6.003: Signal Processing

Discrete Fourier Transform

- Discrete Fourier Transform (DFT)
- Relations to Discrete-Time Fourier Transform (DTFT)
- Relations to Discrete-Time Fