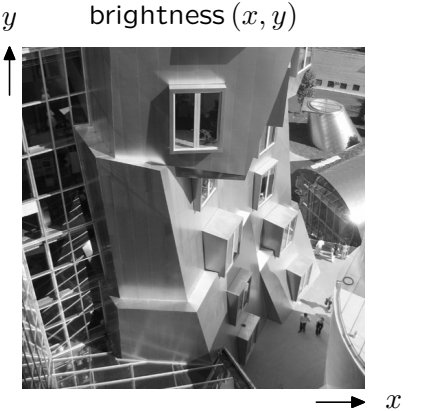
- no lecture, no recitation, no office hours 4-5pm
- regular office hours 7-9pm

- HW 10 is posted
  - includes lab 10
  - no check-in

Signals

Signals are functions that are used to convey information.
– may have 1 or 2 or 3 or even more **independent variables**





A 1D signal has a one-dimensional domain.
We usually thought of it as time t or discrete time n.

A 2D signal has a two-dimensional domain.
We will usually think of the domain as x and y or n_x and n_y.

Fourier Representations

From "Continuous Time" to "Continuous Space."

One dimensional CTFT:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Two dimensional CTFT:

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-j(\omega_x x + \omega_y y)}\, dx\, dy$$

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega_x, \omega_y)\, e^{j(\omega_x x + \omega_y y)}\, d\omega_x\, d\omega_y$$

integrals $\rightarrow$ double integrals; sum of x and y exponents in kernal function.

Fourier Representations

"Discrete Time" to "Discrete Space."

One dimensional DTFT:

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]\, e^{-j\Omega n}$$

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega)\, e^{j\Omega n} d\Omega$$

Two dimensional DTFT:

$$F(\Omega_x, \Omega_y) = \sum_{n_x=-\infty}^{\infty} \sum_{n_y=-\infty}^{\infty} f[n_x, n_y]\, e^{-j(\Omega_x n_x + \Omega_y n_y)}$$

$$f[n_x, n_y] = \frac{1}{4\pi^2} \int_{2\pi} \int_{2\pi} F(\Omega_x, \Omega_y)\, e^{j(\Omega_x n_x + \Omega_y n_y)} d\Omega_x d\Omega_y$$

double integrals; double sums; sum of x and y exponents in kernal function.

Fourier Representations

1D DFT to 2D DFT.

One dimensional DFT:

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n]\, e^{-j\frac{2\pi k}{N} n}$$

$$f[n] = \sum_{k=0}^{N-1} F[k]\, e^{j\frac{2\pi k}{N} n}$$

Two dimensional DFT:

$$F[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f[n_x, n_y]\, e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

$$f[n_x, n_y] = \sum_{k_x=0}^{N_x-1} \sum_{k_y=0}^{N_y-1} F[k_x, k_y]\, e^{j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

double sums; sum of x and y exponents in kernal function.

Importance of Orthogonality

Fourier series represent periodic signals as weighted sum of **basis functions**.

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi}{N}kn}$$

We "sifted" out the l^{th} component by multiplying both sides by $e^{-j\frac{2\pi}{N}ln}$ and summing over a period.

$$\sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}ln} = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} F[k] e^{j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}ln} = \sum_{k=0}^{N-1} F[k] \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n}$$

$$= \begin{cases} NF[l] & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

This sifting provided an explicit "analysis" formula for the coefficients:

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}kn}$$

Orthogonality of the basis functions is key to Fourier decomposition.

Orthogonality

The form of the 2D Fourier kernal preserves orthogonality.

1D DFT basis functions: $\phi_k[n] = e^{j\frac{2\pi}{N}kn}$

"Inner product" of 1D basis functions:

$$\sum_n \phi_k^*[n]\phi_l[n] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} e^{j\frac{2\pi}{N}ln} = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-l)n} = N\delta(k-l)$$

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2D DFT basis functions: $\phi_{kx,ky}[n_x, n_y] = e^{j\frac{2\pi}{N_x}k_x n_x} e^{j\frac{2\pi}{N_y}k_y n_y}$

"Inner product" of 2D basis functions:

$$\sum_{nx,ny} \phi_{kx,ky}^*[n_x, n_y]\phi_{lx,ly}[n_x, n_y] = \sum_{nx,ny} e^{-j\left(\frac{2\pi}{Nx}kxnx + \frac{2\pi}{Ny}kyny\right)} e^{j\left(\frac{2\pi}{Nx}lxnx + \frac{2\pi}{Ny}lyny\right)}$$

$$= \left(\sum_{nx} e^{-j\frac{2\pi}{Nx}(kx-lx)nx}\right)\left(\sum_{ny} e^{-j\frac{2\pi}{Ny}(ky-ly)ny}\right) = N_x N_y \delta[k_x - l_x]\delta[k_y - l_y]$$

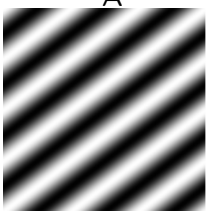
Check Yourself

The 2D Fourier basis functions have the following form.

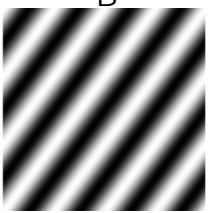
$$\phi_{k_x,k_y}[n_x, n_y] = e^{j\left(\frac{2\pi k_x}{Nx} n_x + \frac{2\pi k_y}{Ny} n_y\right)}$$

Which (if any) of the following images show the real part of one of the basis functions $\phi_{k_x,k_y}[n_x, n_y]$?

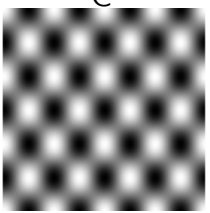
A



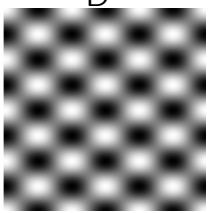
B



C



D



What values of k_x and k_y correspond to basis function?

Check Yourself

The 2D Fourier basis functions have the following form.

$$\begin{aligned}\phi_{kx,ky}[n_x, n_y] &= e^{j\left(\frac{2\pi k_x}{N_x}n_x + \frac{2\pi k_y}{N_y}n_y\right)} \\ &= \cos\left(\frac{2\pi k_x}{N_x}n_x + \frac{2\pi k_y}{N_y}n_y\right) + j\sin\left(\frac{2\pi k_x}{N_x}n_x + \frac{2\pi k_y}{N_y}n_y\right)\end{aligned}$$

If $\frac{2\pi k_x}{N_x}n_x + \frac{k_y}{N_y}n_y$ is constant, the complex exponential will be constant.

Example: Let k_x=3 and k_y= $-$ 4 when $N_x = N_y = 128$.

Then the exponent is $\frac{2\pi 3}{128}n_x - \frac{2\pi 4}{128}n_y$.

This exponent is zero if $3n_x = 4n_y$.

If the exponent is zero, then cosine is at its peak value of 1.

Thus the real part of the 2D basis function is 1 along the line $n_y = \frac{3}{4}n_x$.

$\rightarrow$ panel A (previous slide)

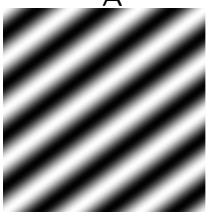
Check Yourself

The 2D Fourier basis functions have the following form.

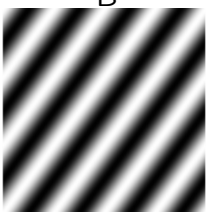
$$\phi_{k_x,k_y}[n_x, n_y] = e^{j(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y)}$$

Which (if any) of the following images show the real part of one of the basis functions $\phi_{k_x,k_y}[n_x, n_y]$? A and B

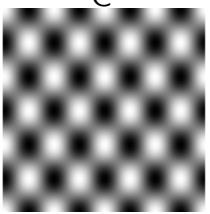
A



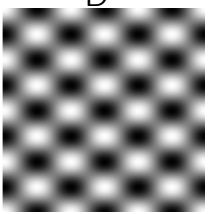
B



C



D



What values of k_x and k_y correspond to basis function?

A: (3,-4) or (-3,4); B: (4,-3) or (-4,3); C: none; D: none

2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.

$$f_0[n_x, n_y] = \delta[n_x]\delta[n_y] = \begin{cases} 1 & n_x = 0 \text{ and } n_y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_0[k_x, k_y] = \frac{1}{N_x N_y} \sum_{nx=0}^{N_x-1} \sum_{ny=0}^{N_y-1} \delta[n_x]\delta[n_y]\, e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

$$= \frac{1}{N_x N_y} \sum_{nx=0} \sum_{ny=0} e^{-j\left(\frac{2\pi k_x}{N_x} 0 + \frac{2\pi k_y}{N_y} 0\right)}$$

$$= \frac{1}{N_x N_y}$$

$$\delta[n_x]\delta[n_y] \overset{\text{DFT}}{\Longleftrightarrow} \frac{1}{N_x N_y}$$

2D Discrete Fourier Transform

Alternatively, implement a 2D DFT as a sequence of 1D DFTs.

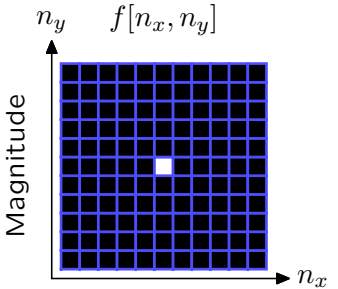
$$F[k_x, k_y] = \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} f[n_x, n_y]\, e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

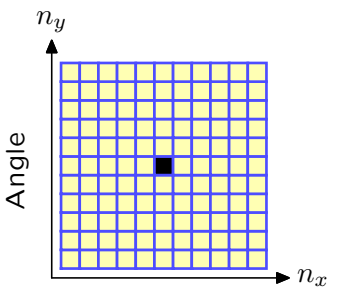
$$= \underbrace{\frac{1}{N_y} \sum_{n_y=0}^{N_y-1} \underbrace{\left(\frac{1}{N_x} \sum_{n_x=0}^{N_x-1} f[n_x, n_y]\, e^{-j\frac{2\pi k_x}{N_x} n_x} \right)}_{\text{first take DFTs of rows}} e^{-j\frac{2\pi k_y}{N_y} n_y}}_{\text{then take DFTs of resulting columns}}$$

Could just as well start with columns and then do rows.

2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.





2D Discrete Fourier Transform

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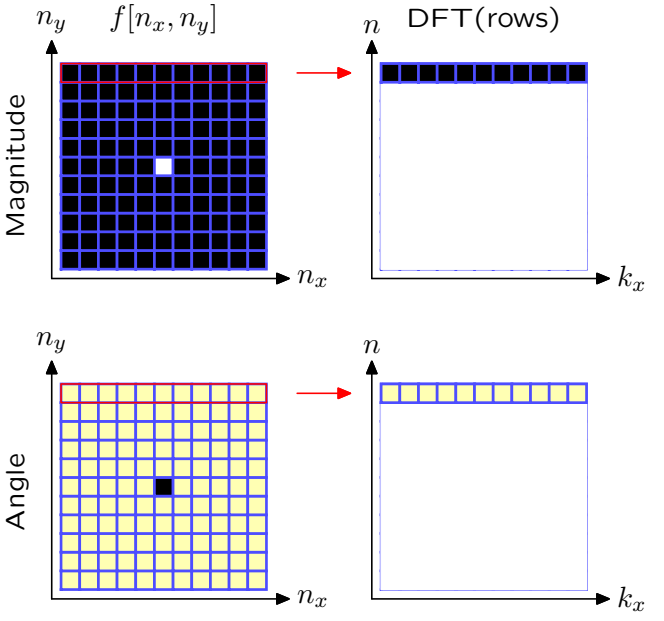

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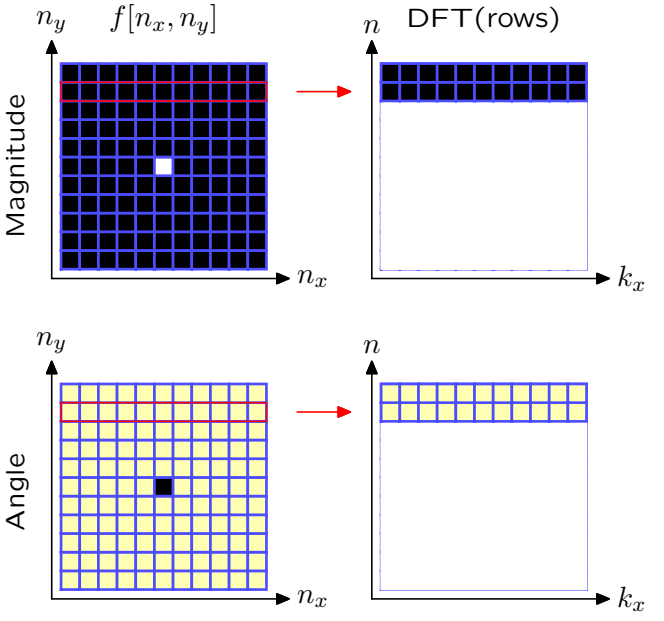

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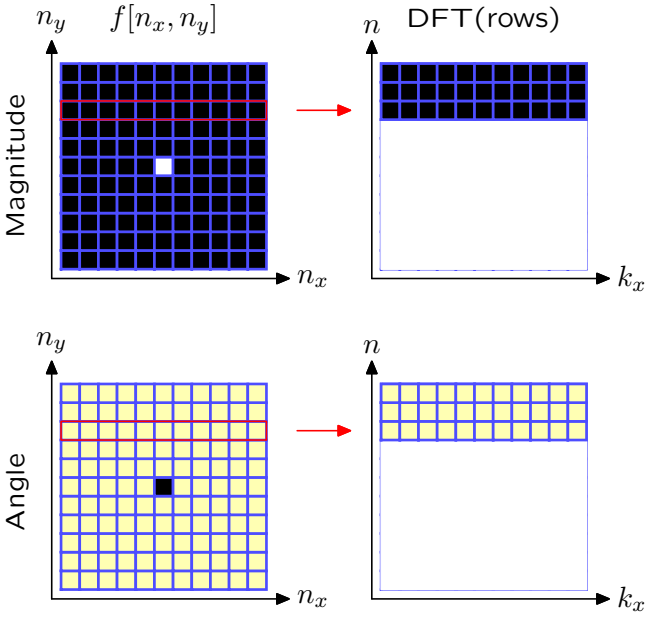

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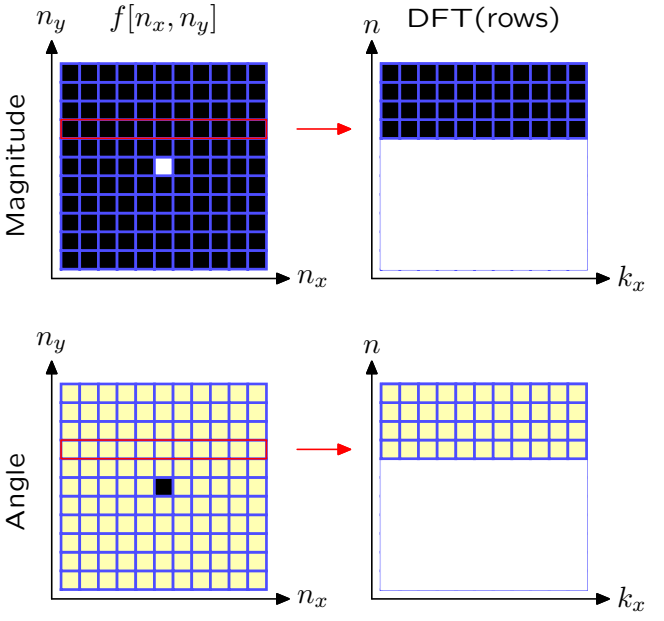

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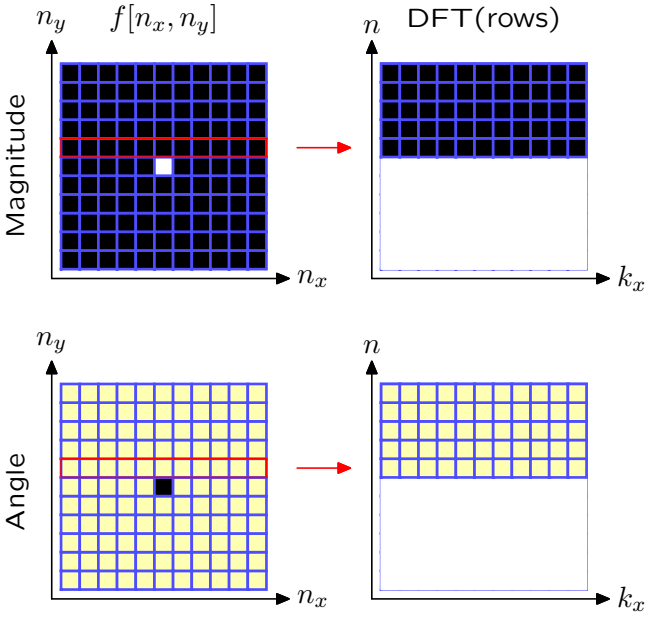

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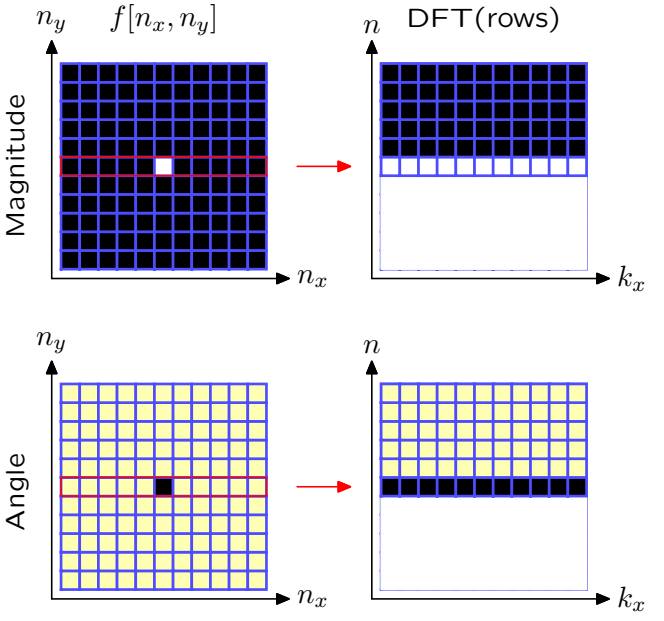

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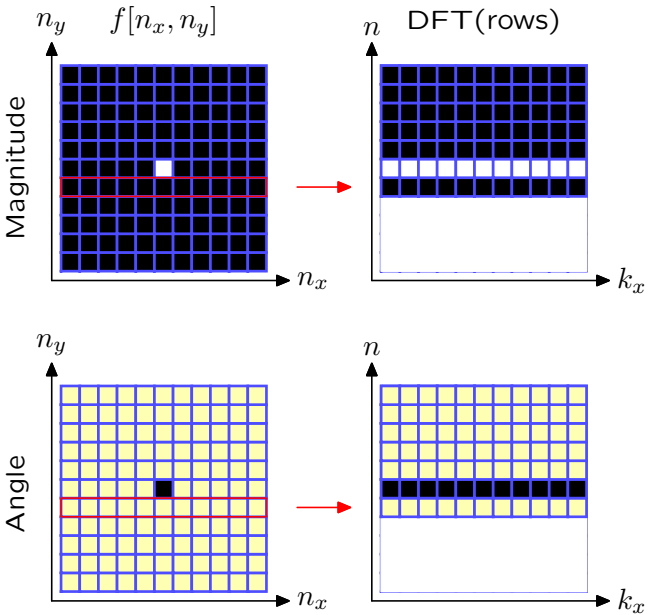

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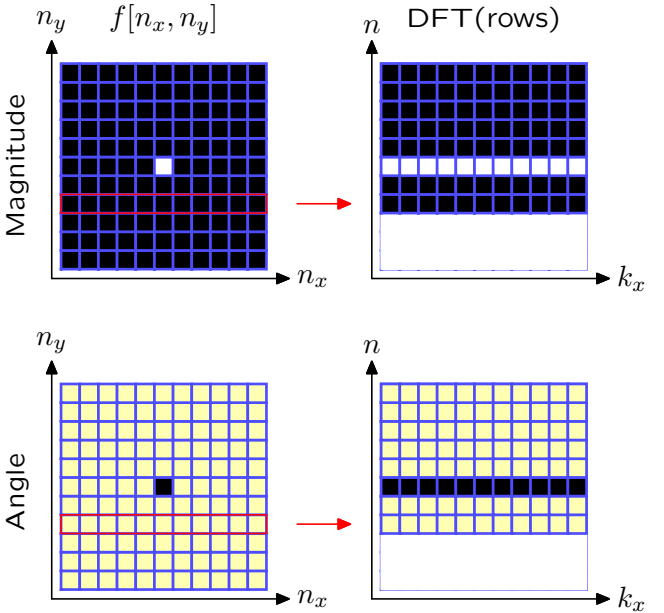

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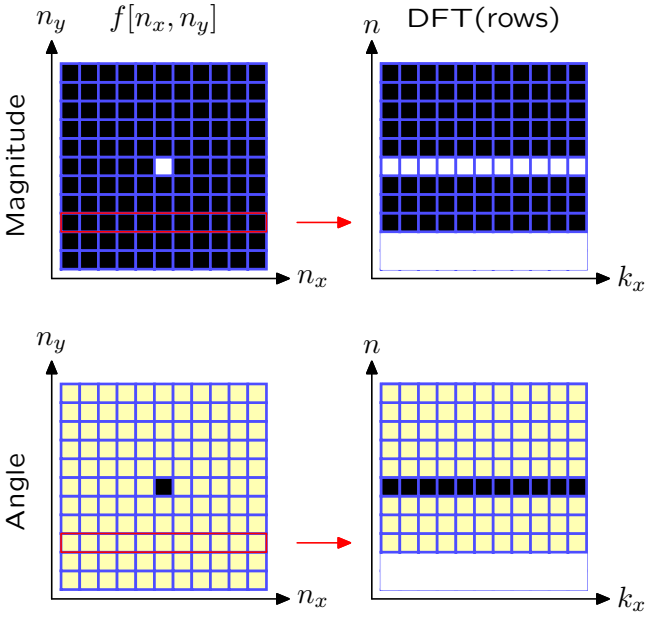

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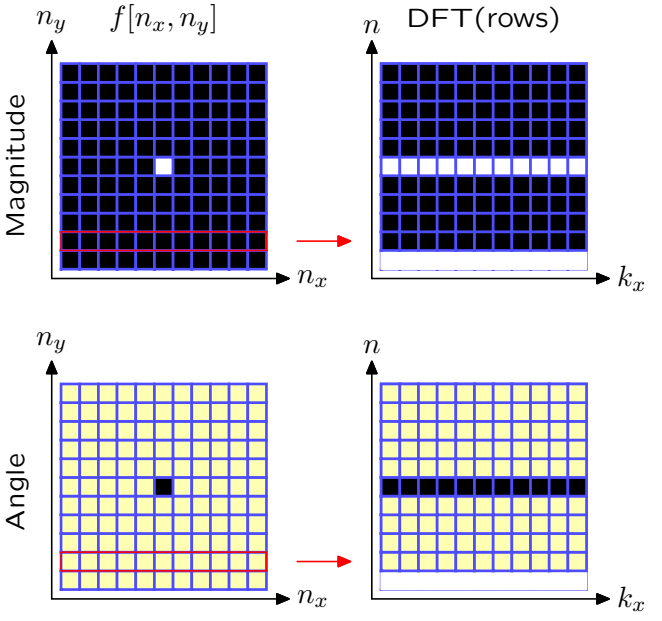

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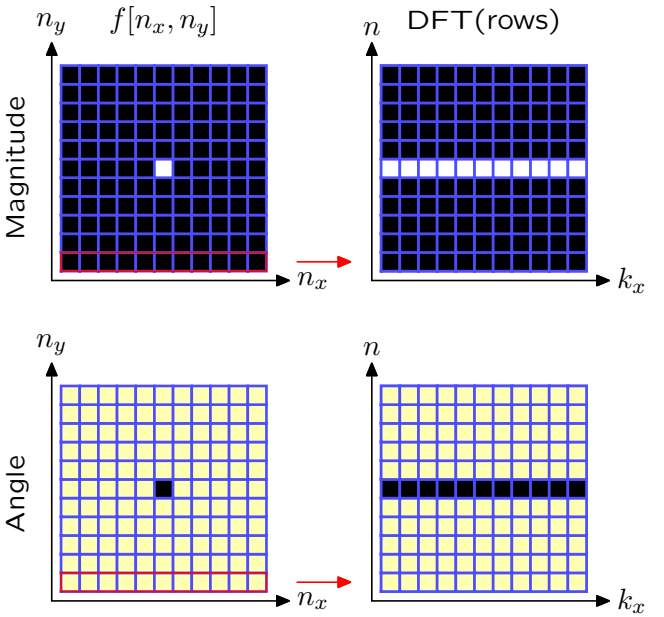

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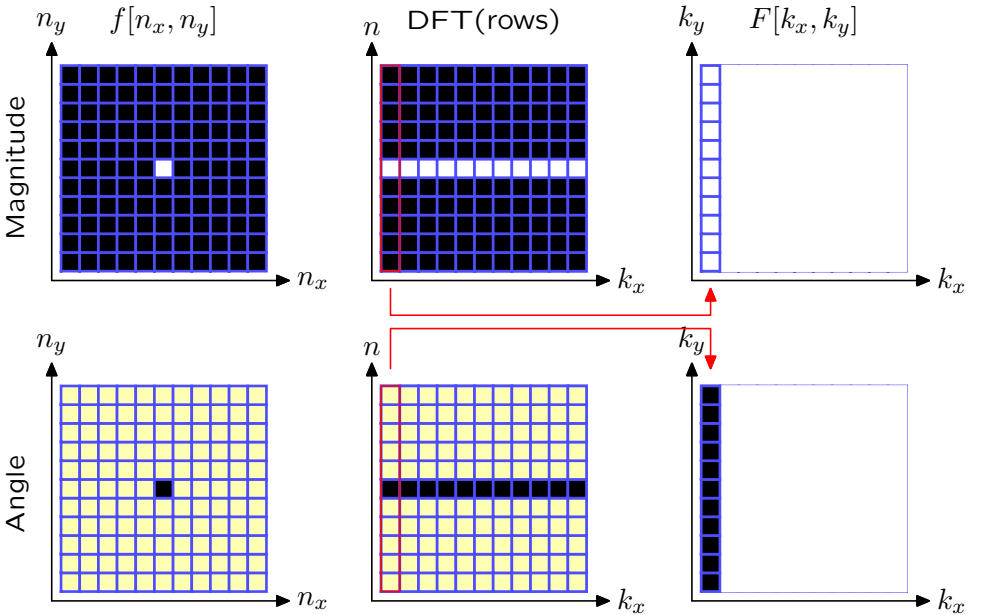

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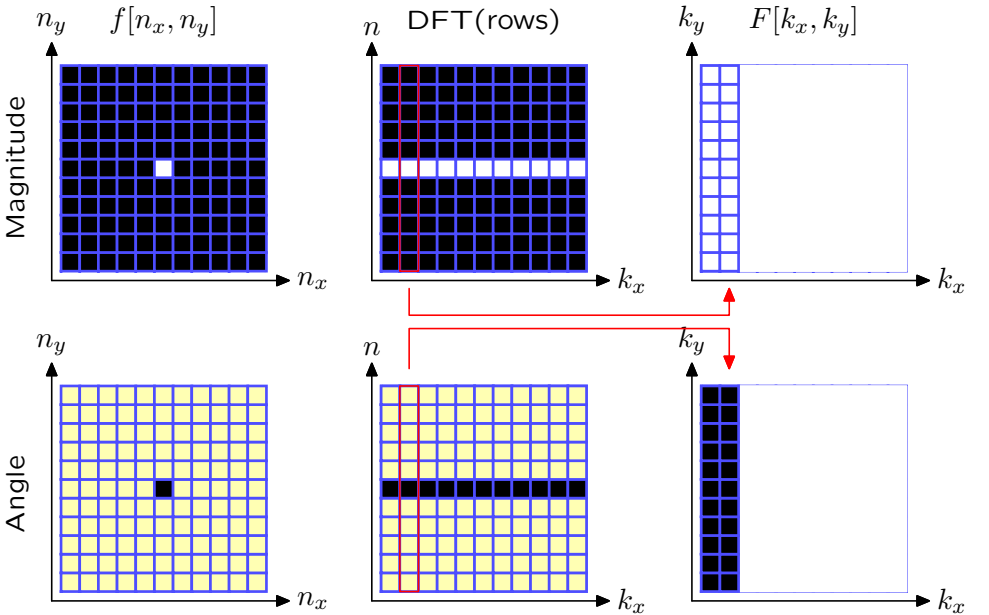

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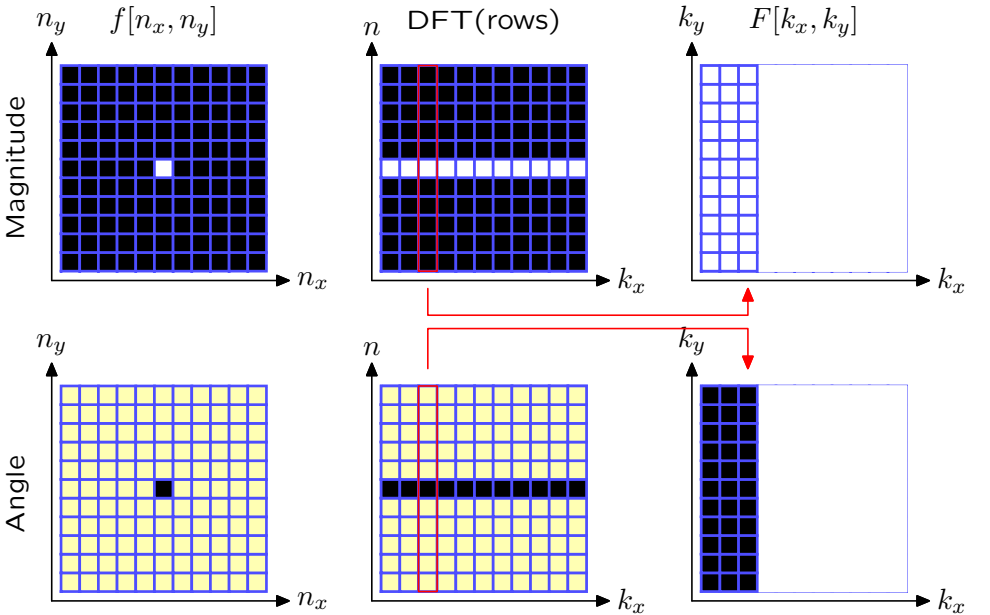

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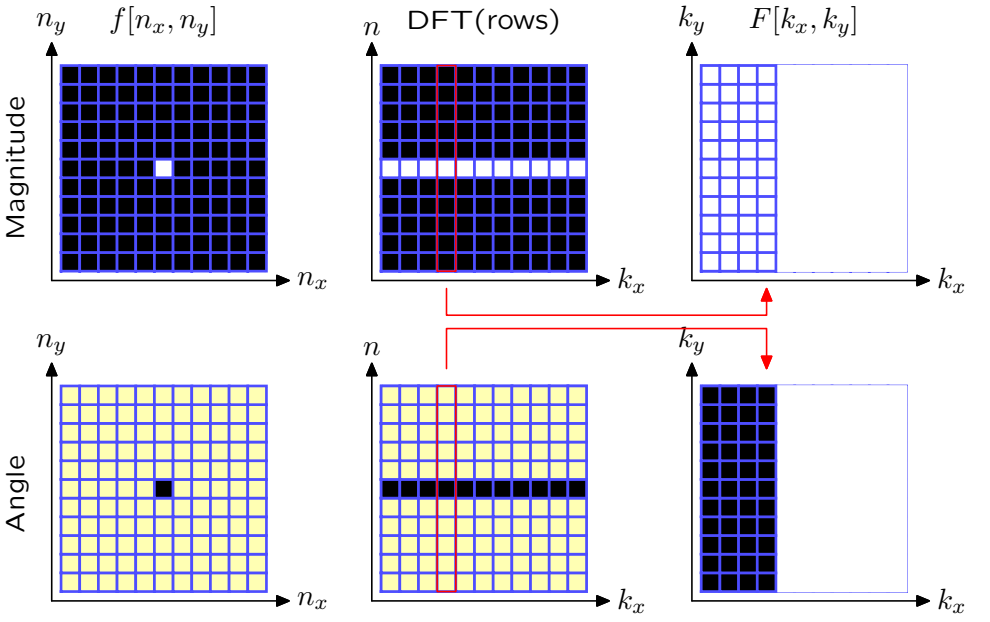

2D Discrete Fourier Transform

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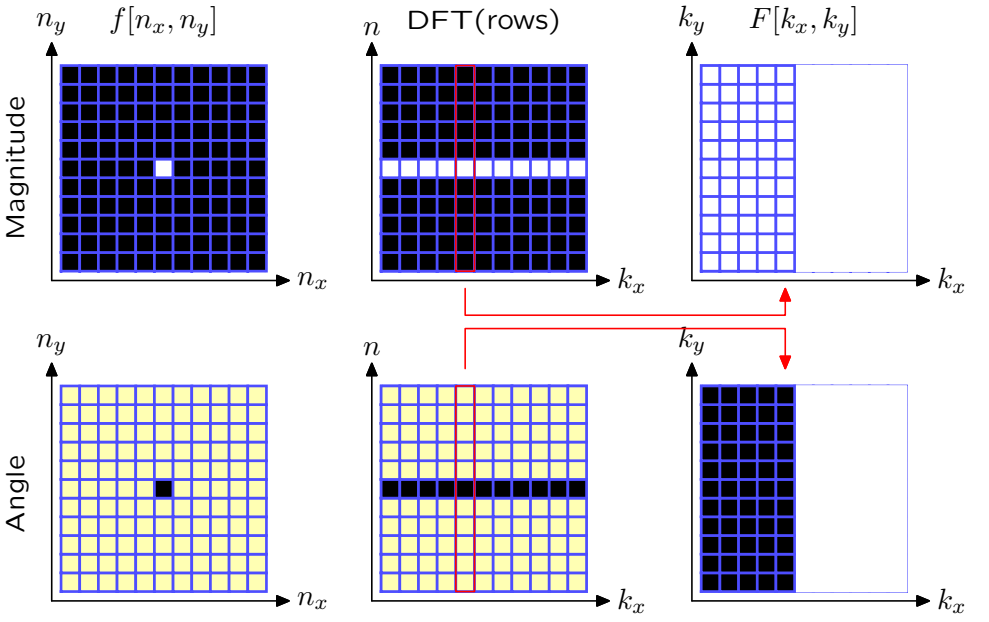

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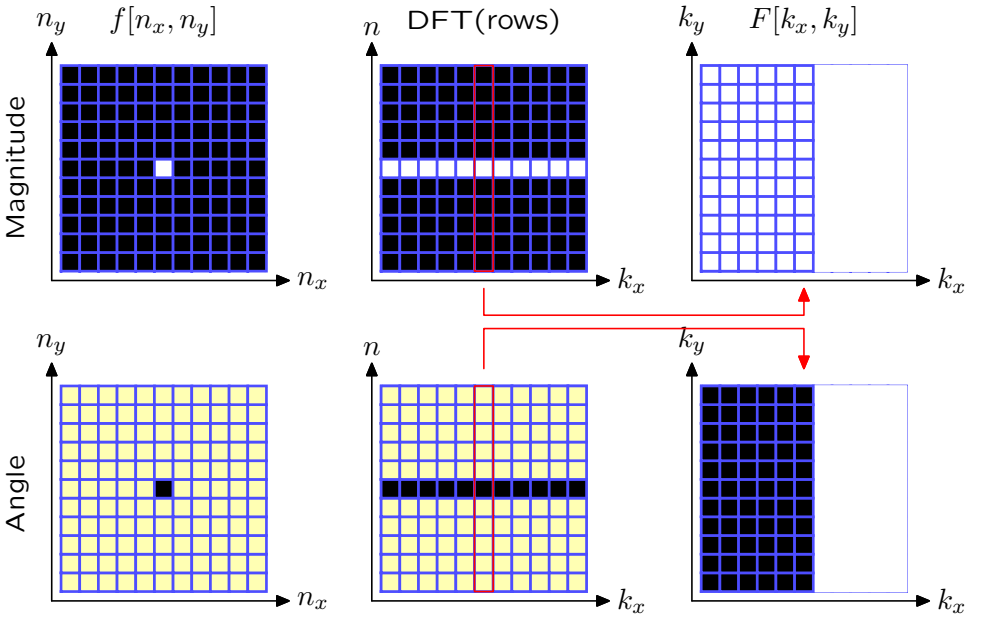

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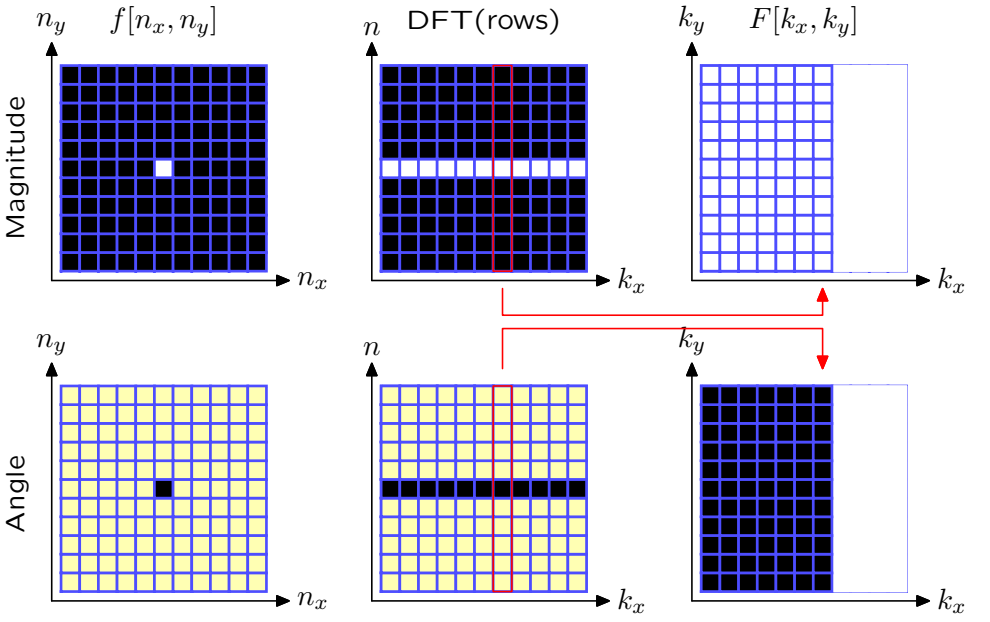

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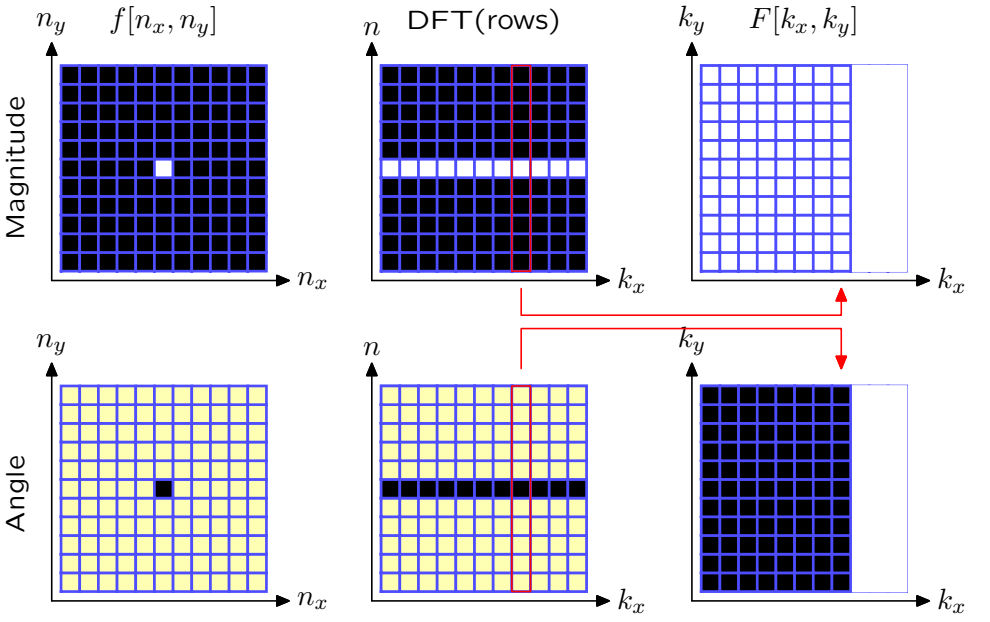
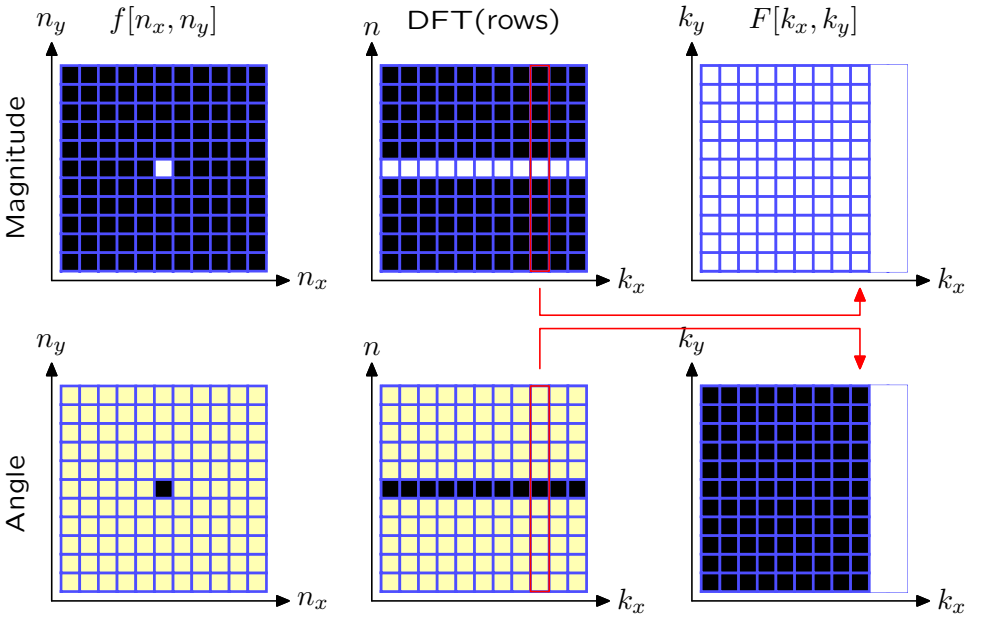


2D Discrete Fourier Transform

Example: Find the DFT of a 2D unit sample.



2D Discrete Fourier Transform

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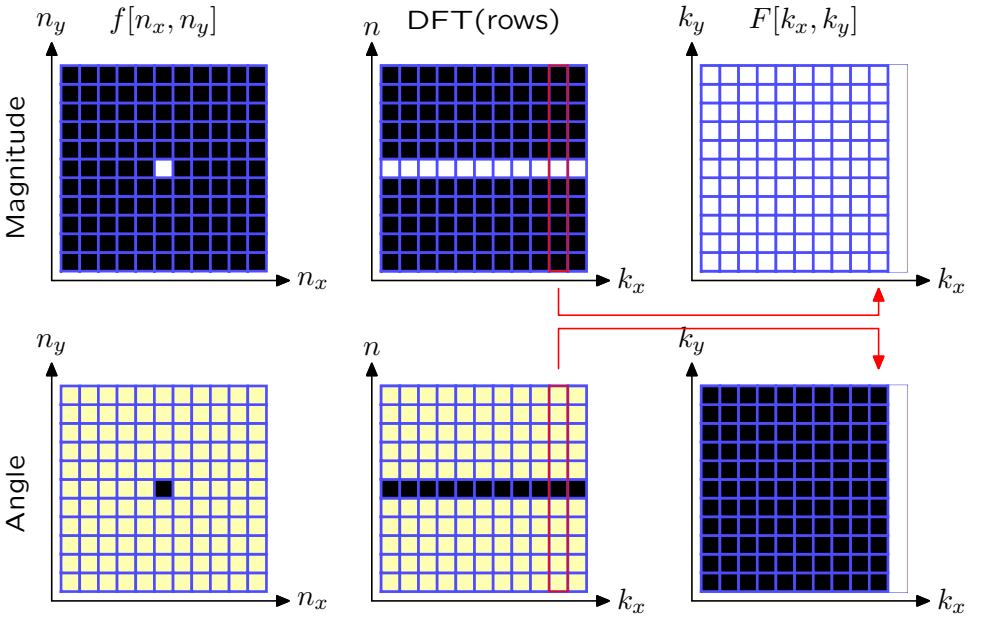

2D Discrete Fourier Transform

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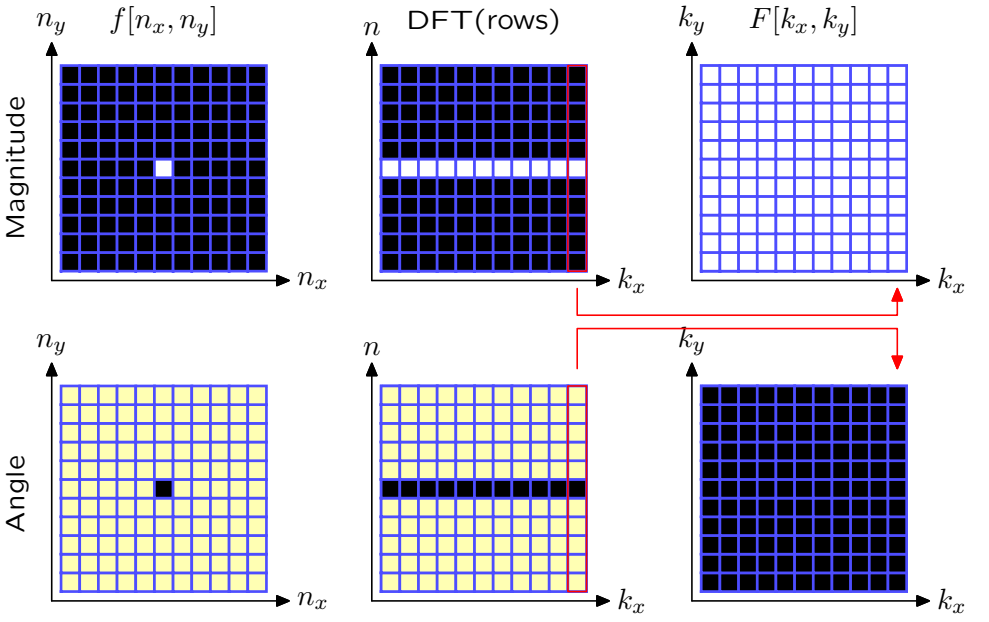

2D Discrete Fourier Transform

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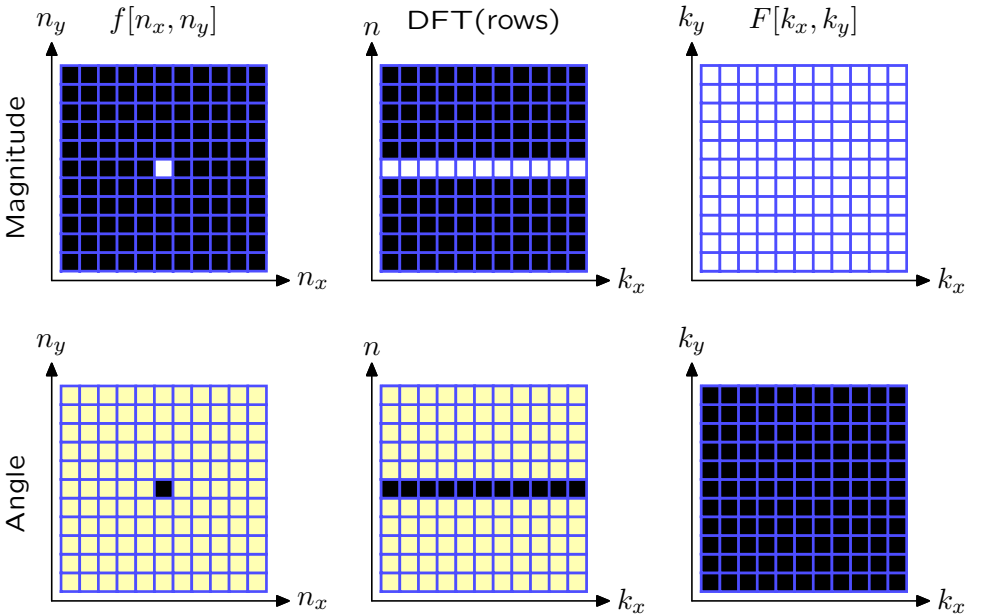

2D Discrete Fourier Transform

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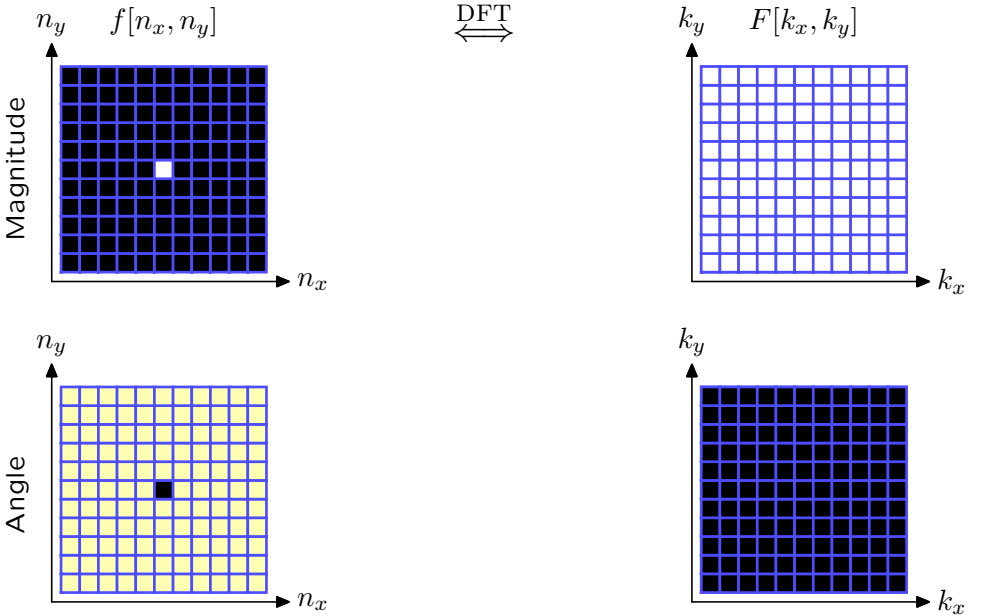


2D Discrete Fourier Transform

Example: Find the DFT of a constant.

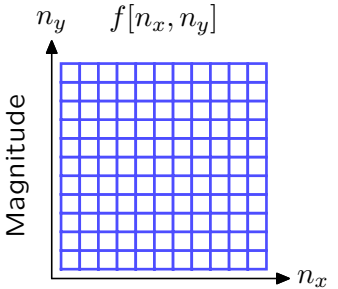
$$f_1[n_x, n_y] = 1$$

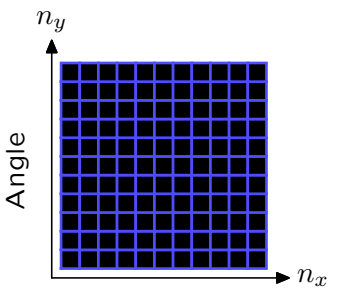
$$\begin{aligned} F_1[k_x, k_y] &= \frac{1}{N_x N_y} \sum_{n_x=0}^{N_x-1} \sum_{n_y=0}^{N_y-1} e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)} \\ &= \left(\frac{1}{N_x} \sum_{n_x=0}^{N_x-1} e^{-j\frac{2\pi k_x}{N_x} n_x} \right) \left(\frac{1}{N_y} \sum_{n_y=0}^{N_y-1} e^{-j\frac{2\pi k_y}{N_y} n_y} \right) \\ &= \delta[k_x]\delta[k_y] \end{aligned}$$

$$1 \quad \overset{\text{DFT}}{\Longleftrightarrow} \quad \delta[k_x]\delta[k_y]$$

2D Discrete Fourier Transform

Example: Find the DFT of a constant.





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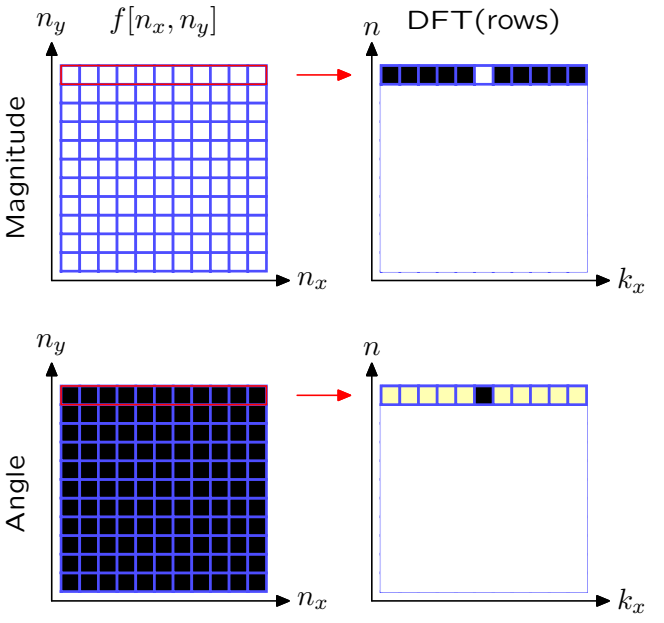

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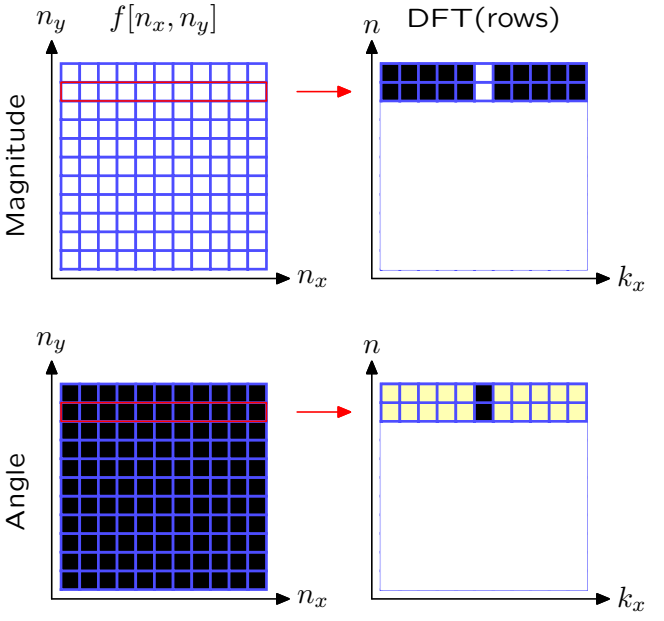

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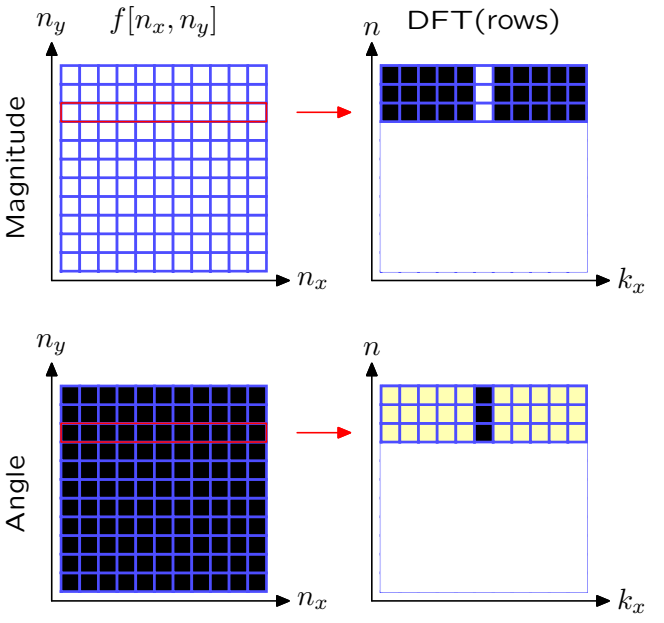

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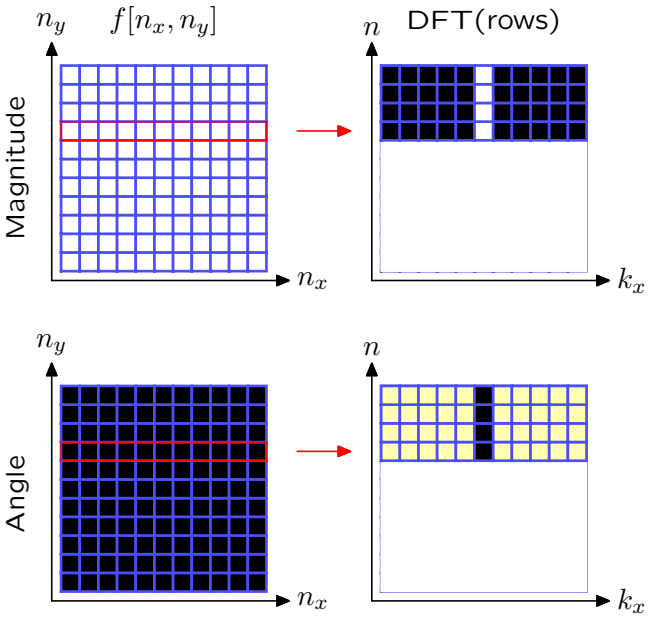

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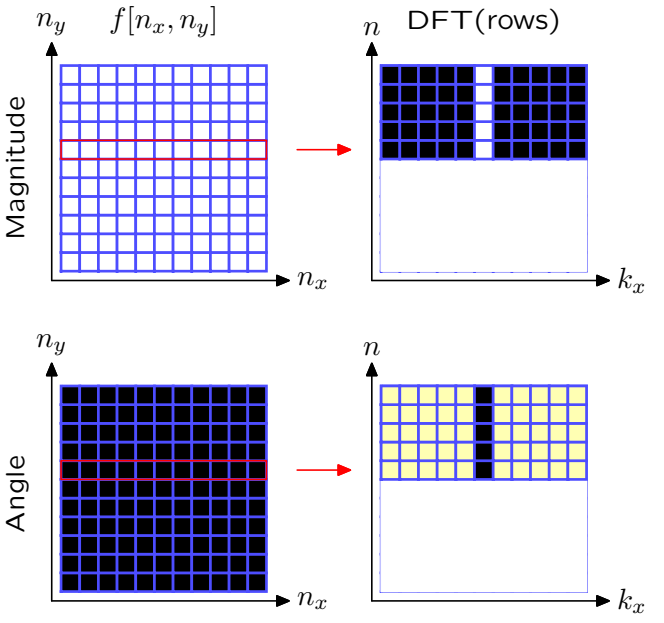

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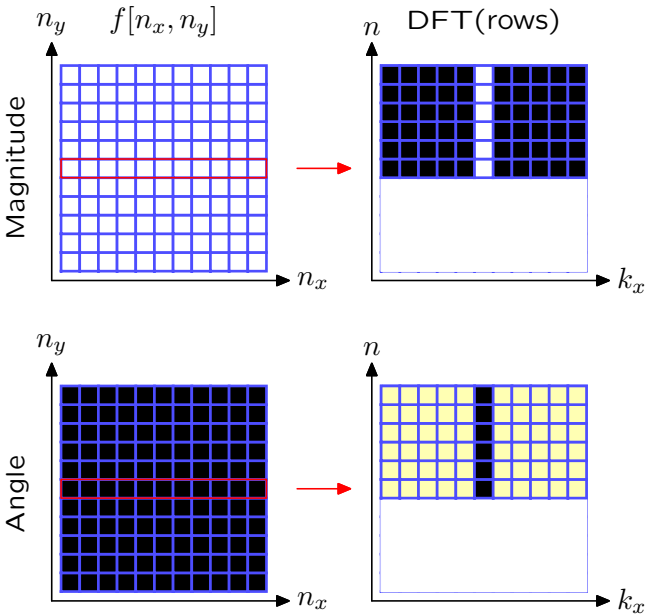

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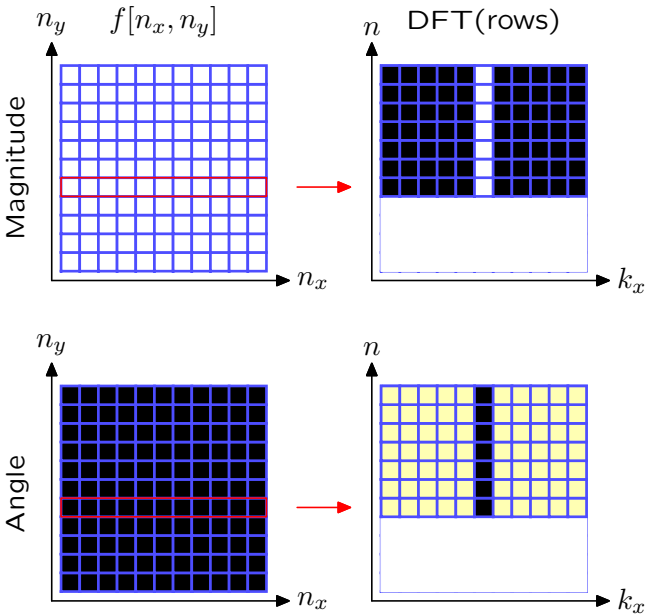

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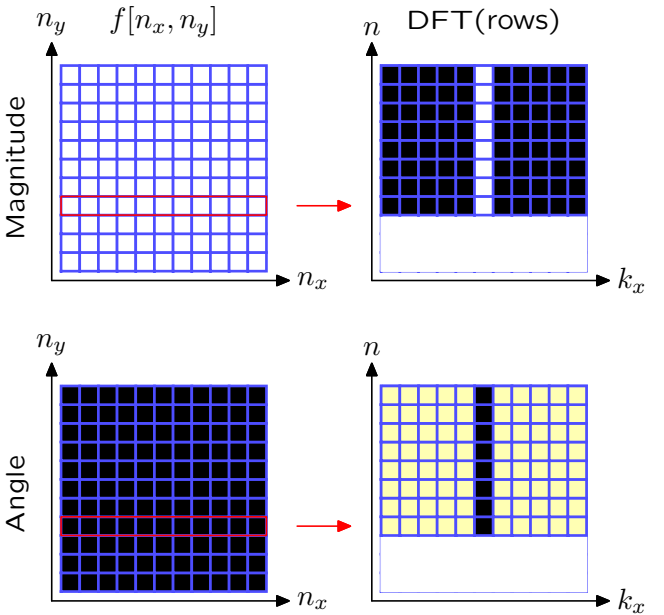

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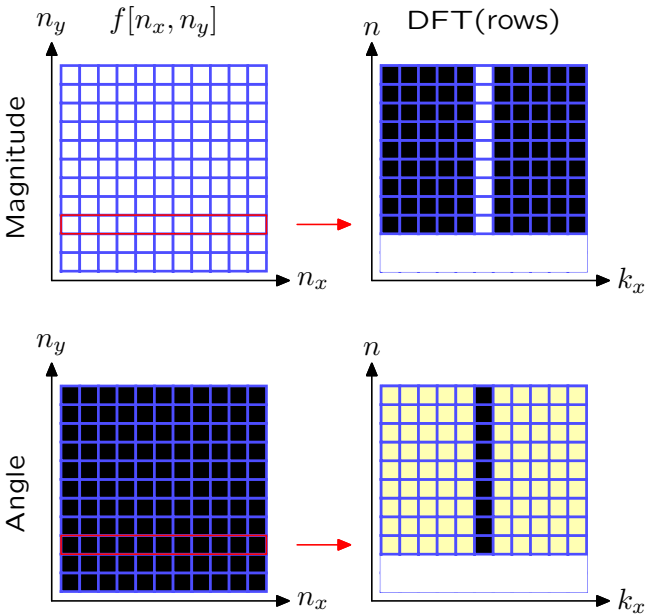

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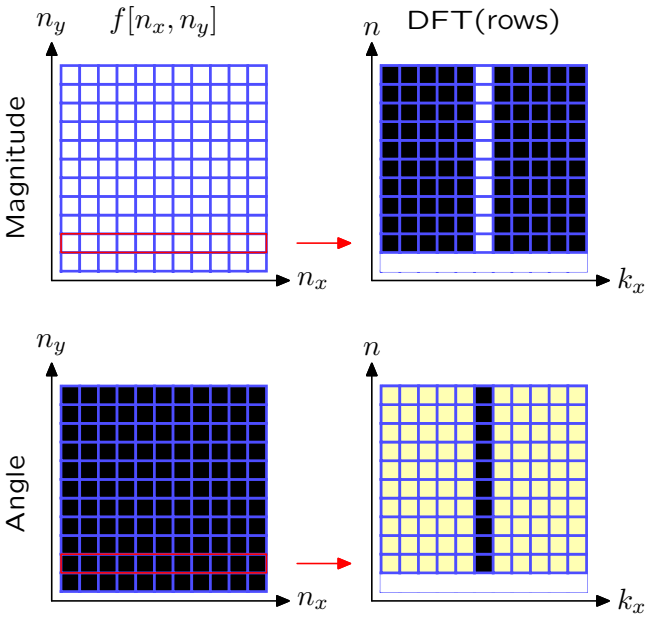

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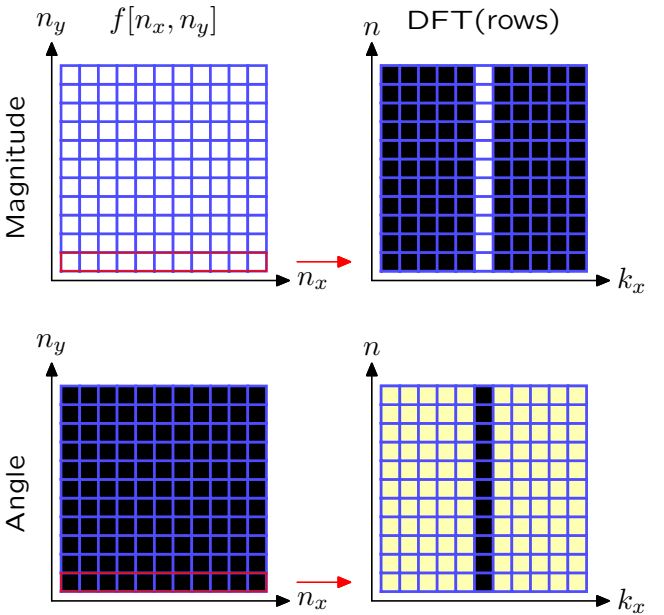

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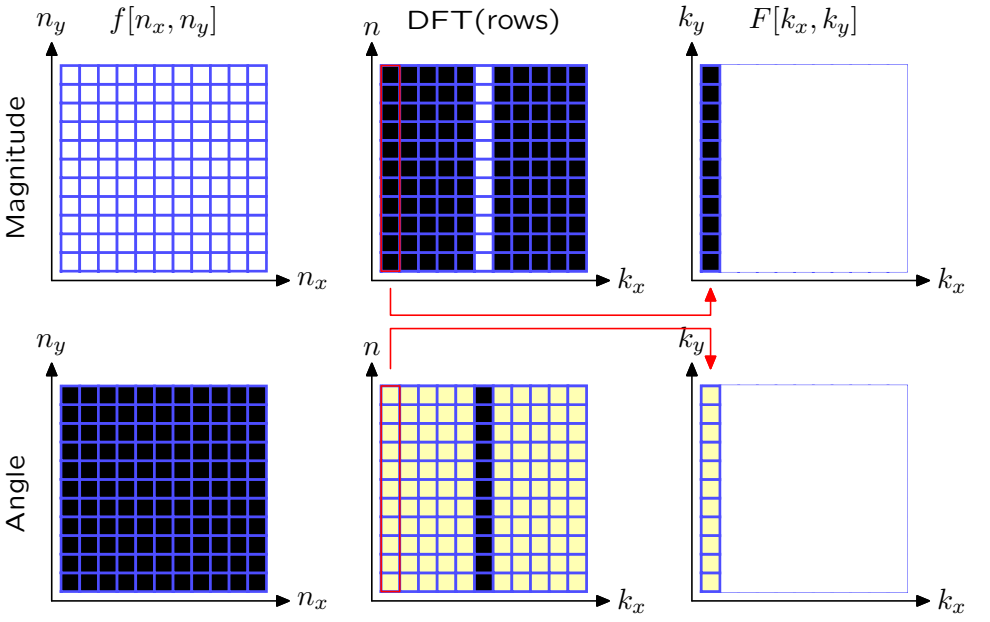

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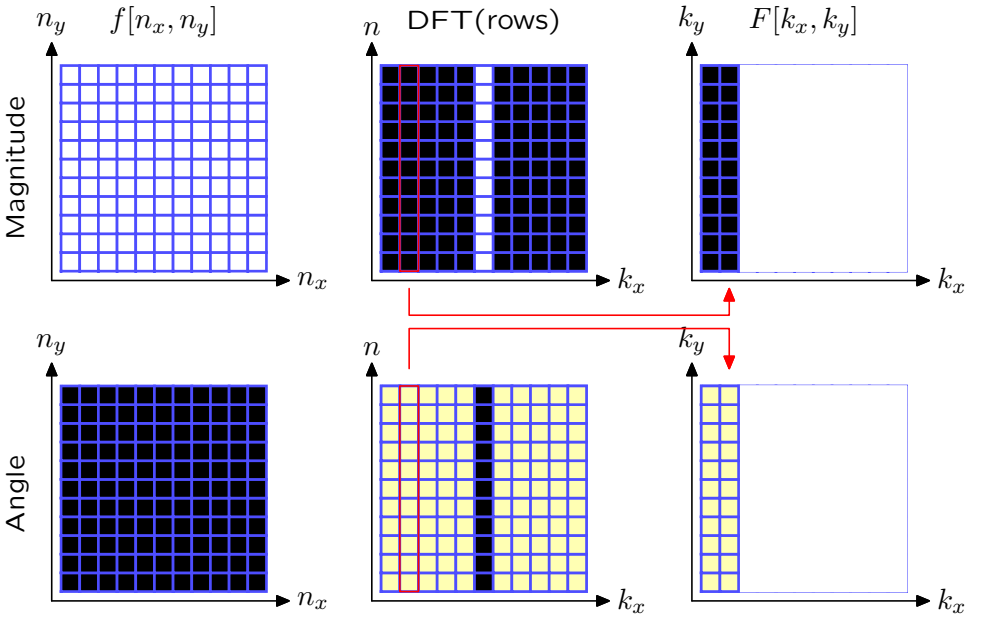

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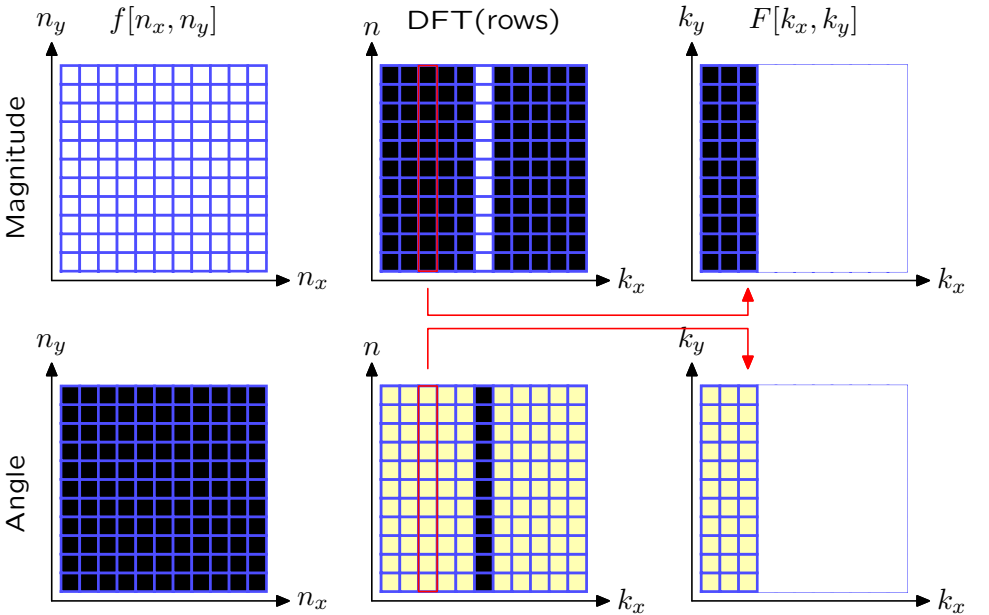

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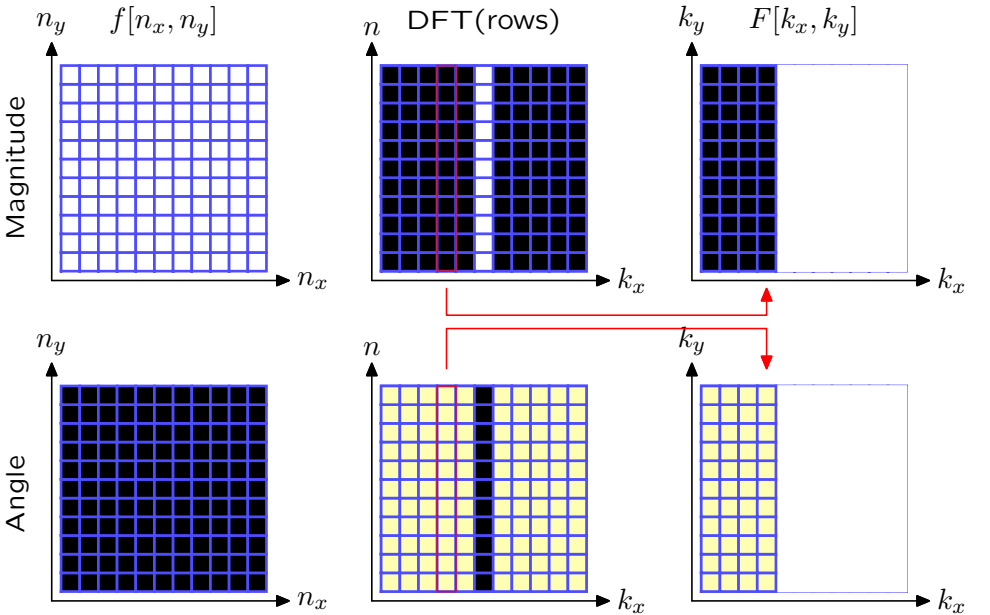

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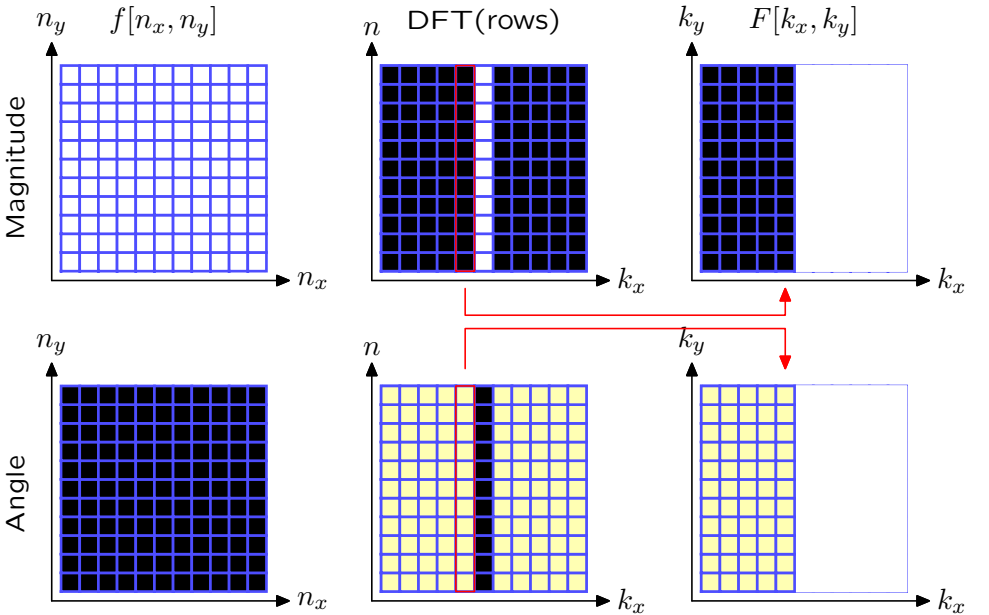

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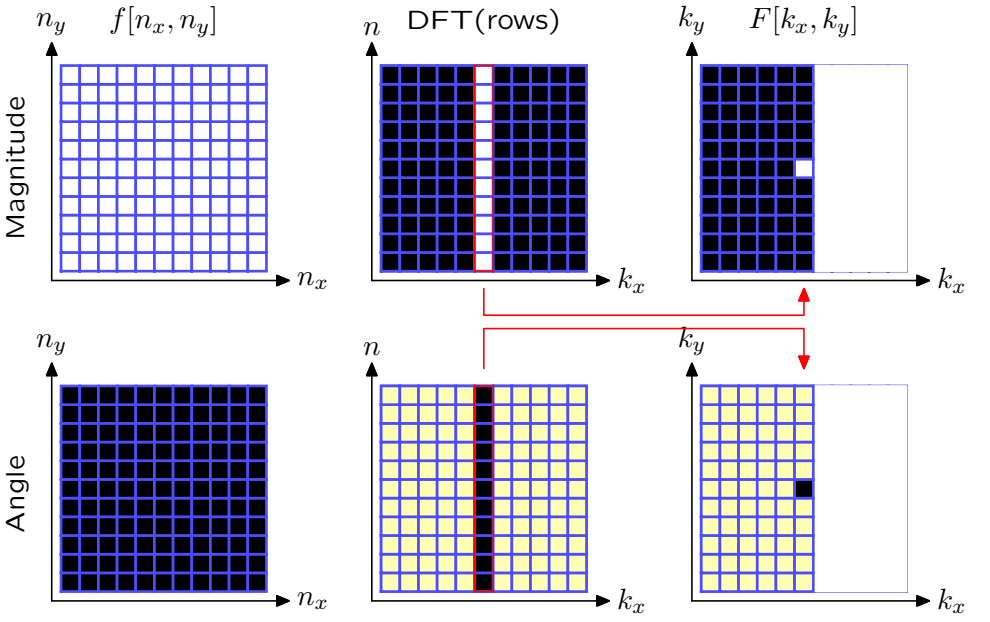

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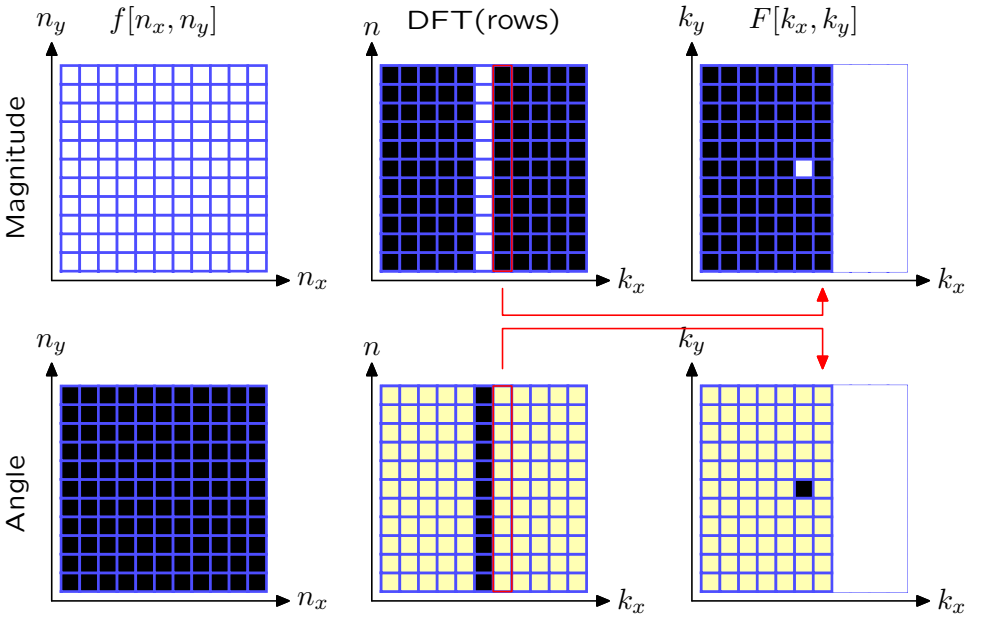

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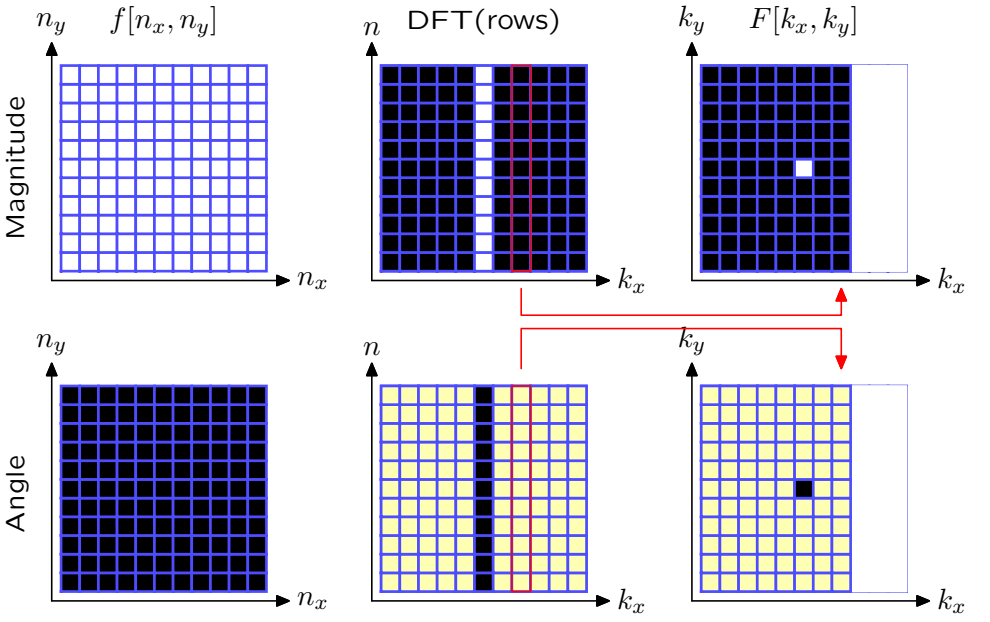

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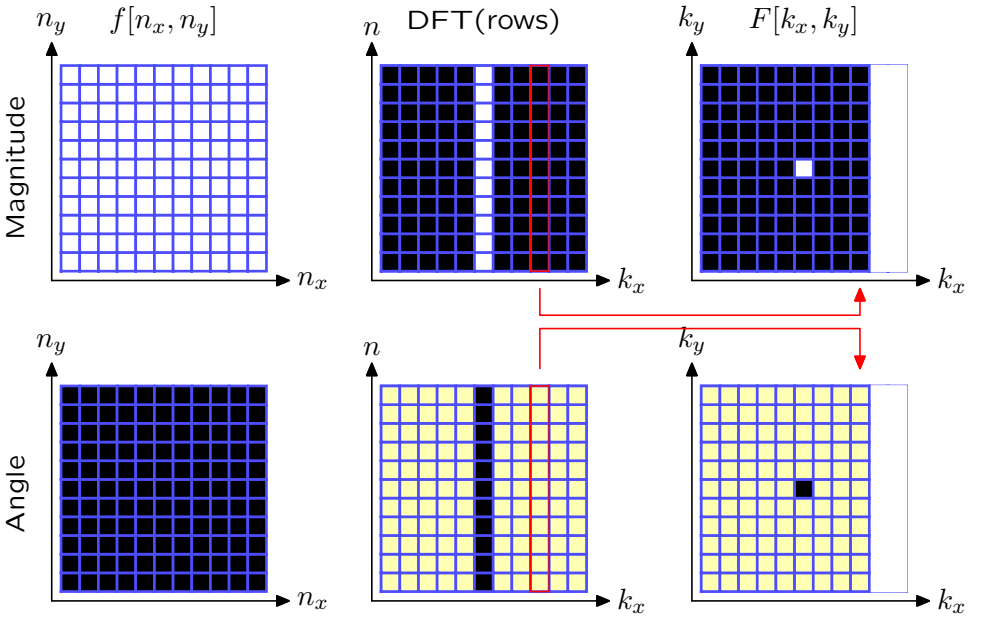

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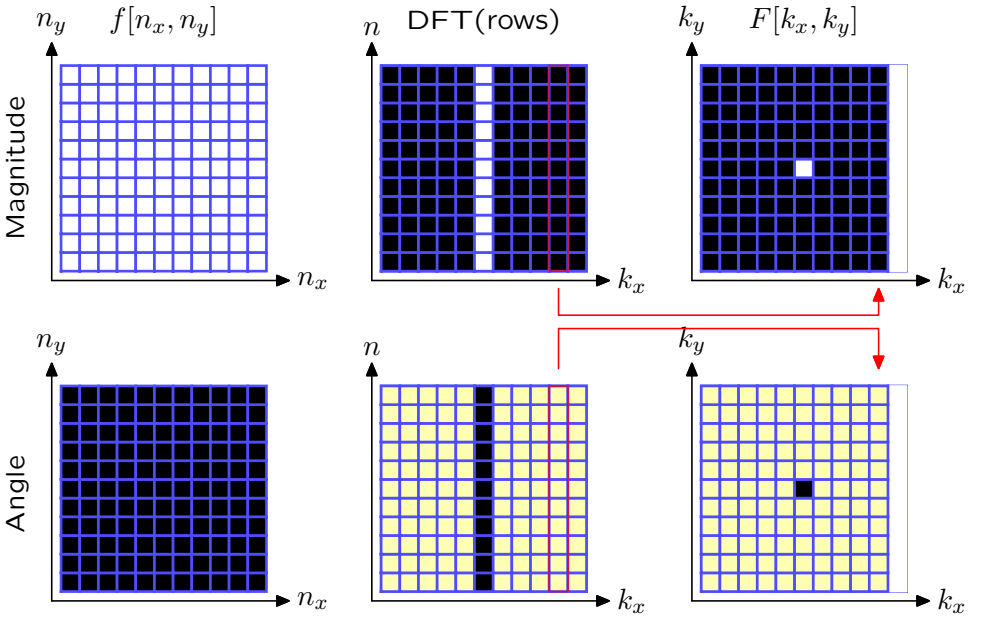

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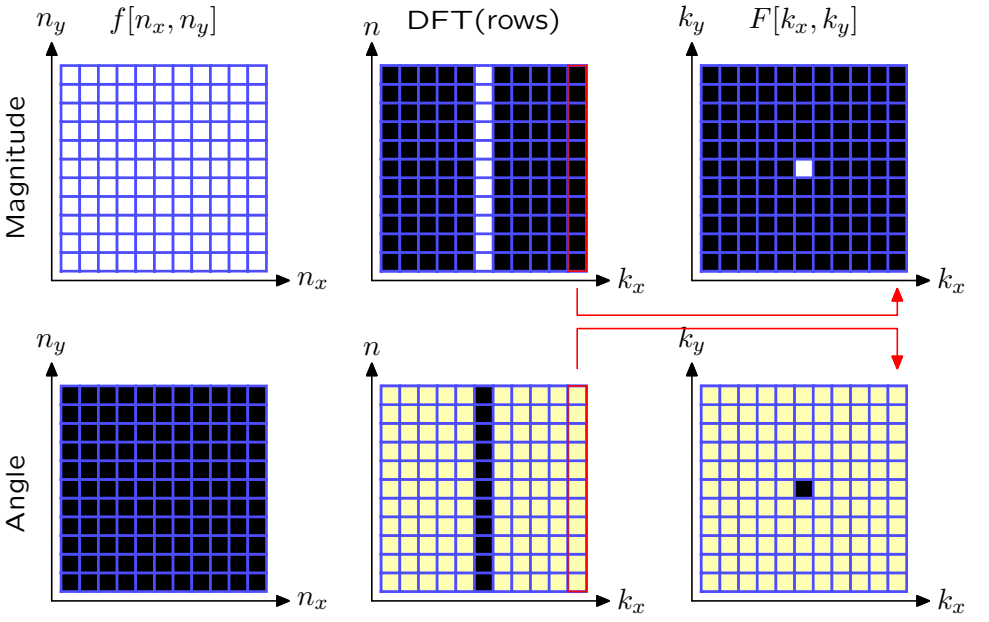

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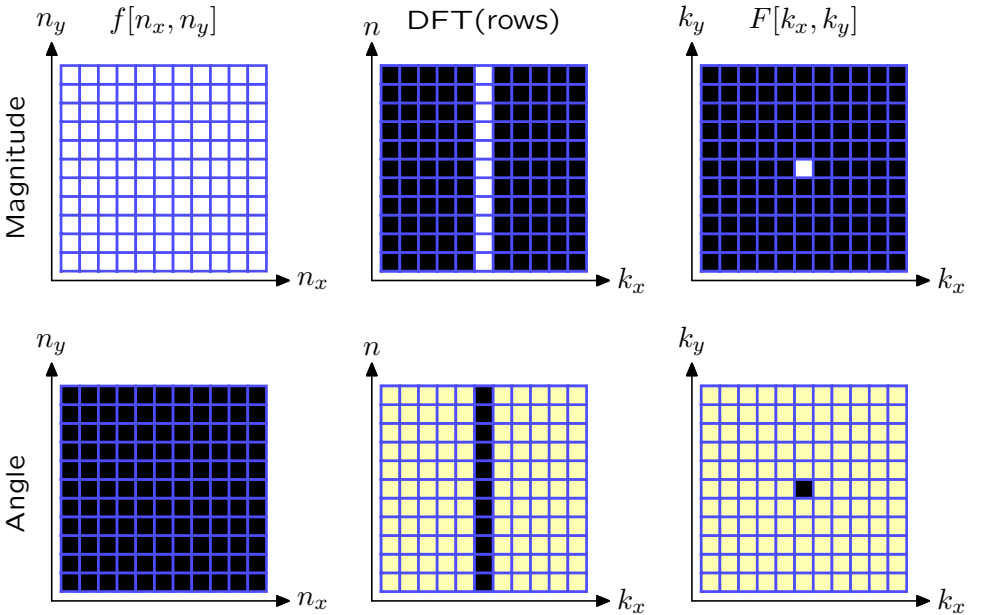

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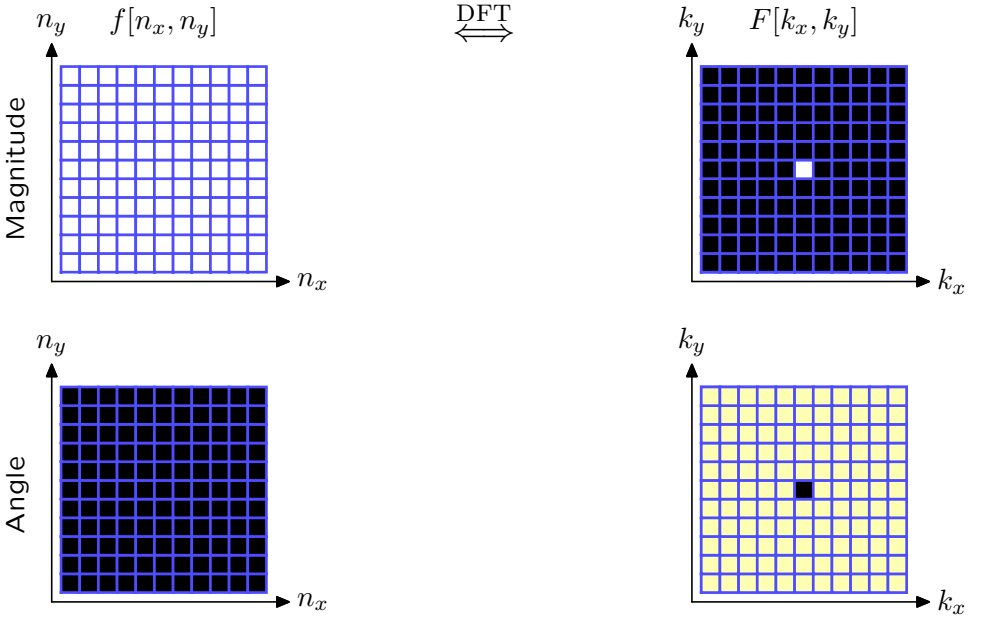

2D Discrete Fourier Transform

Example: Find the DFT of a vertical line.

$$f_v[n_x, n_y] = \delta[n_x] = \begin{cases} 1 & n_x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_v[k_x, k_y] = \frac{1}{N_x N_y} \sum_{nx=0}^{N_x-1} \sum_{ny=0}^{N_y-1} \delta[n_x]\, e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

$$= \frac{1}{N_x N_y} \sum_{nx=0}^{0} \sum_{ny=0}^{N_y-1} e^{-j\left(\frac{2\pi k_x}{N_x} 0 + \frac{2\pi k_y}{N_y} n_y\right)} = \frac{1}{N_x N_y} \sum_{ny=0}^{N_y-1} e^{-j\frac{2\pi k_y}{N_y} n_y}$$

But $\displaystyle\sum_{ny=0}^{N_y-1} e^{-j\frac{2\pi k_y}{N_y} n_y} = \begin{cases} N_y & k_y = 0 \\ 0 & \text{otherwise} \end{cases}$

$$F_v[k_x, k_y] = \frac{1}{N_x N_y} N_y \delta[k_y] = \frac{1}{N_x} \delta[k_y]$$

$$\delta[n_x] \quad \overset{\text{DFT}}{\Longleftrightarrow} \quad \frac{1}{N_x} \delta[k_y]$$

2D Discrete Fourier Transform

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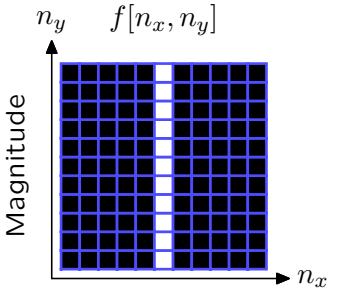

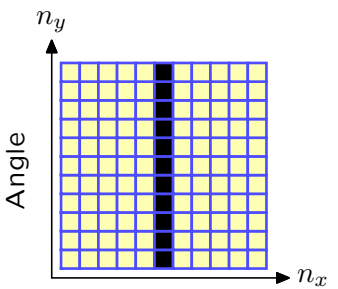

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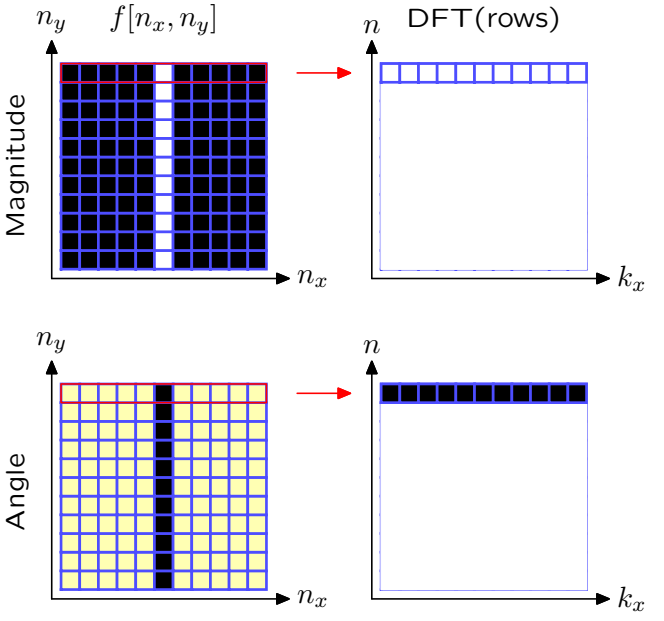

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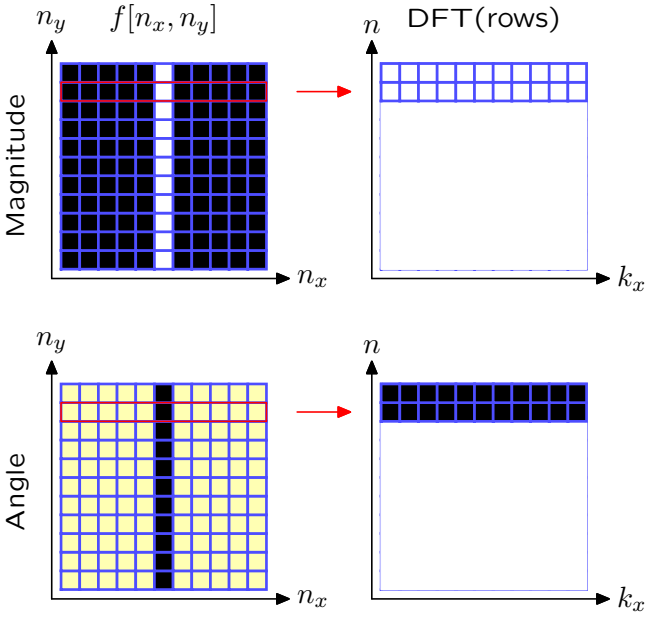

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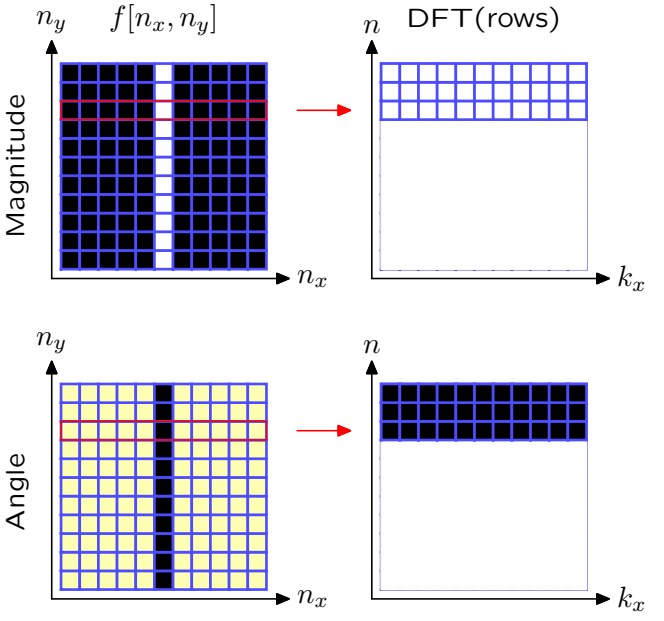

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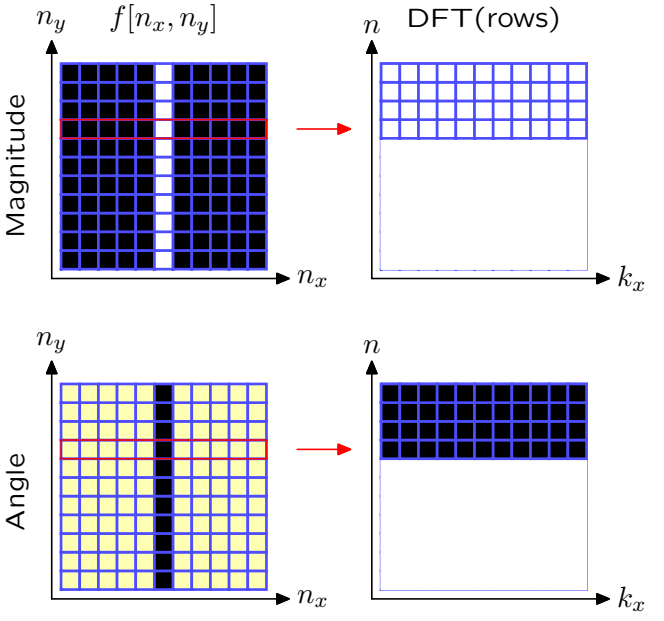

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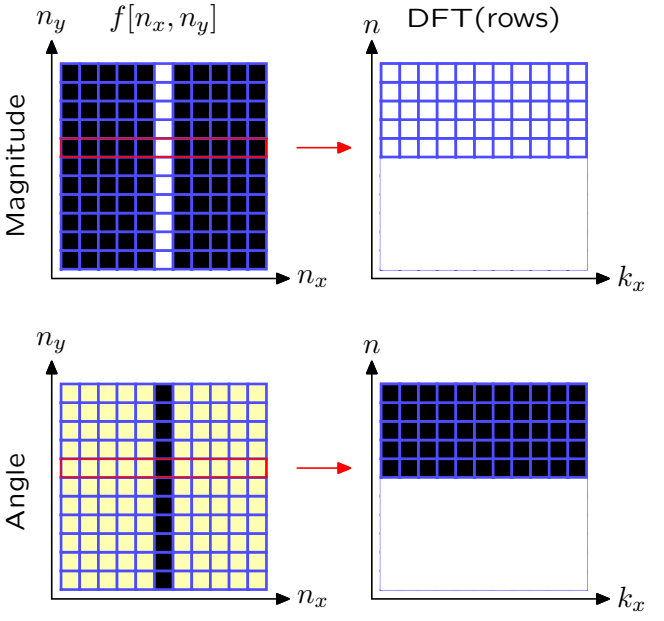

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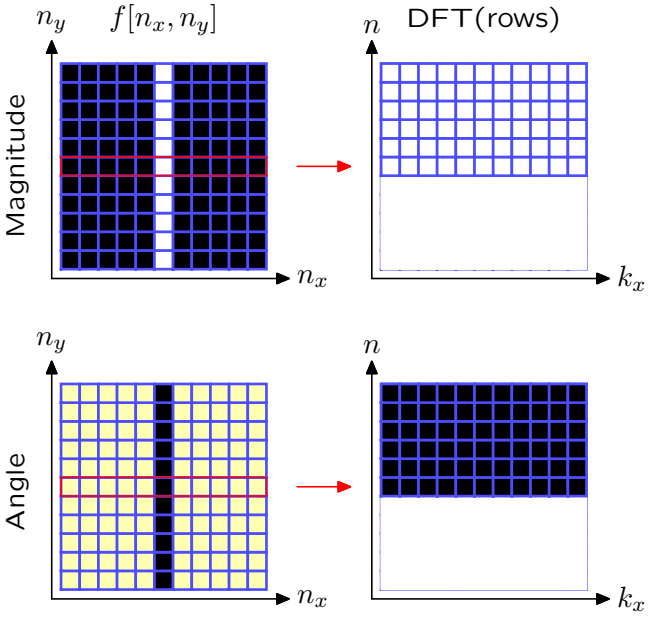

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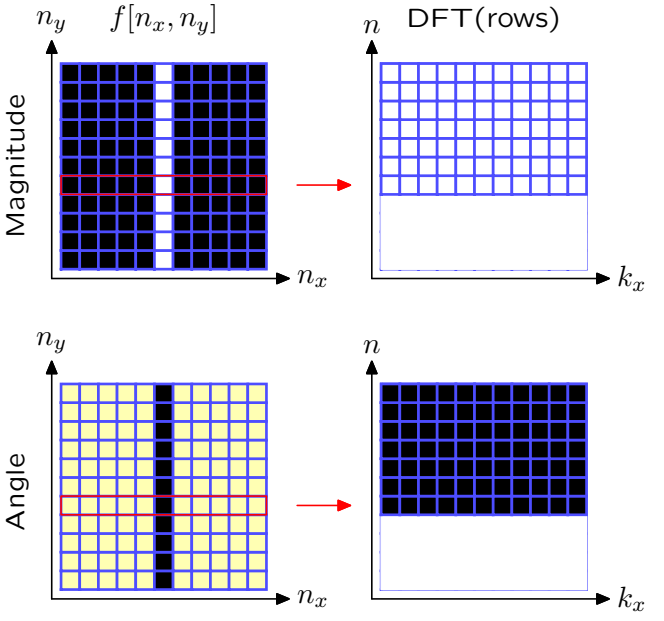

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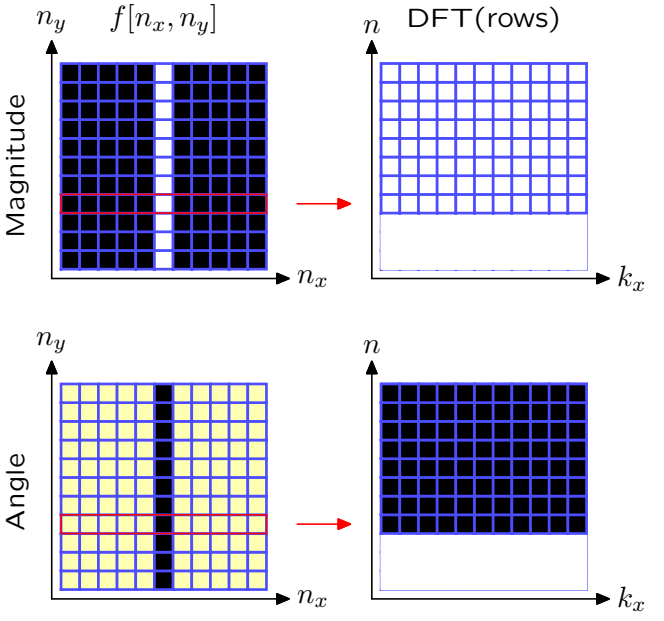

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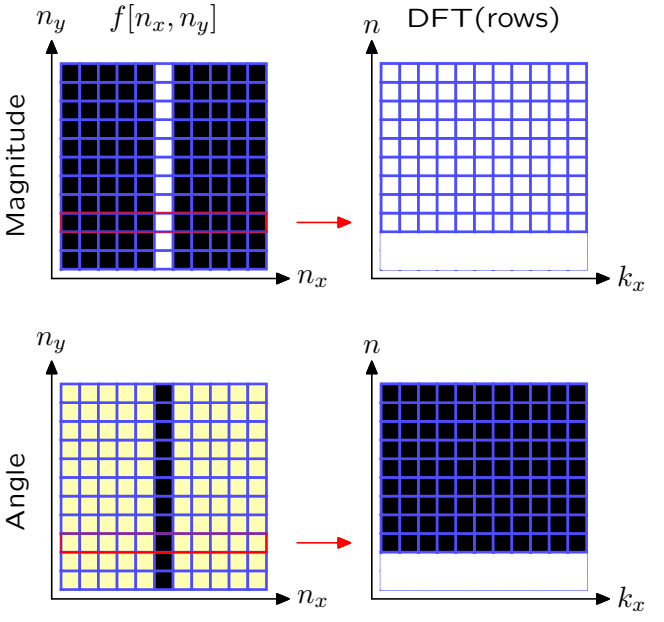

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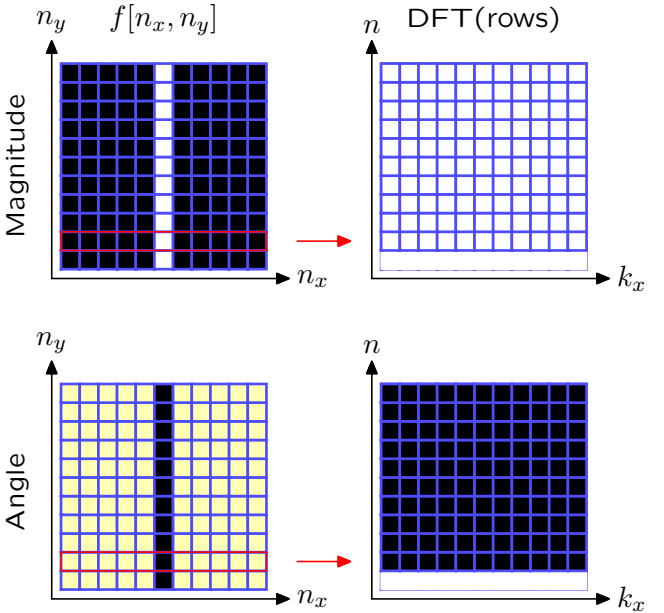

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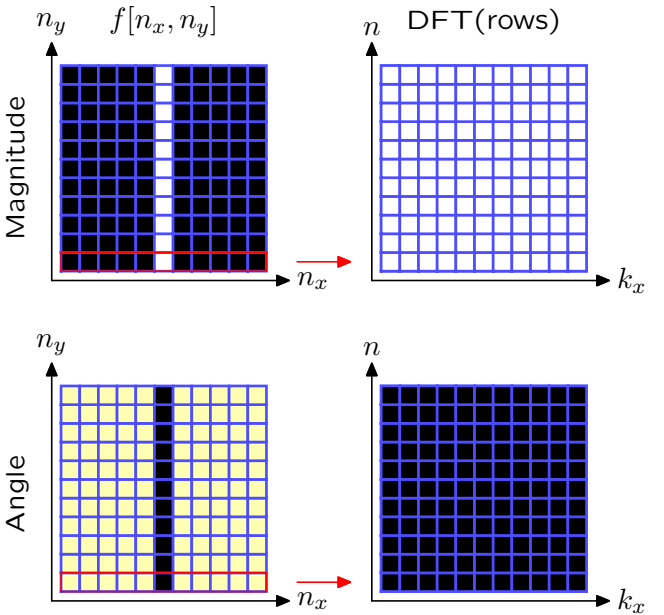

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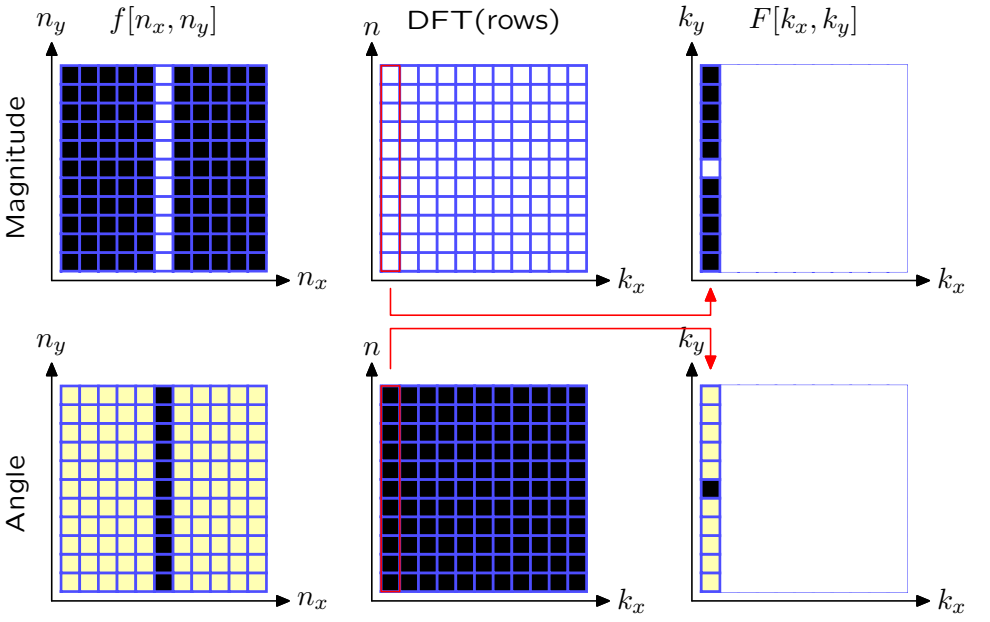

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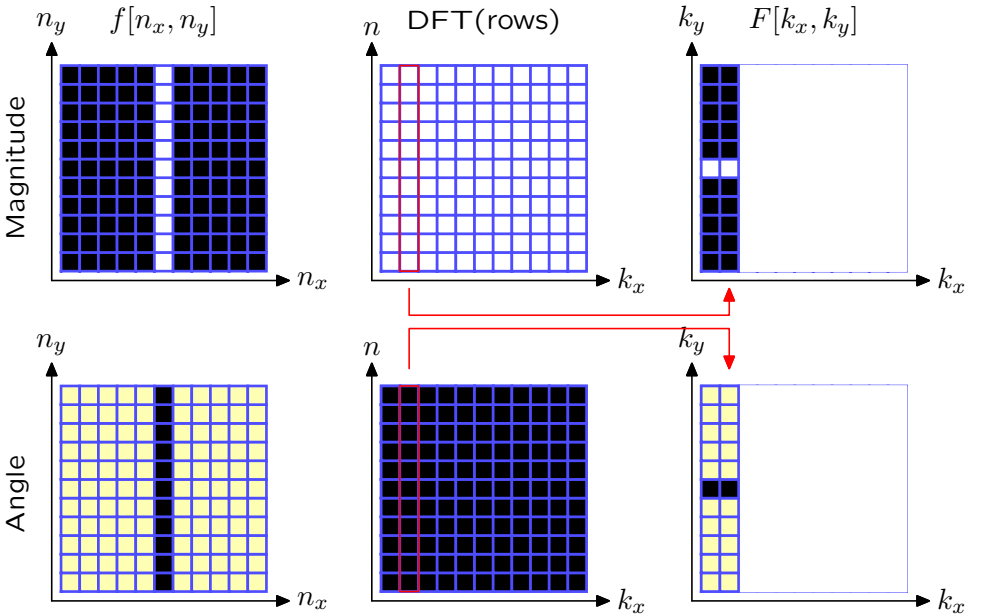

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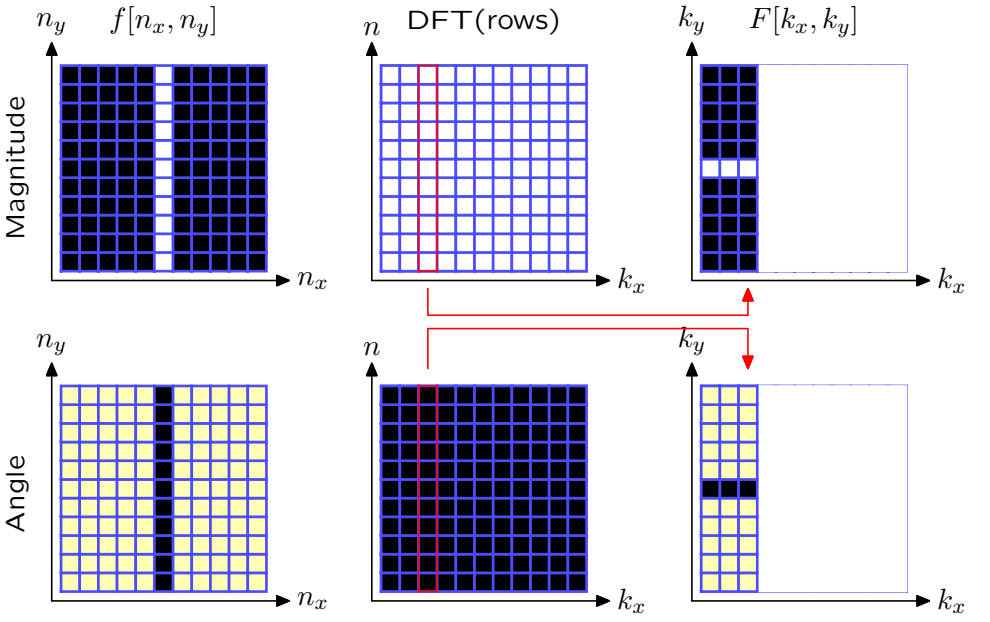

2D Discrete Fourier Transform

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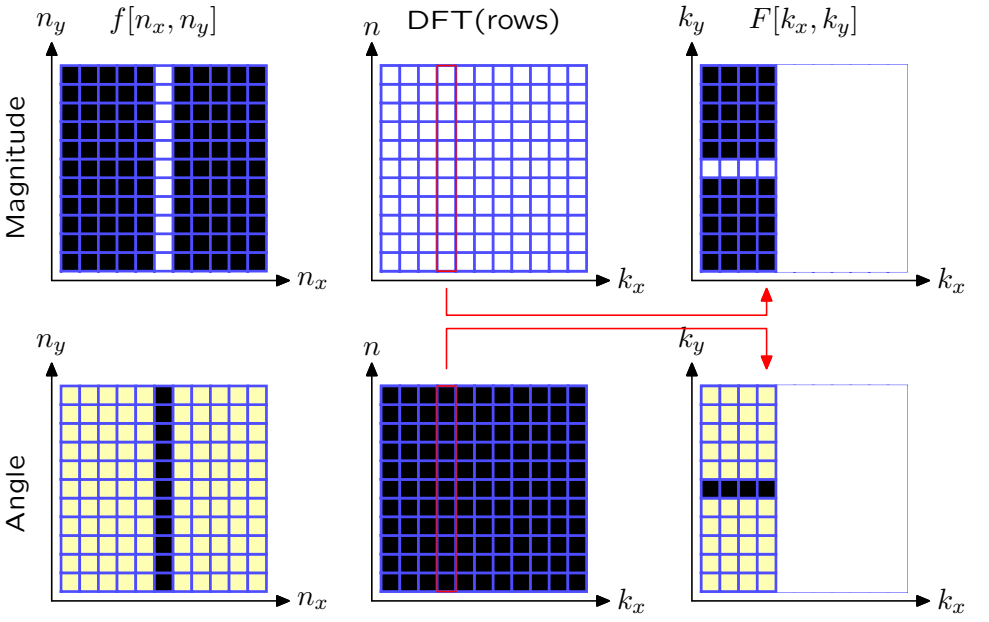

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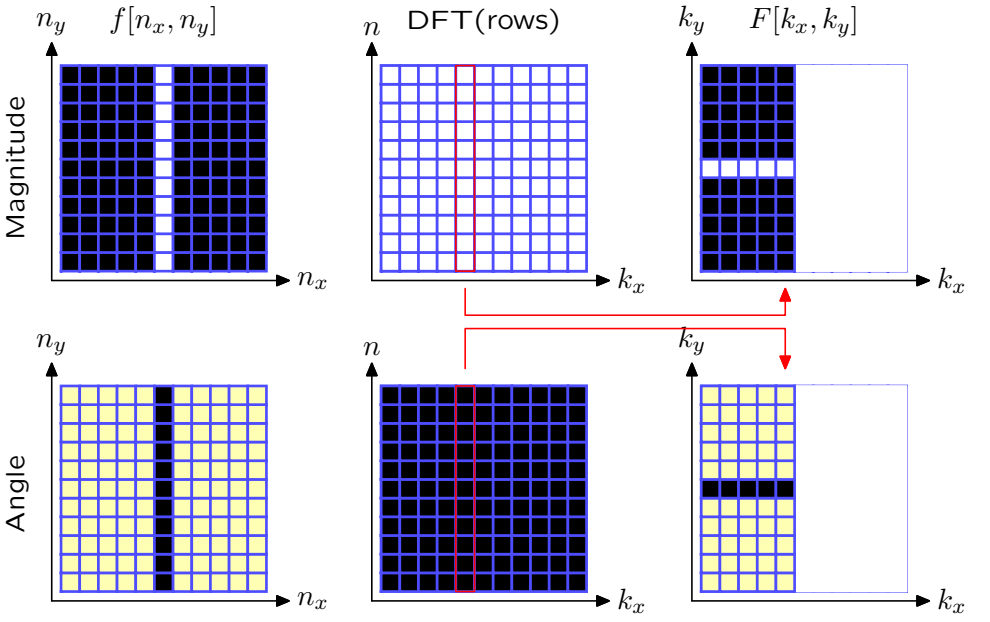

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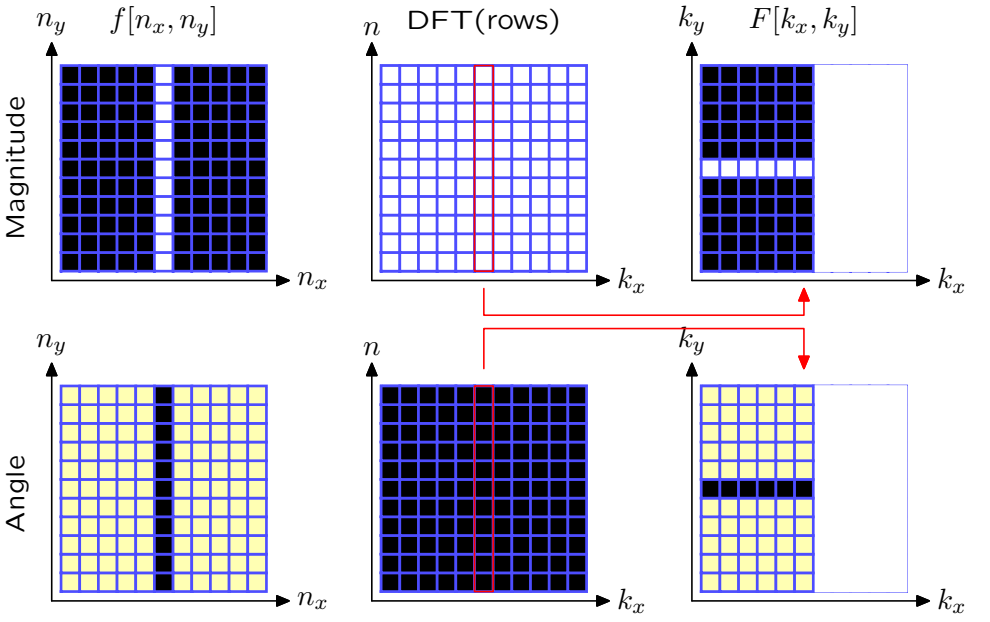

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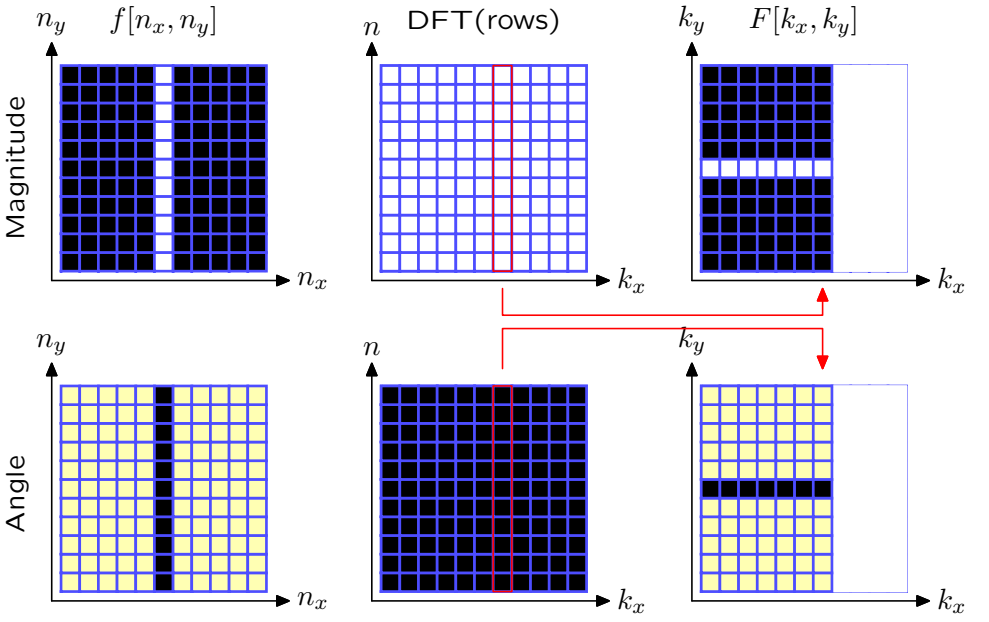

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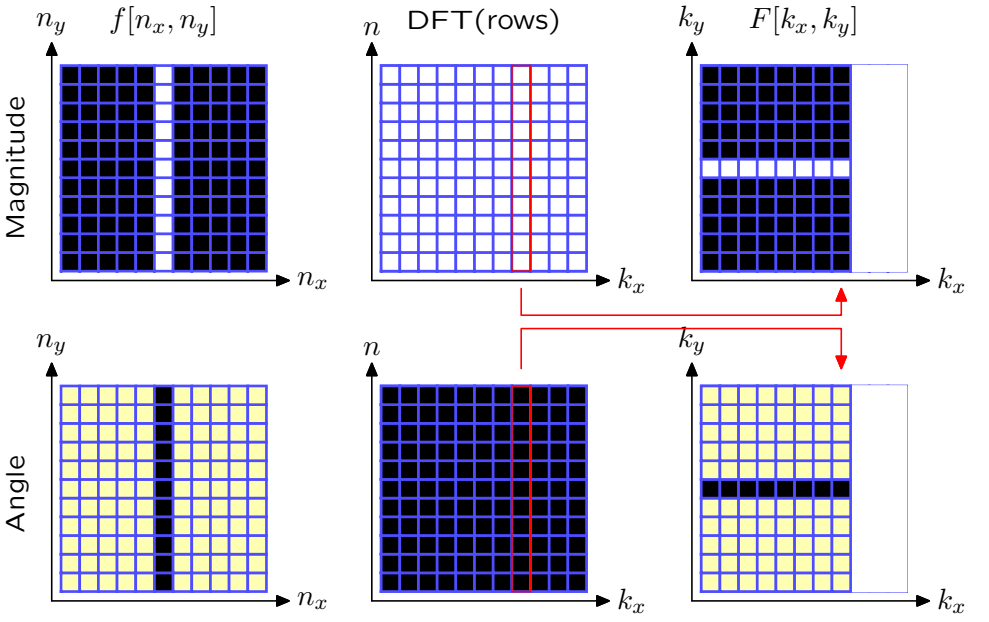

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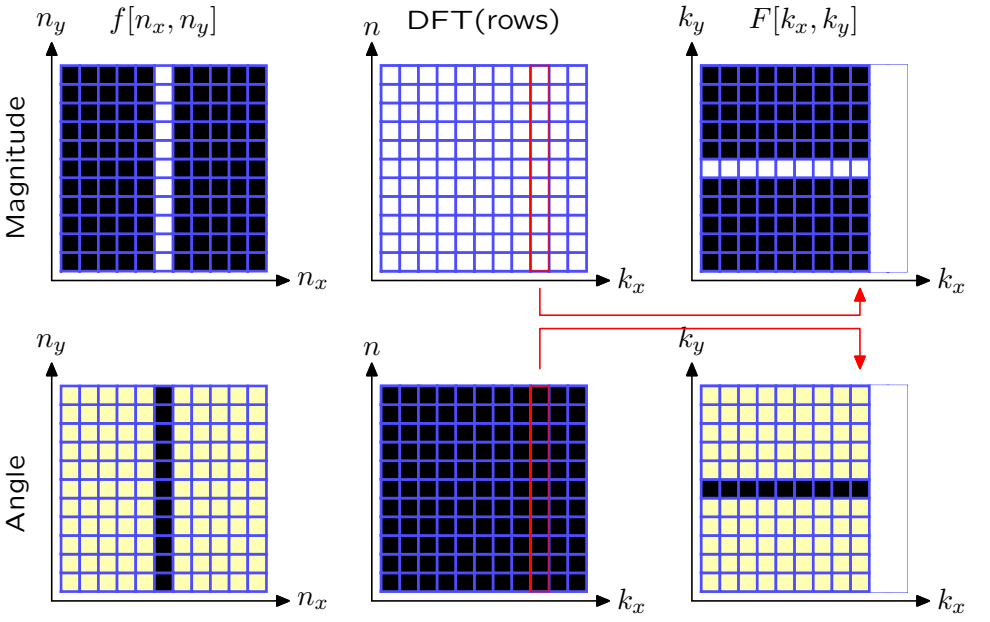

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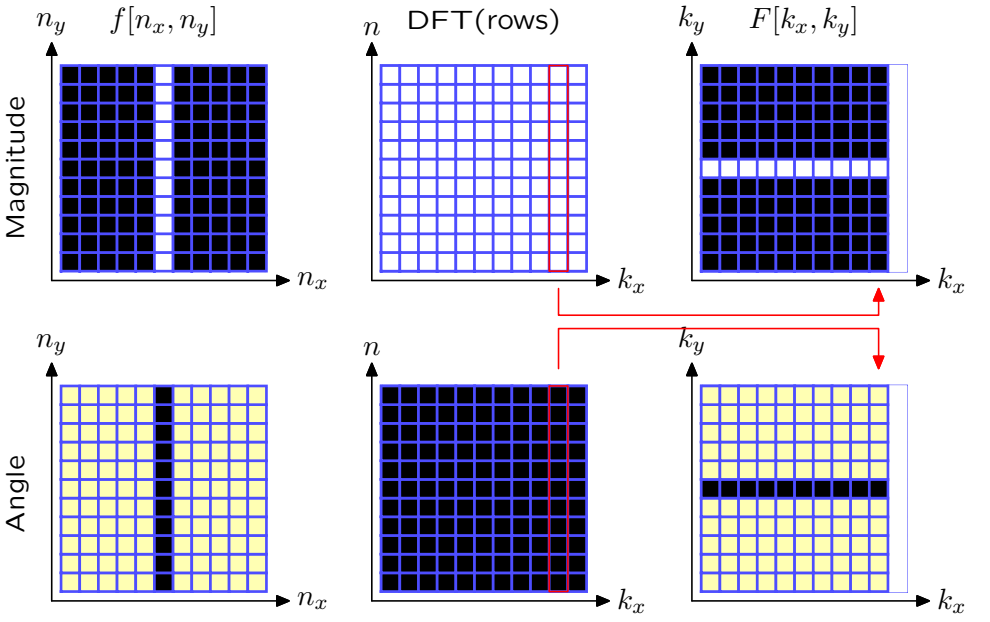

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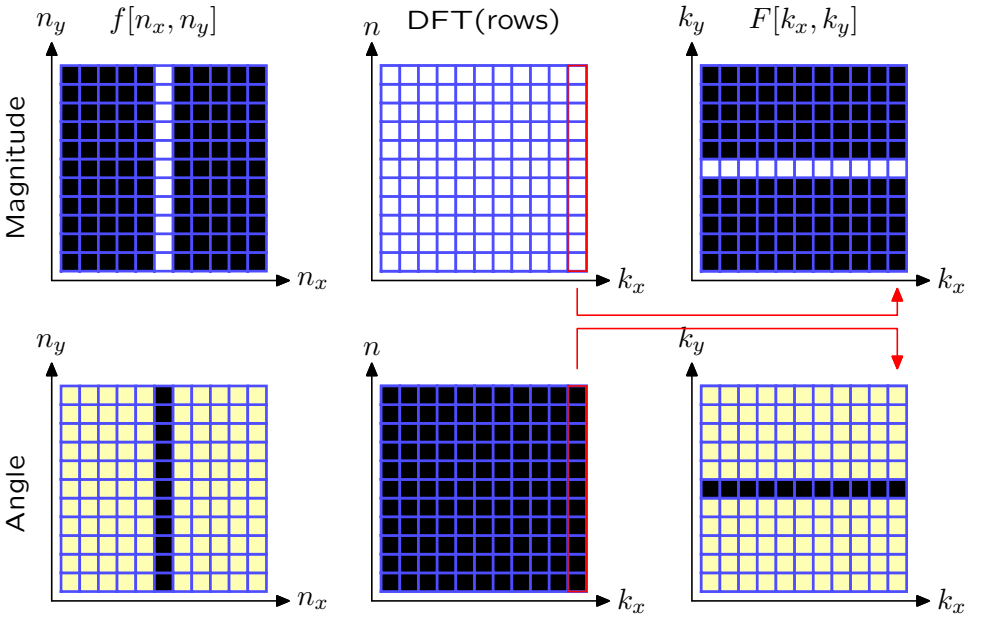

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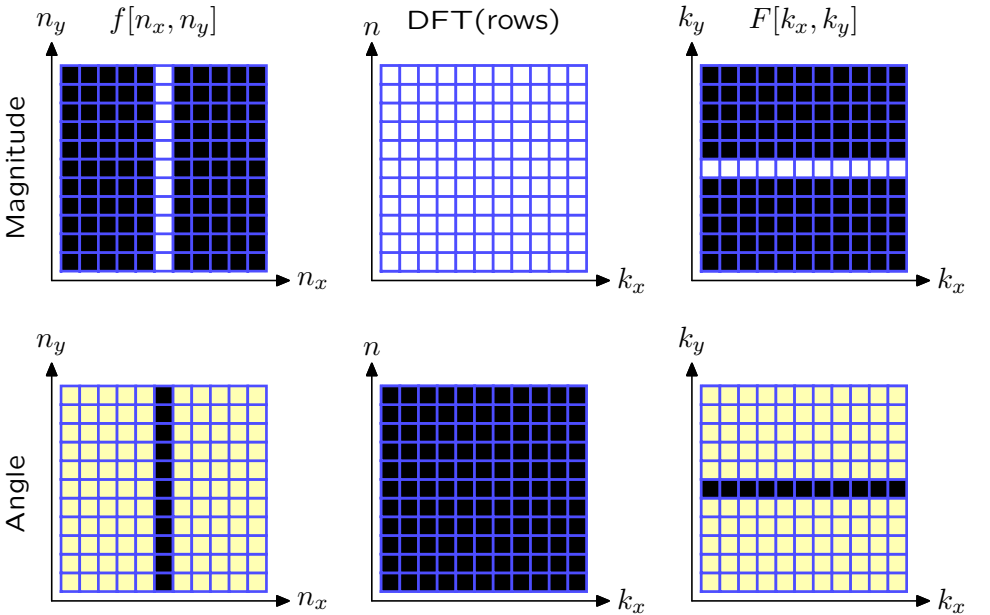

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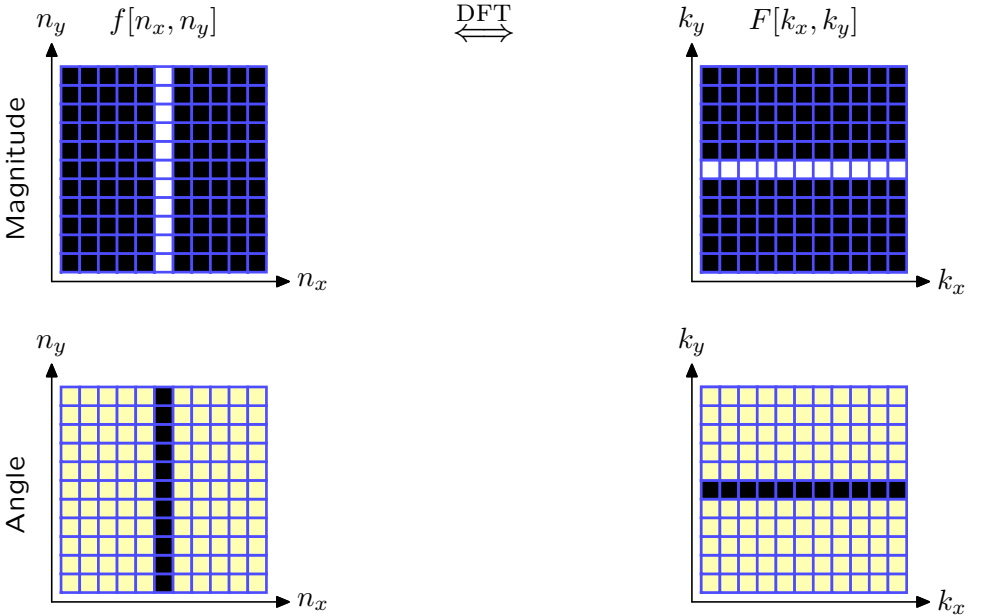


2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.

$$f_h[n_x, n_y] = \delta[n_y] = \begin{cases} 1 & n_y = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_h[k_x, k_y] = \frac{1}{N_x N_y} \sum_{nx=0}^{N_x-1} \sum_{ny=0}^{N_y-1} \delta[n_y]\, e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} n_y\right)}$$

$$= \frac{1}{N_x N_y} \sum_{nx=0}^{N_x-1} \sum_{ny=0}^{0} e^{-j\left(\frac{2\pi k_x}{N_x} n_x + \frac{2\pi k_y}{N_y} 0\right)} = \frac{1}{N_x N_y} \sum_{nx=0}^{N_x-1} e^{-j\frac{2\pi k_x}{N_x} n_x}$$

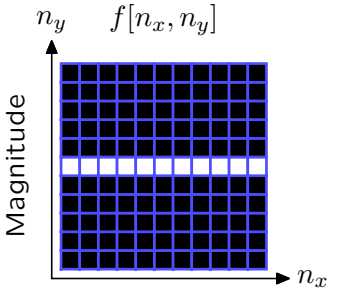
But $\displaystyle\sum_{nx=0}^{N_x-1} e^{-j\frac{2\pi k_x}{N_x} n_x} = \begin{cases} N_x & k_x = 0 \\ 0 & \text{otherwise} \end{cases}$

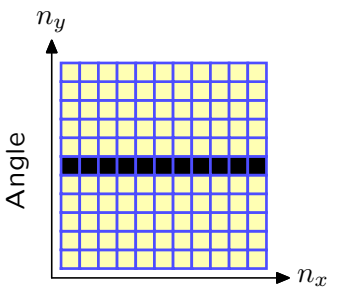
$$F_h[k_x, k_y] = \frac{1}{N_x N_y} N_x \delta[k_x] = \frac{1}{N_y} \delta[k_x]$$

$$\delta[n_y] \quad \overset{\text{DFT}}{\Longleftrightarrow} \quad \frac{1}{N_y} \delta[k_x]$$

2D Discrete Fourier Transform

Example: Find the DFT of a horizontal line.





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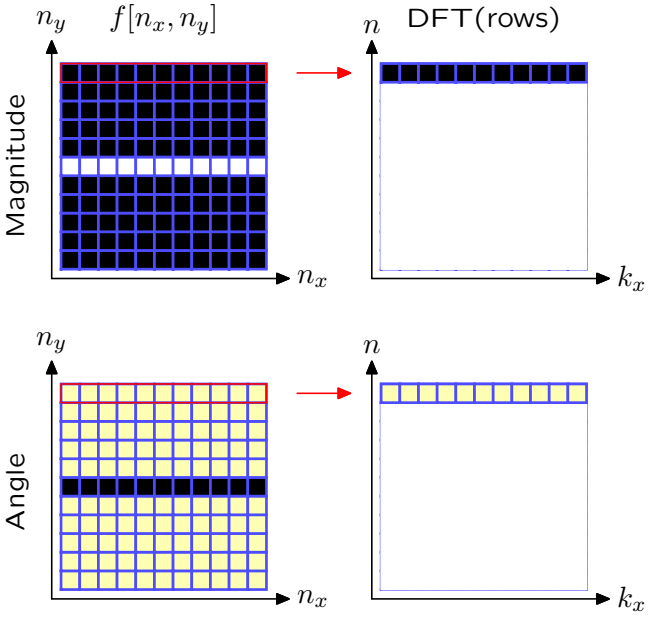

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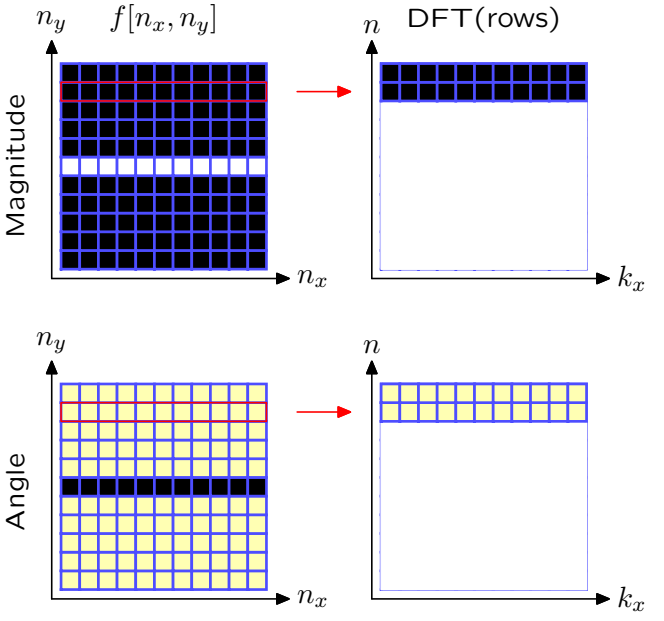

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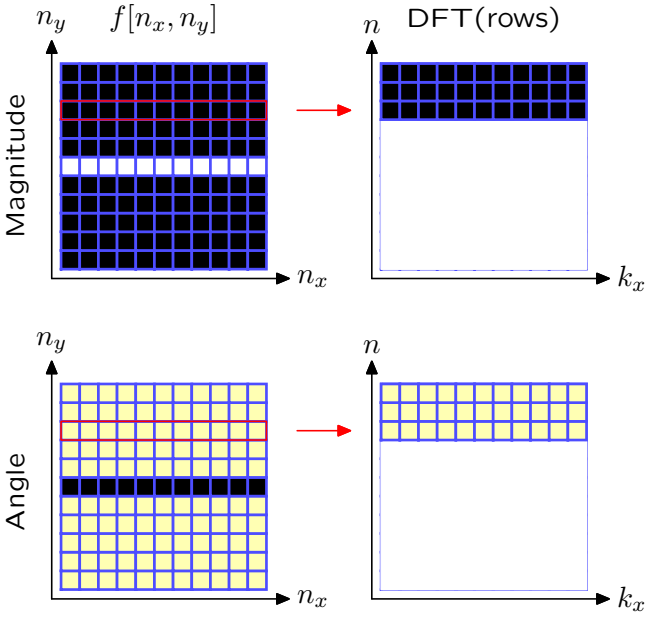

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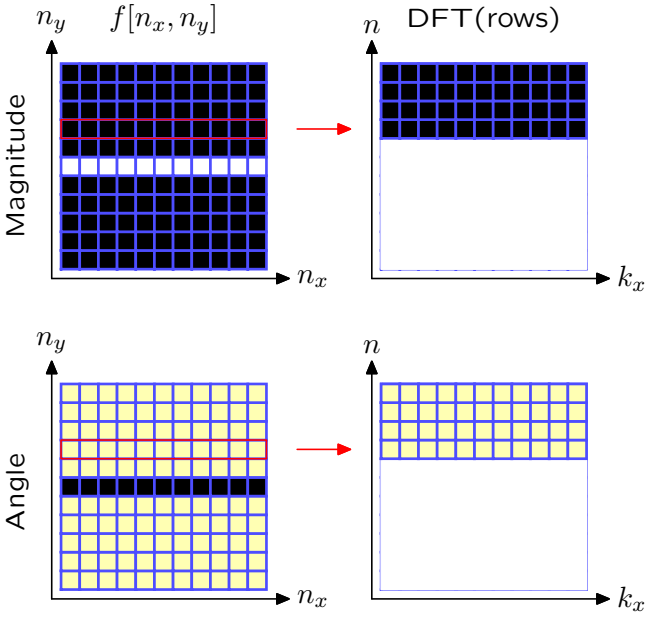

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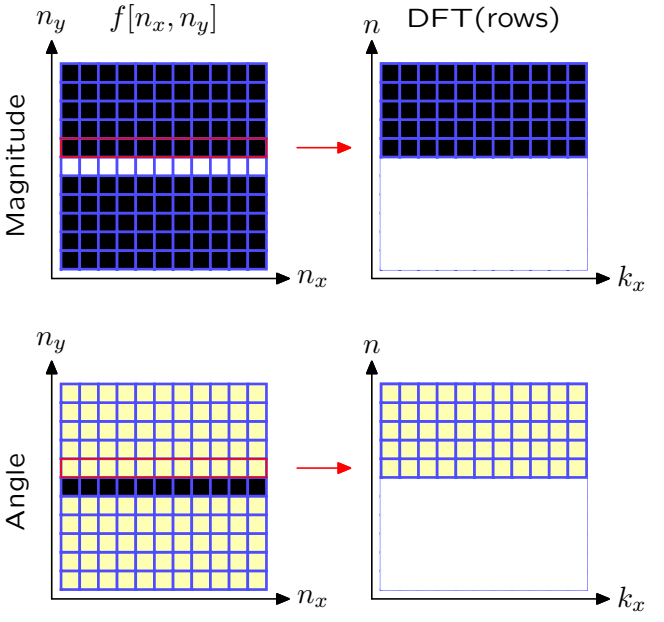

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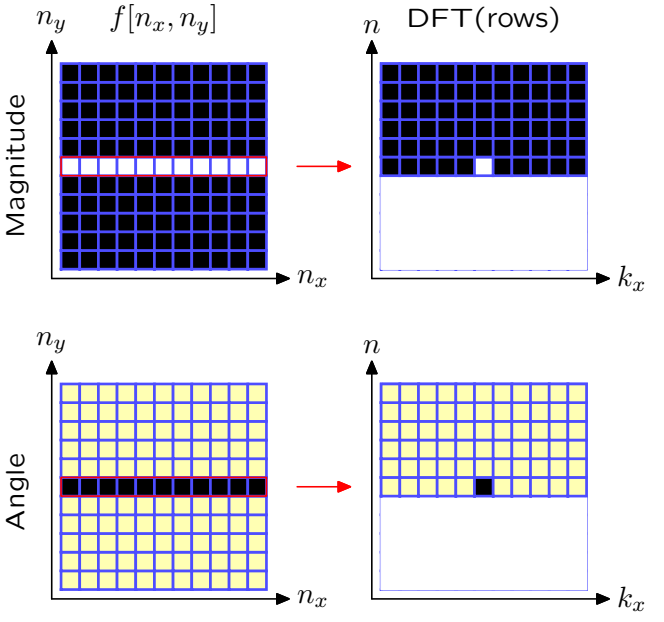

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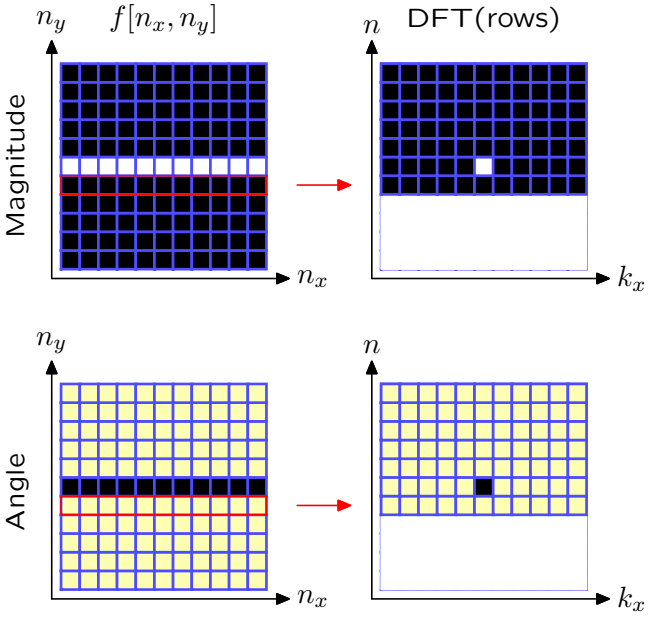

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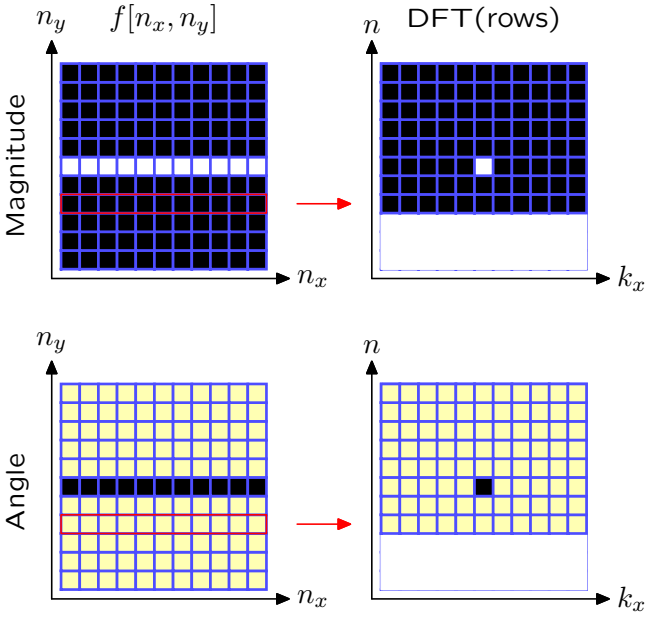

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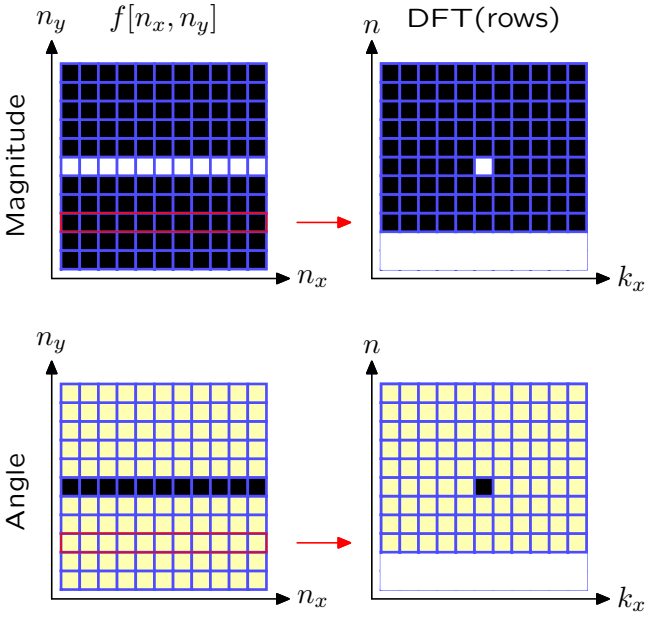

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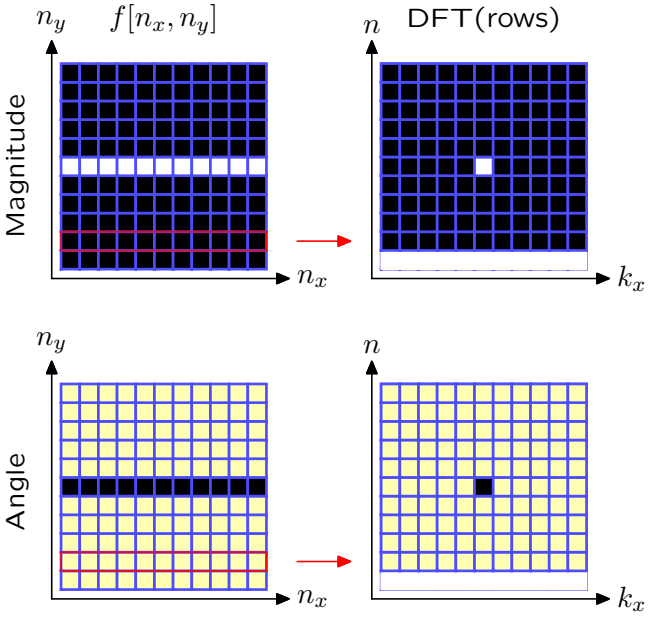

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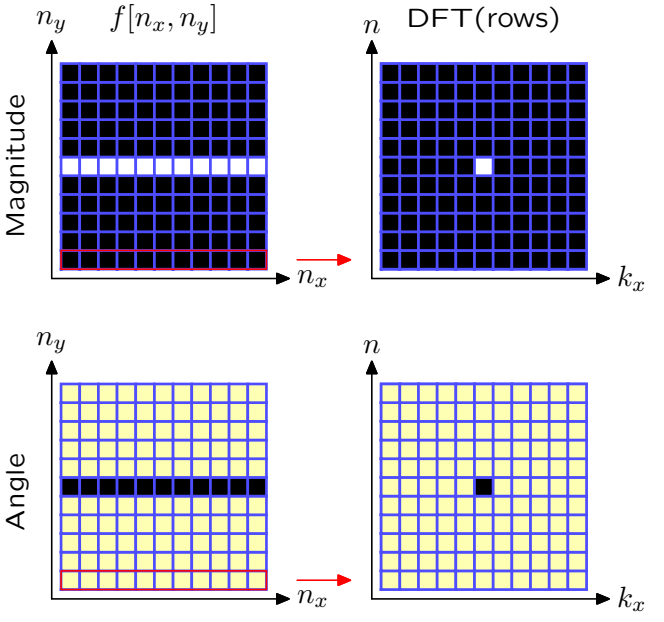

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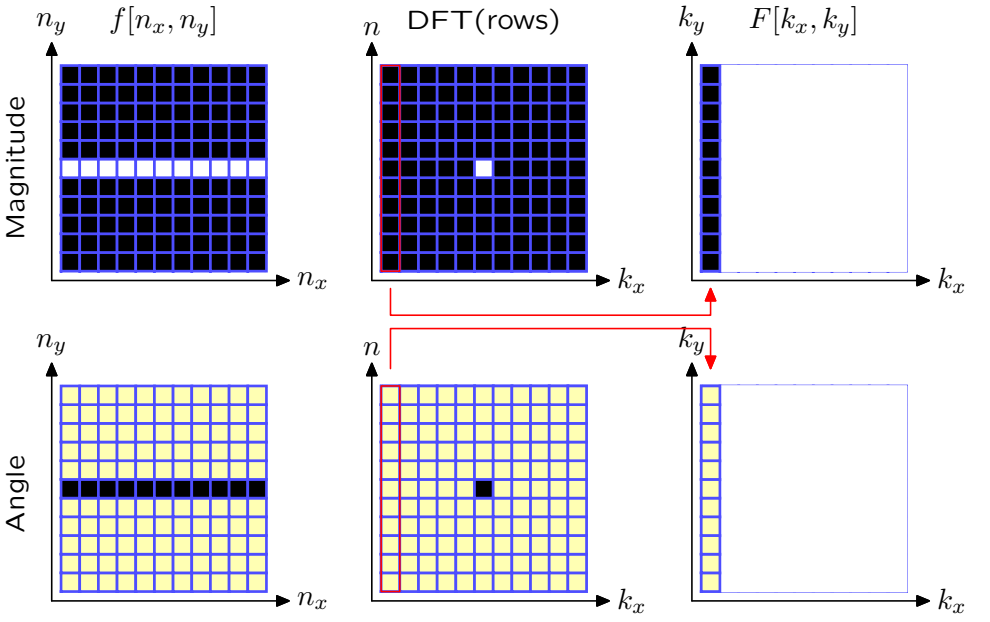

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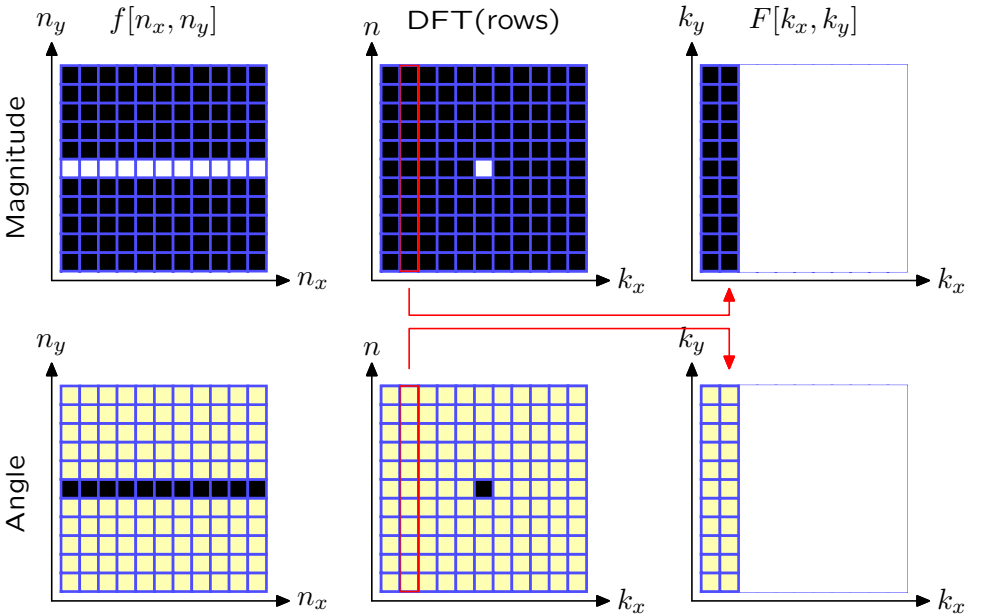

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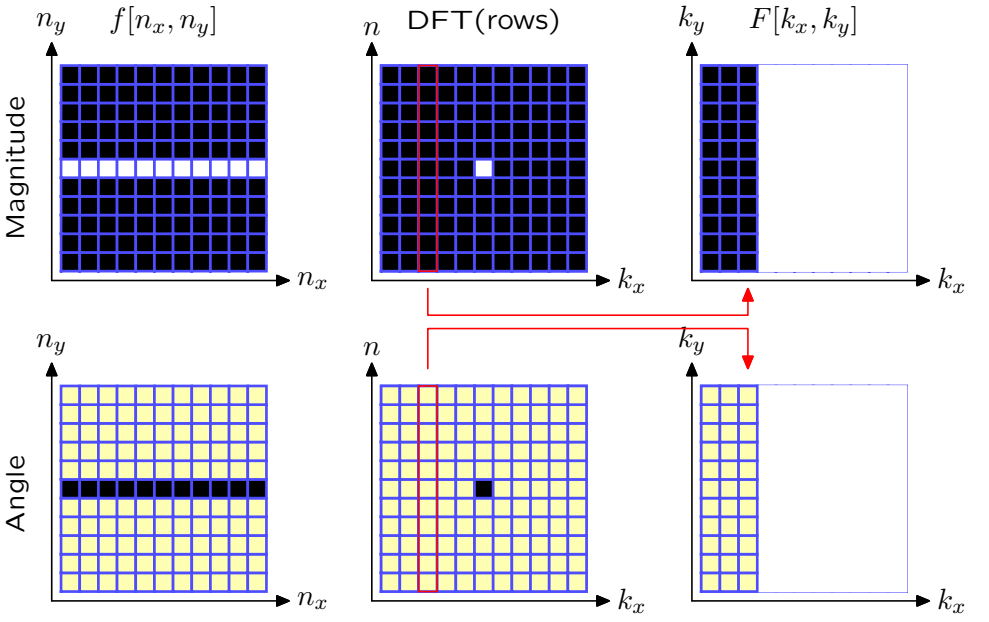

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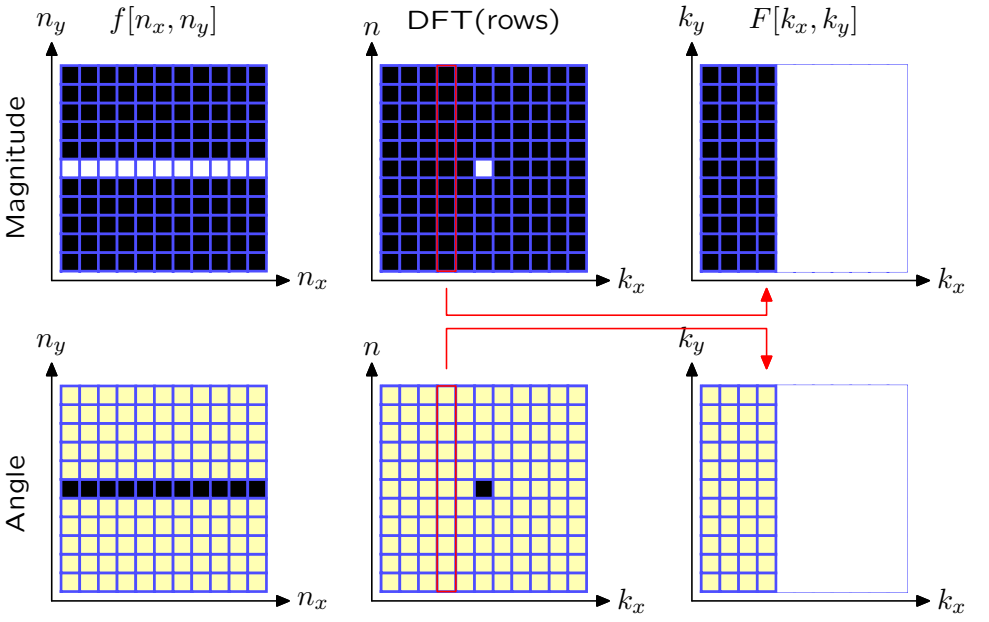

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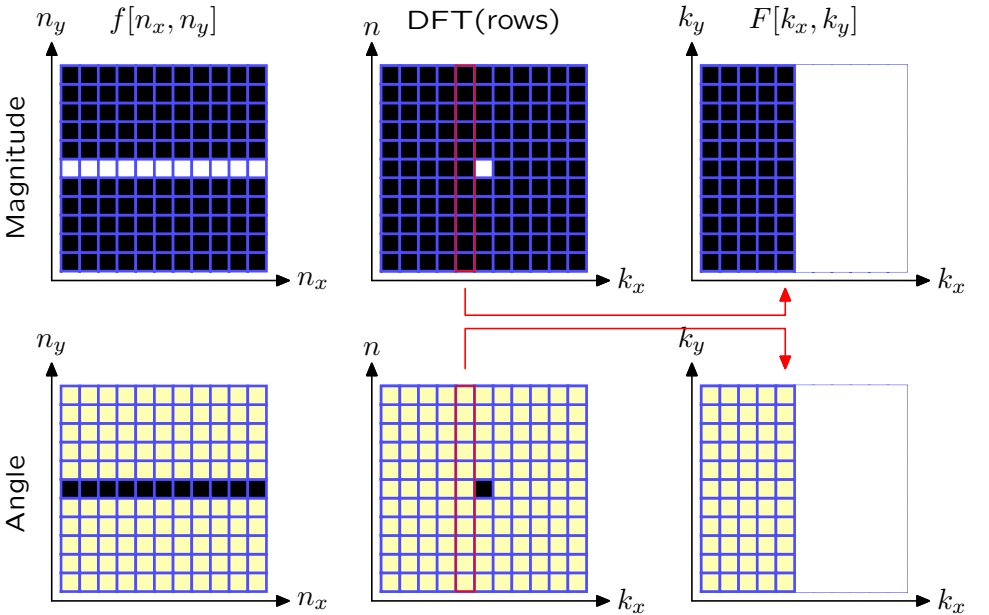

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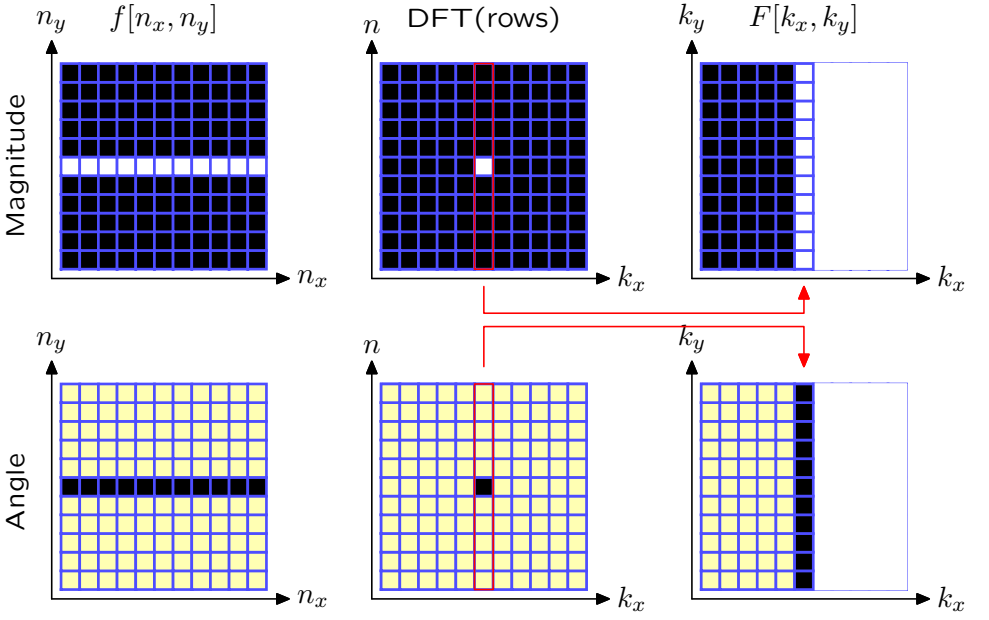

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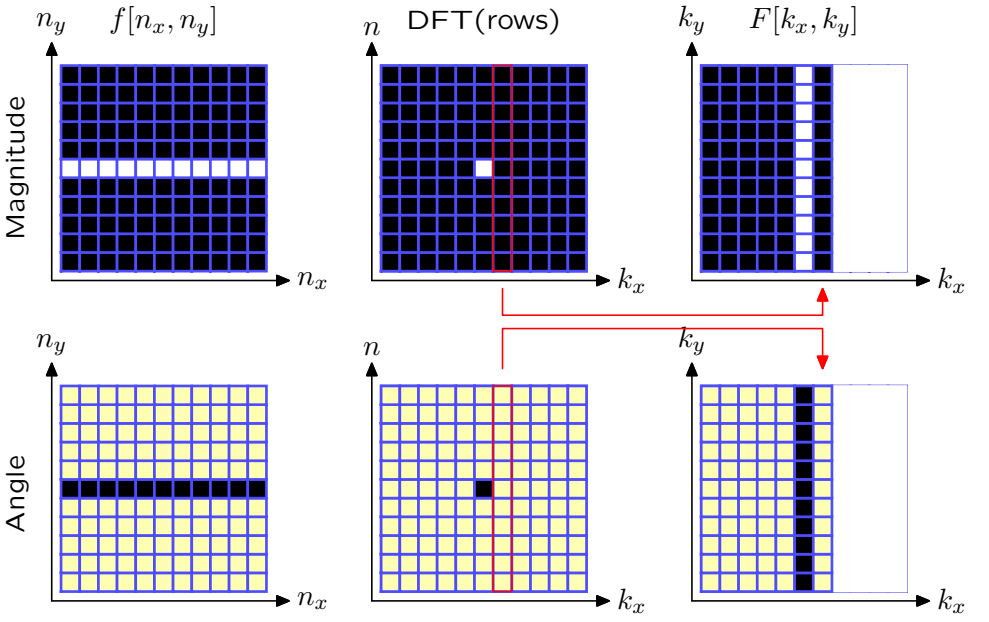

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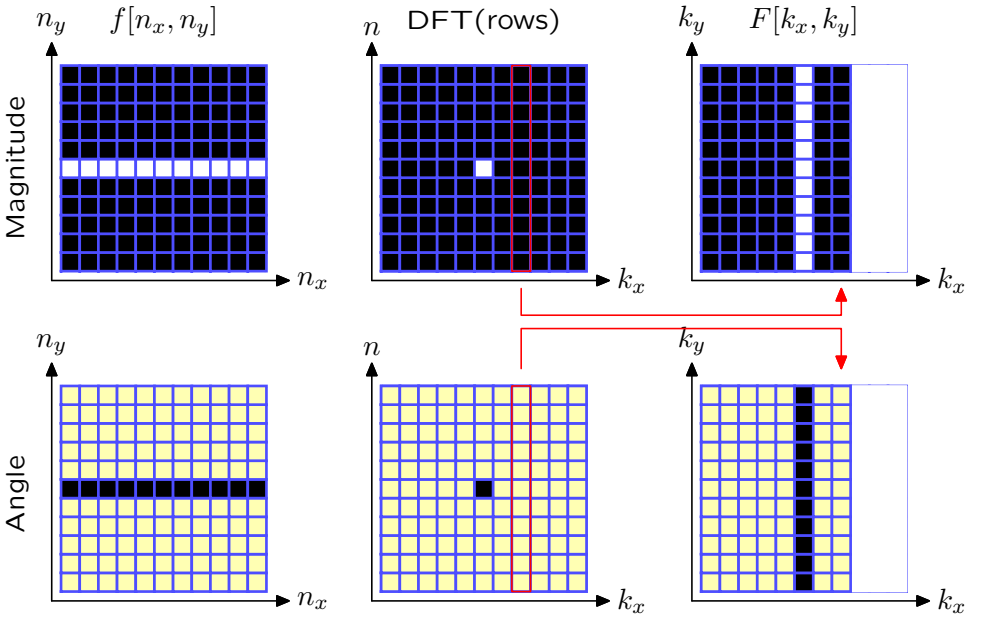

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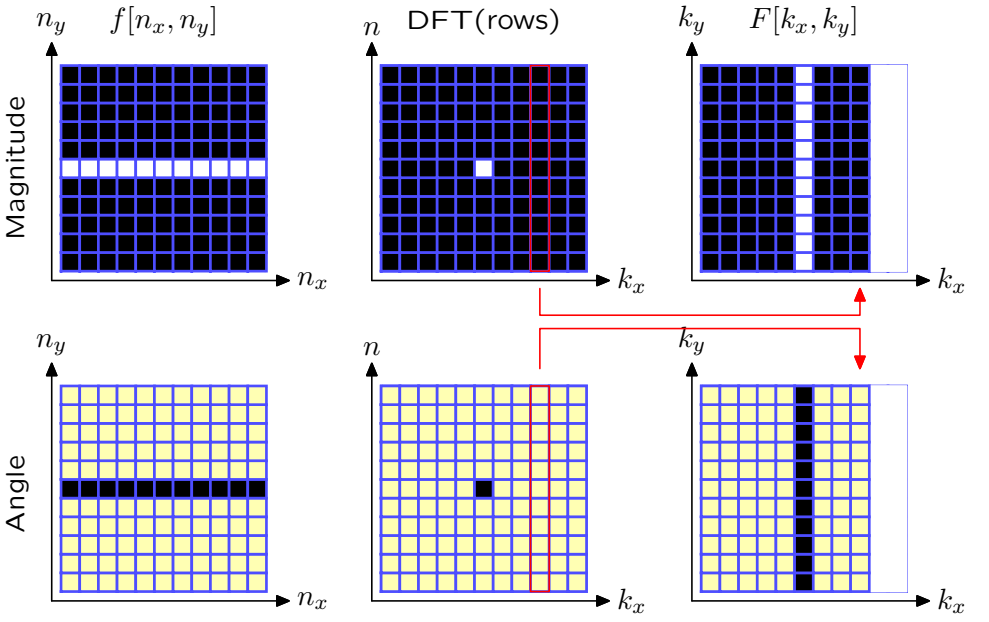

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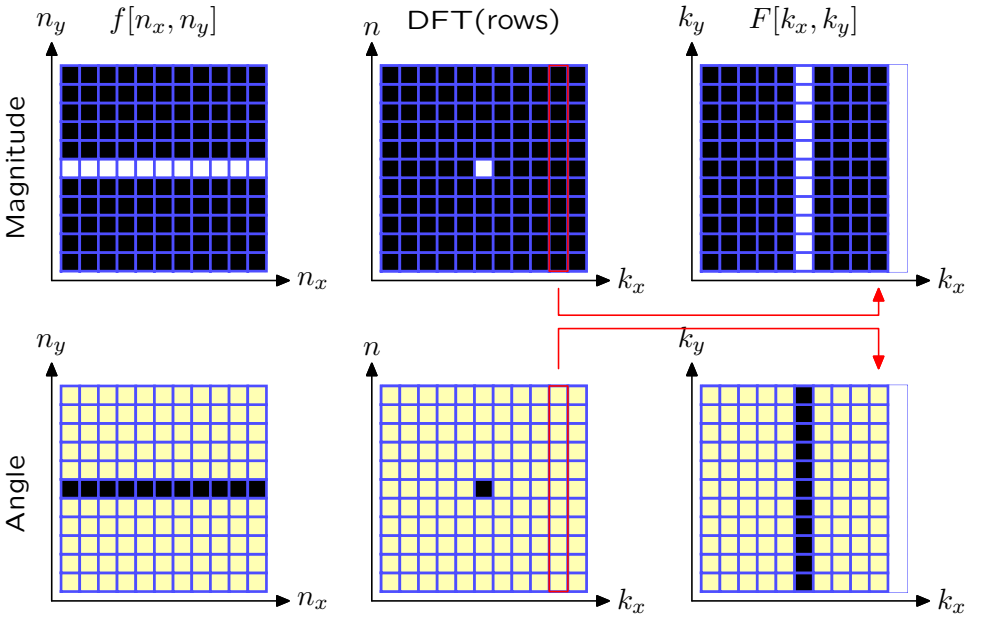

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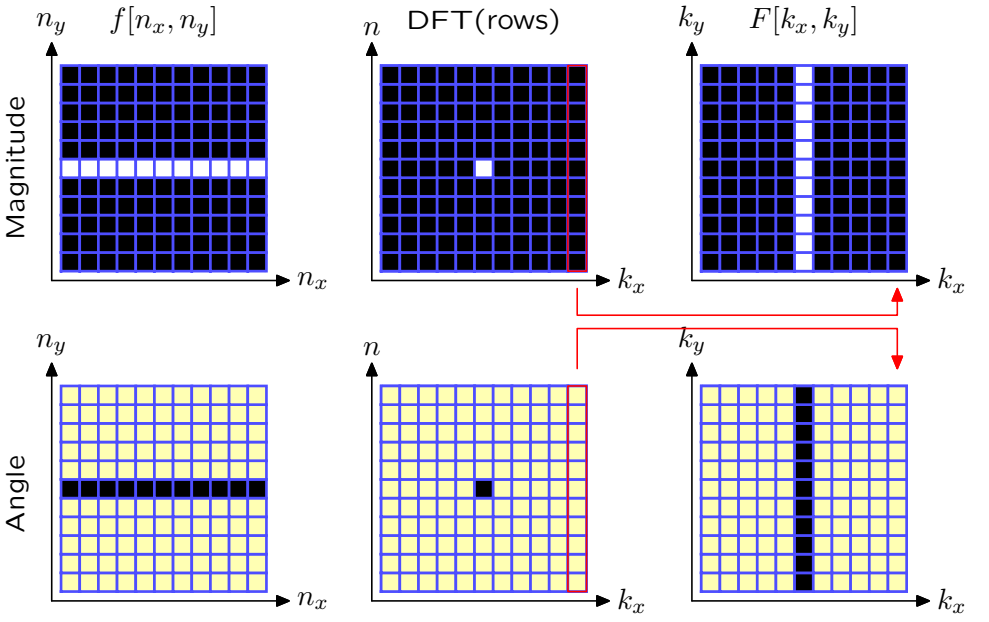

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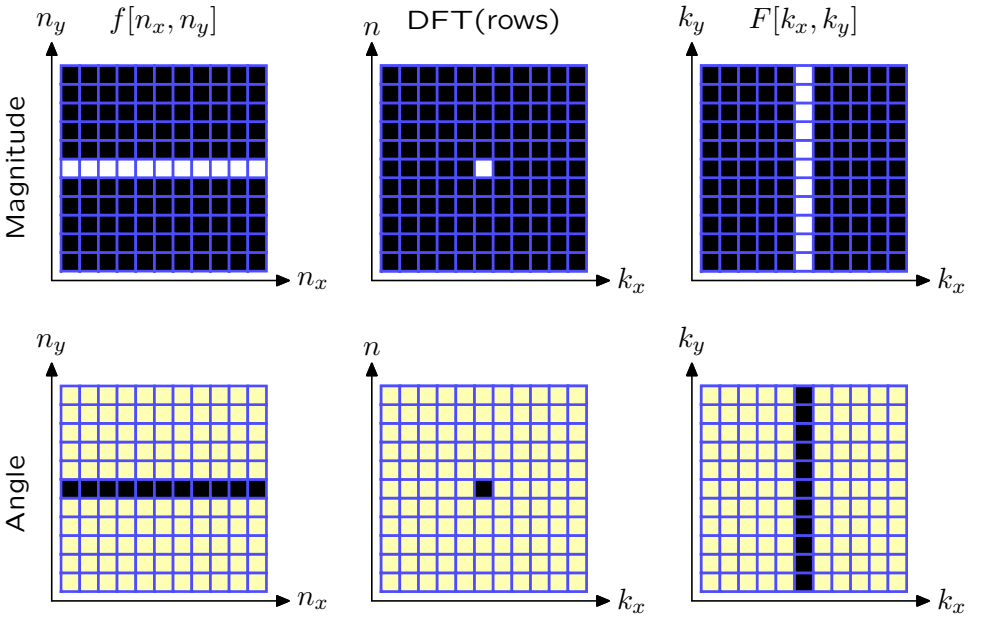

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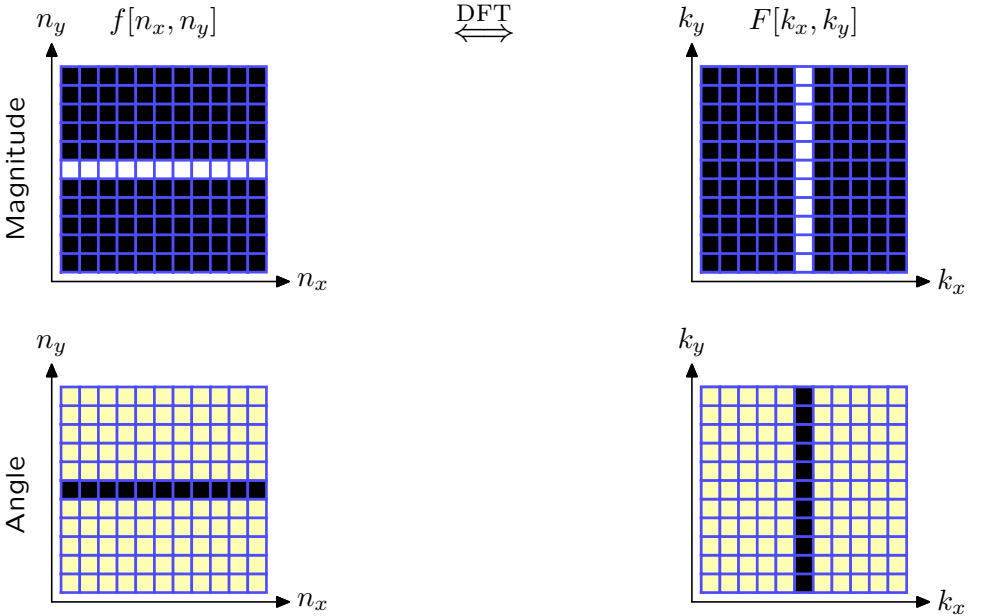


Translating (Shifting) an Image

Effect of image translation (shifting) on its Fourier transform.

Assume that $f_0[n_x, n_y] \overset{\text{DFT}}{\Longleftrightarrow} F_0[k_x, k_y]$.

Find the 2D DFT of $f_1[n_x, n_y] = f_0[n_x - n_{x0}, n_y - n_{y0}]$

$$F_1[k_x, k_y] = \sum_{kx} \sum_{ky} f_1[n_x, n_y] e^{-j\frac{2\pi k_x}{Nx} n_x} e^{-j\frac{2\pi k_y}{Ny} n_y}$$

$$= \sum_{kx} \sum_{ky} f_0[n_x - n_{x0}, n_y - n_{y0}] e^{-j\frac{2\pi k_x}{Nx} n_x} e^{-j\frac{2\pi k_y}{Ny} n_y}$$

Let $l_x = n_x - n_{x0}$ and $l_y = n_y - n_{y0}$. Then

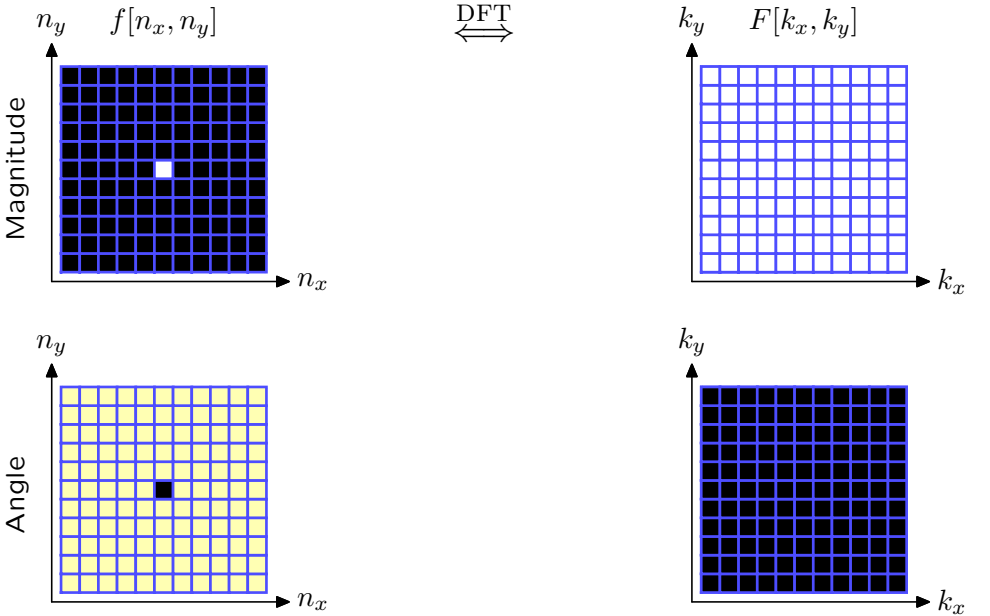
$$F_1[k_x, k_y] = \sum_{lx} \sum_{ly} f_0[l_x, l_y] e^{-j\frac{2\pi k_x}{Nx}(l_x + n_{x0})} e^{-j\frac{2\pi k_y}{Ny}(l_y + n_{y0})}$$

$$= e^{-j\frac{2\pi k_x}{Nx} n_{x0}} e^{-j\frac{2\pi k_y}{Ny} n_{y0}} F_0[k_x, k_y]$$

Translating an image adds linear (in k_x, k_y) **phase** to its transform.

2D Discrete Fourier Transform

Example: Find the DFT of a shifted 2D unit sample.



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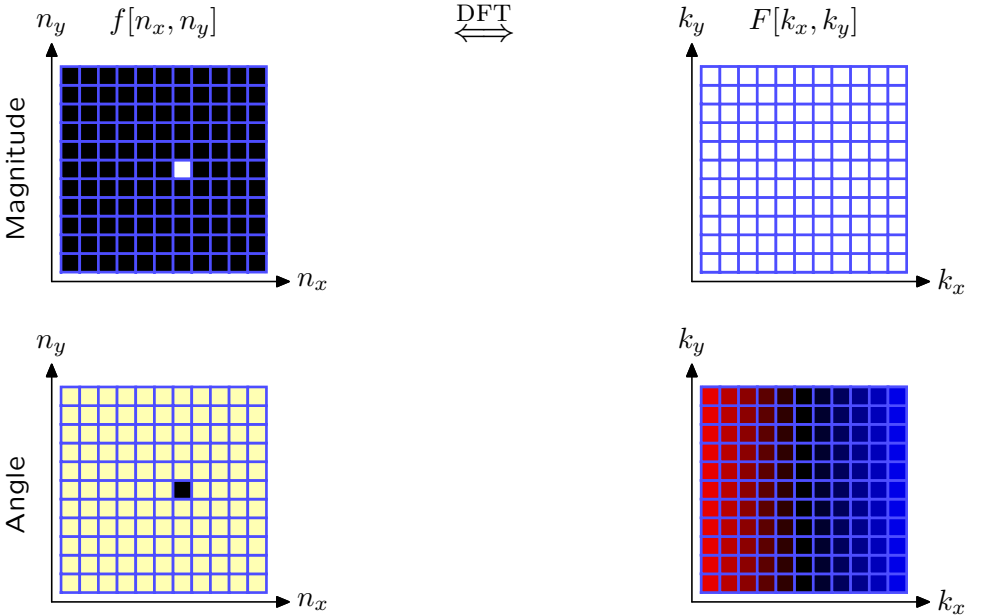

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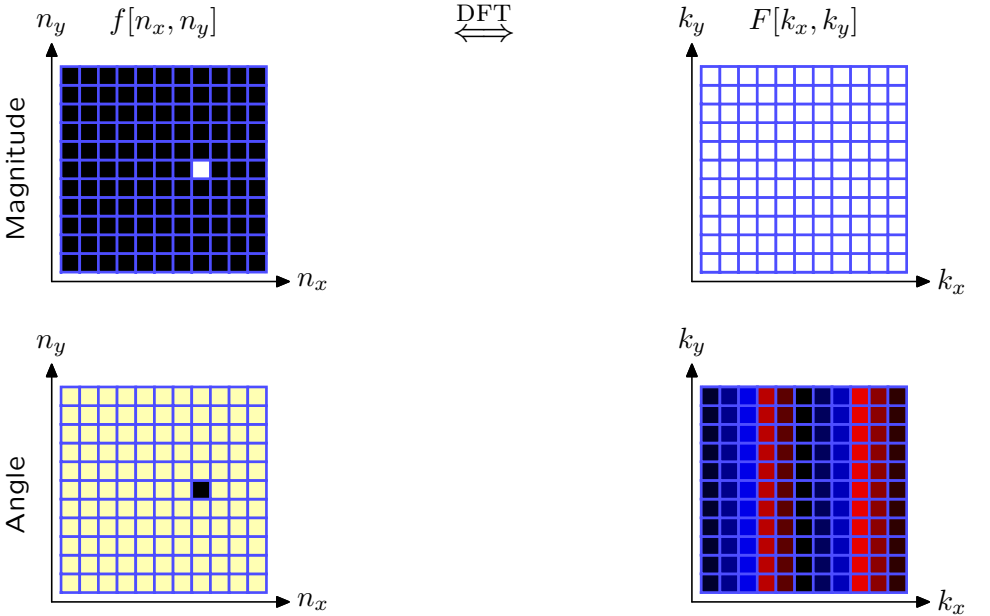

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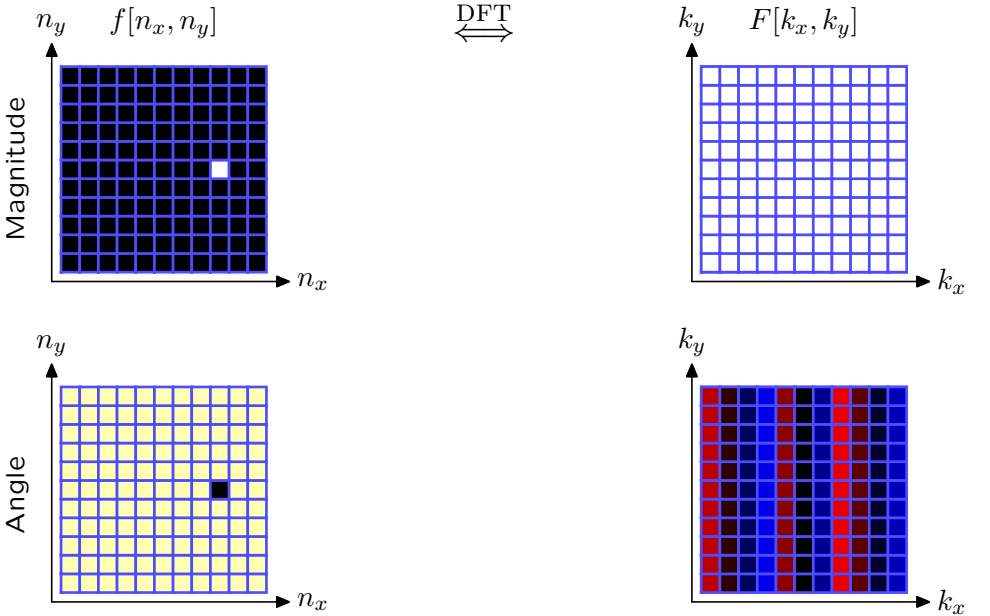

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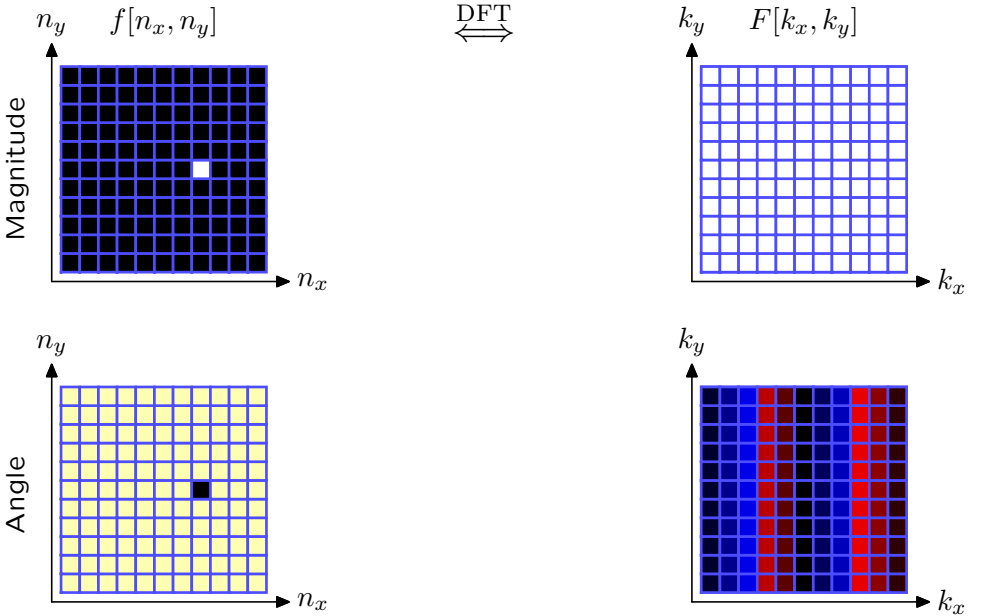

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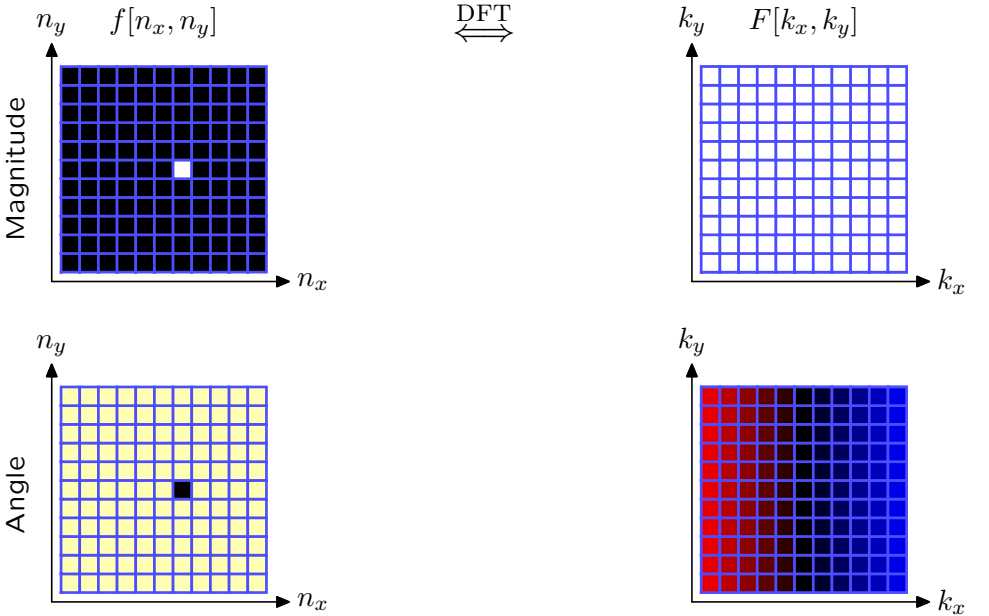

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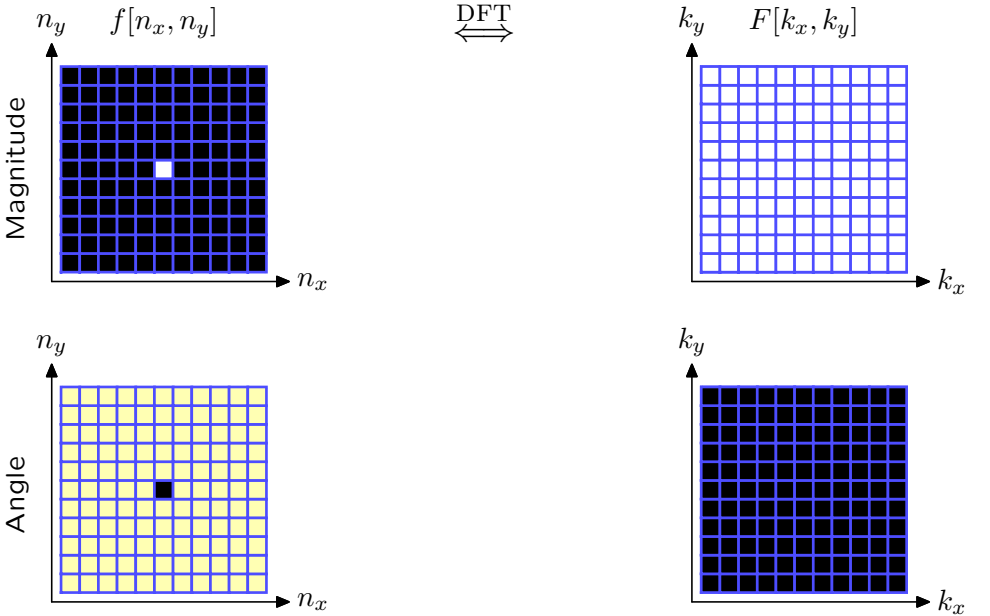

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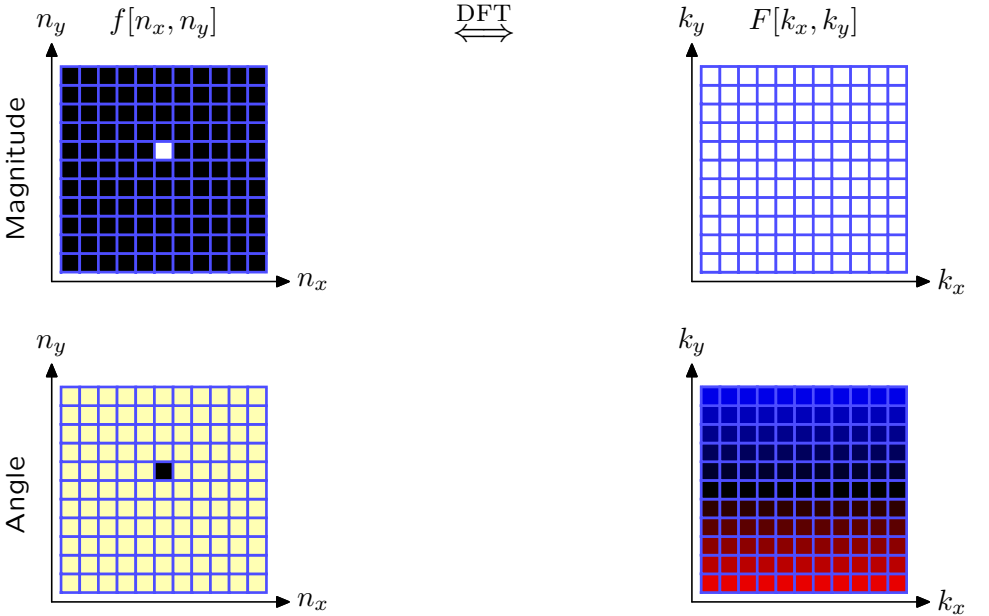

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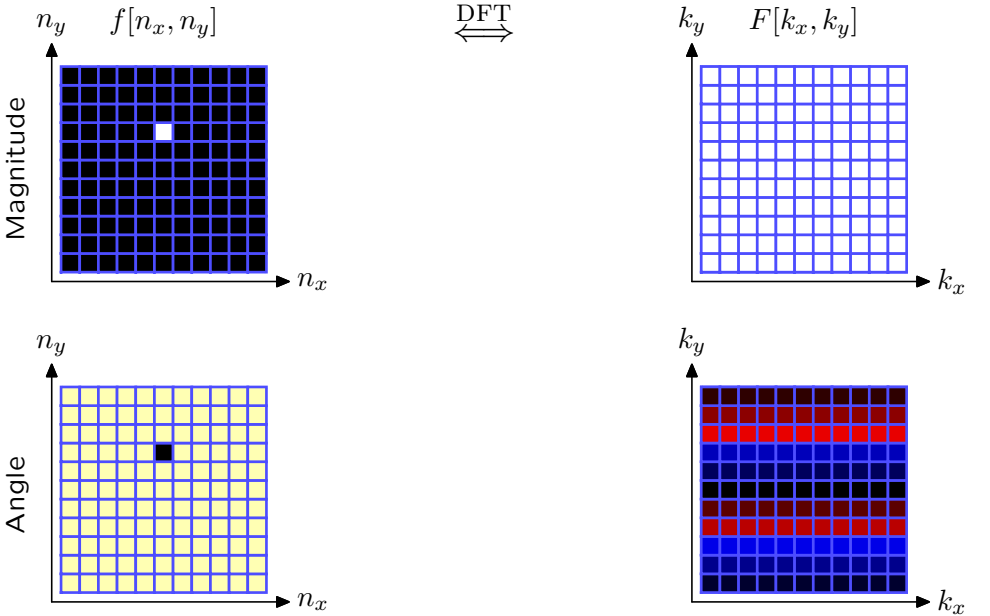
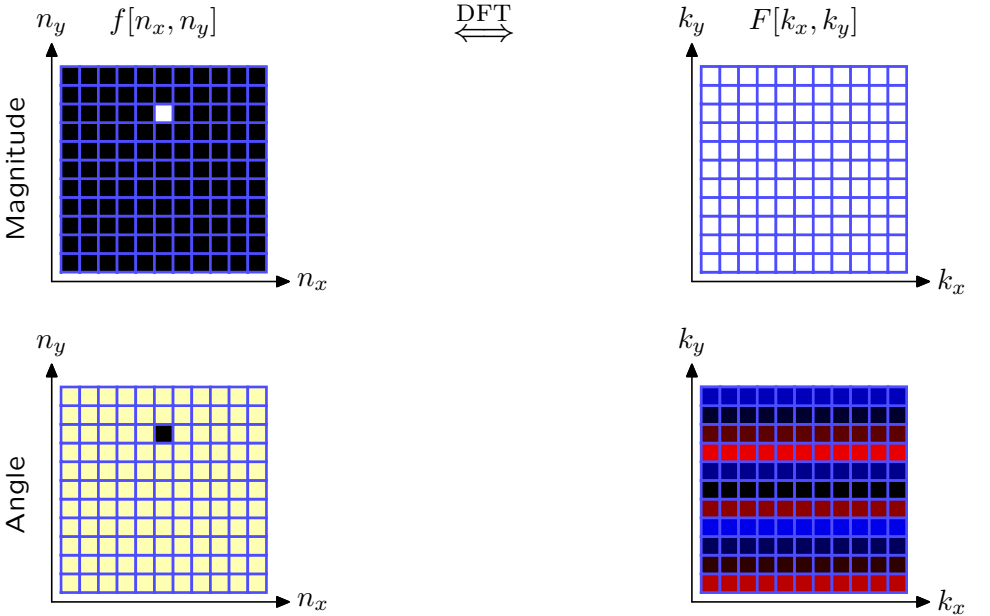


2D Discrete Fourier Transform

Example: Find the DFT of a shifted 2D unit sample.



Using Python

Calculating DFTs is most efficient in NumPy (Numerical Python).

- NumPy arrays are **homogeneous**: their elements are of the same type
- Numpy operators (`+`, `-`, `abs`, `.real`, `.imag`) combine **elements** to create new arrays. e.g., `(f+g)[n]` is `f[n]+g[n]`.
- 2D Numpy arrays can be **indexed by tuples**: e.g., `f[r,c] = f[r][c]`.
- 2D Numpy arrays support **negative indices**: e.g., `f[-1] = f[len(a)-1]`
- 2D indices address **row then column**.

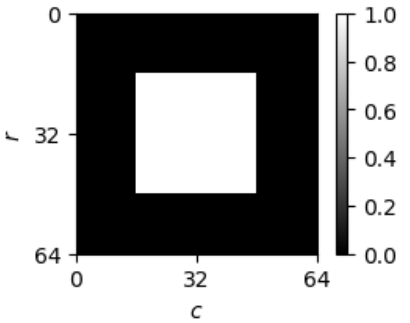
$$\begin{array}{ccccc} f[0,0] & f[0,1] & f[0,2] & f[0,3] & \cdots \\ f[1,0] & f[1,1] & f[1,2] & f[1,3] & \cdots \\ f[2,0] & f[2,1] & f[2,2] & f[2,3] & \cdots \\ f[3,0] & f[3,1] & f[3,2] & f[3,3] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

NumPy indexing is consistent with **linear algebra** (row first then column with rows increasing downward and columns increasing to the right). But it differs from **physical mathematics** (x then y with x increasing to the right and y increasing upward). You may do calculations either way, but row,column is often less confusing.

Numpy Example

Make a white square on a black background.

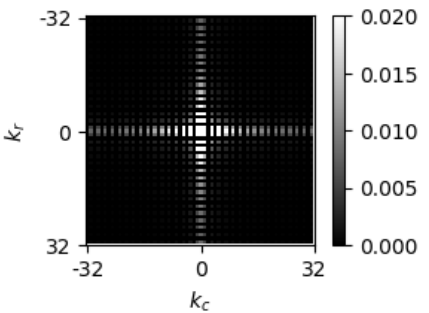
```
import numpy
from lib6003.image import show_image
f = numpy.zeros((64,64))
for r in range(16,48):
    for c in range(16,48):
        f[r,c] = 1
show_image(f,zero_loc='topleft')
```



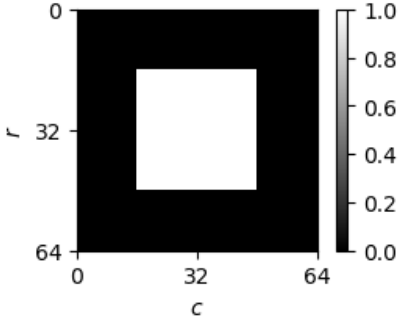
Numpy Example

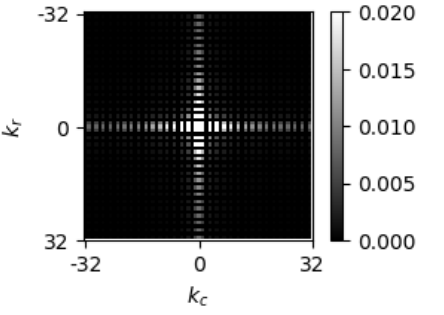
Find the 2D DFT of the square.

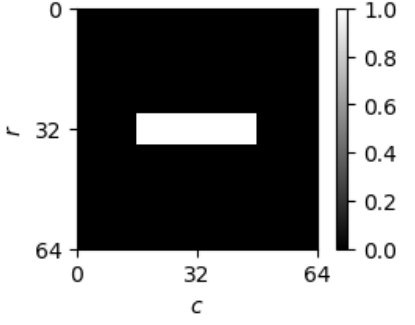
```
import numpy
from lib6003.image import show_image
from lib6003.fft import fft2

F = fft2(f)
show_image(numpy.abs(F),zero_loc='center',vmin=0,vmax=0.02)
```

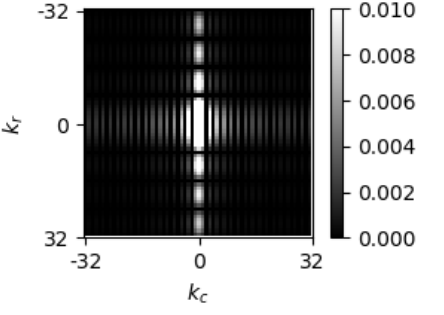


Big and Small



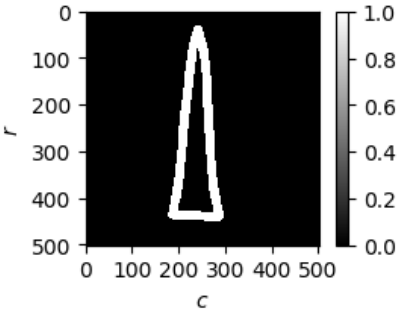






Triangle

What are the dominant features of the magnitude of the DFT of a triangle?



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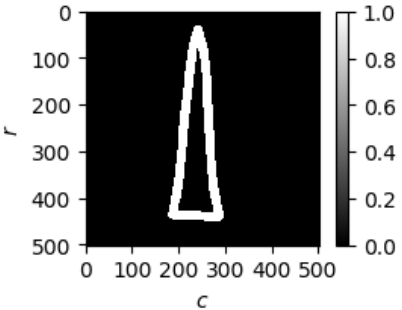

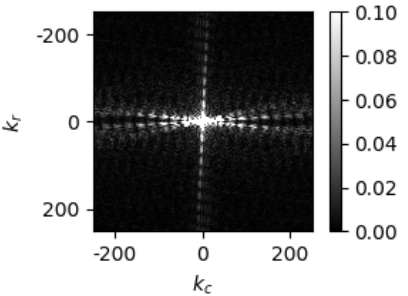

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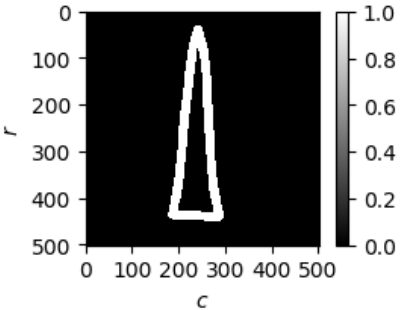
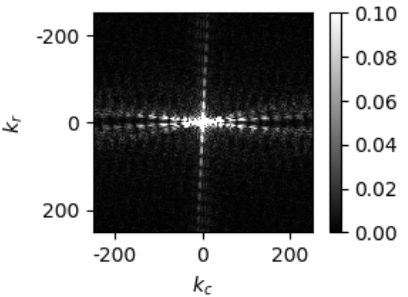
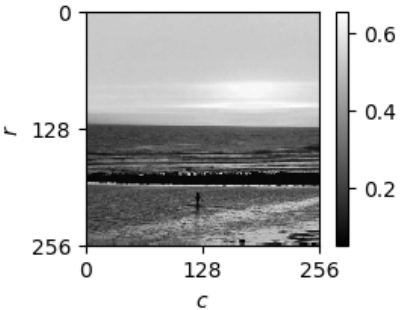




The DFT has three nearly linear features, one for each edge of the triangle. Lines in the frequency domain are perpendicular to those in space domain.

Ocean

What are the dominant features of the DFT magnitude of an ocean view?



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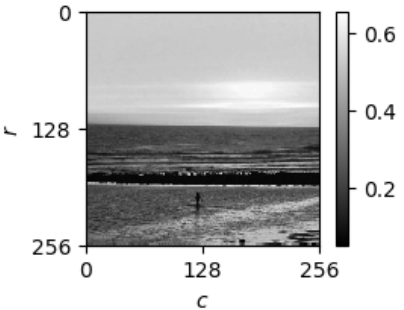

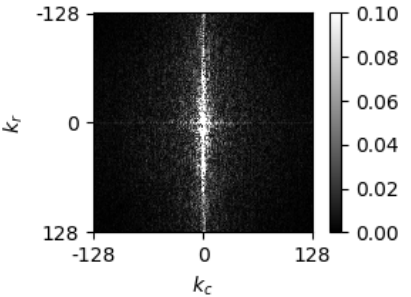

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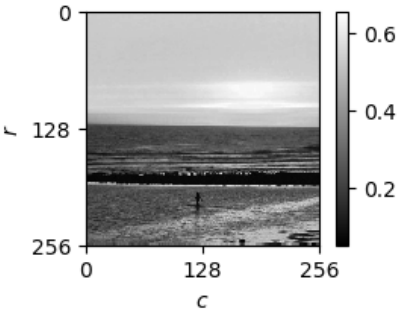
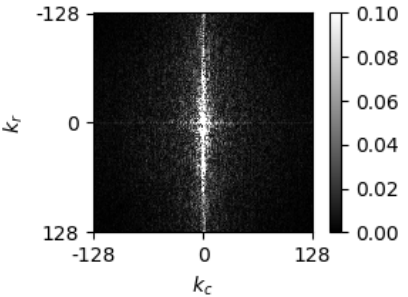
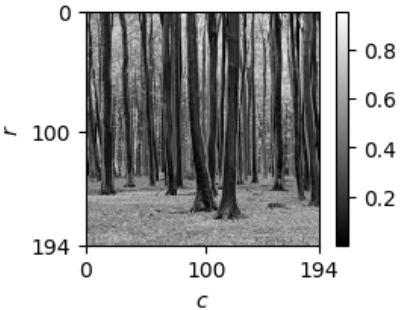




The horizontal features in the ocean view show up as a strong vertical line in the DFT.

Trees

What are the dominant features of the DFT magnitude of these trees?



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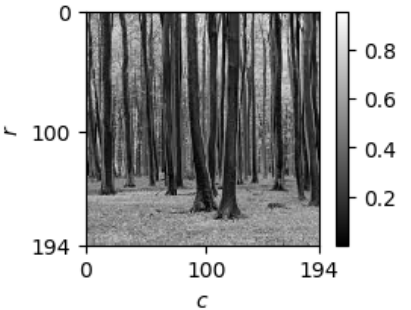

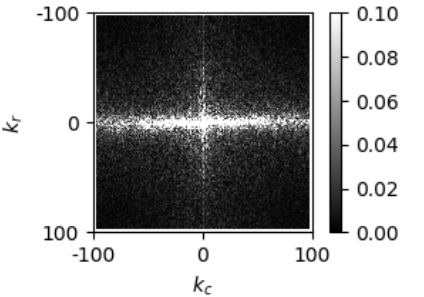

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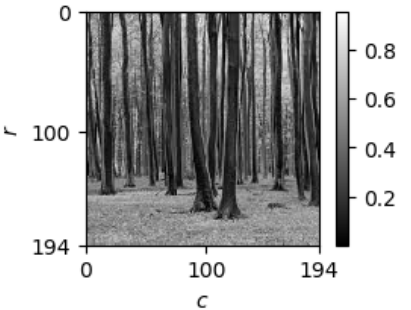
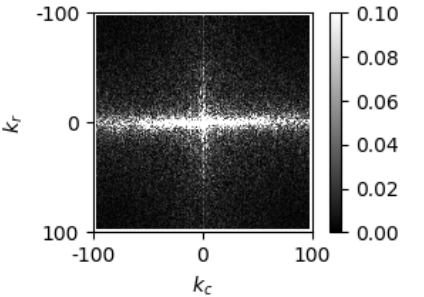
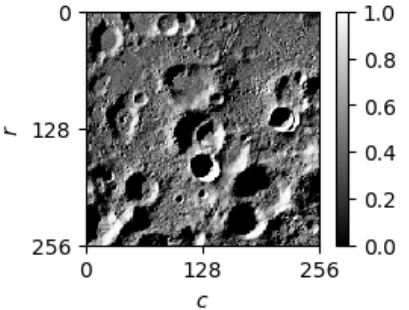




Now there is a strong horizontal line in the DFT.

Moon

What are the dominant features of the DFT magnitude of the moon?



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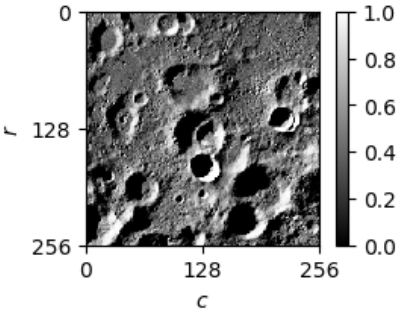

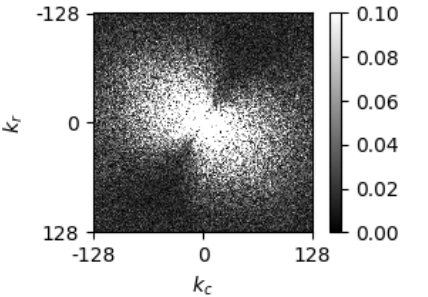

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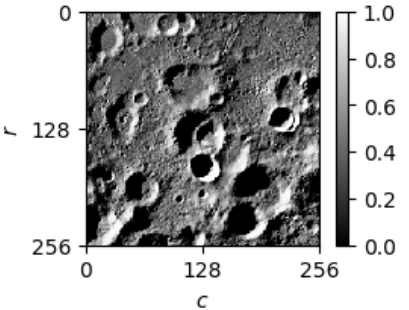
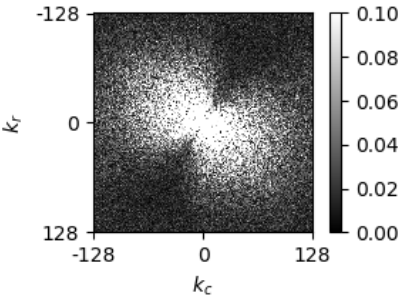
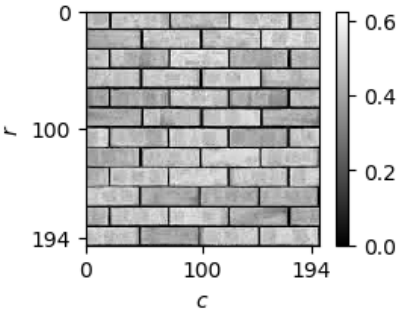




Large distribution of frequencies. Concentration along $r = c$ axis due to illumination from upper left.

Bricks

What are the dominant features of the DFT magnitude of this brick wall?



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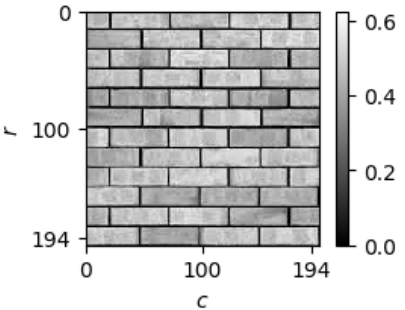

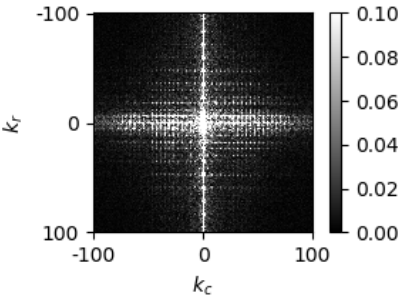

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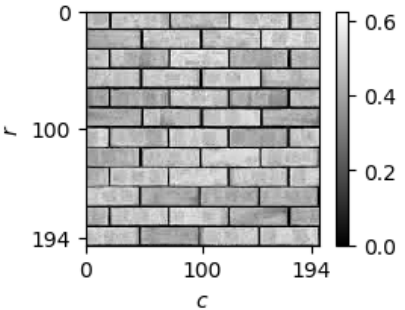
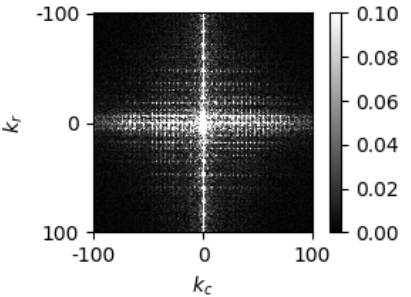


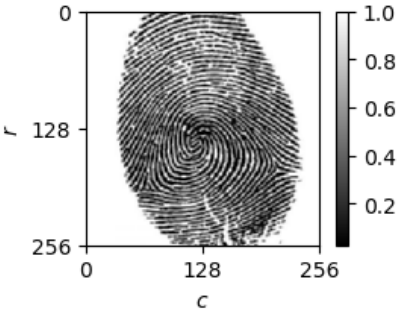


Strong horizontal "layer" in space $\rightarrow$ strong vertical line in frequency.

Strong vertical features but broken up periodically $\rightarrow$ periodicity in frequency.

Fingerprint

What are the dominant features of the DFT magnitude of this fingerprint?



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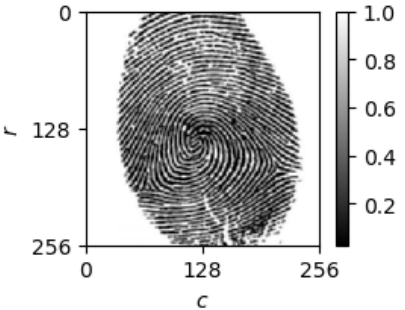

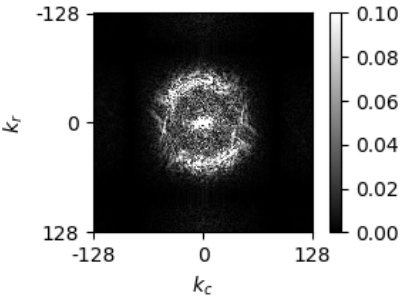

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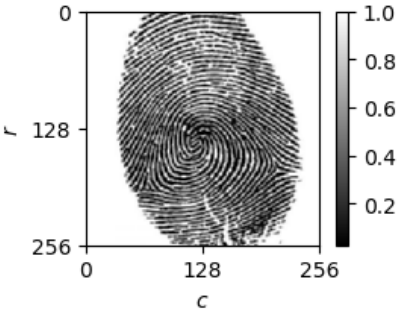

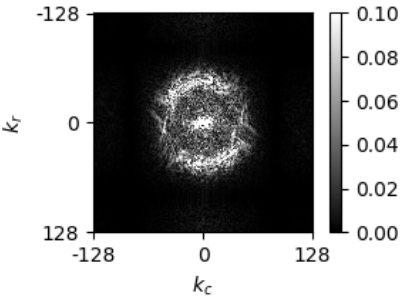

About 40 fingerprint ridges $\rightarrow k \approx 40$.

Summary

Introduced 2D signal processing.

- generally simple extensions of 1D ideas
- some differences (e.g., bi-linear interpolation)

Introduced 2D Fourier representations.

- Fourier kernal comprises the sum of an x part and a y part
- basis functions are complex exponentials

Properties of 2D DFT

- transform all of the rows then transform all of the columns
- transform all of the columns then transform all of the rows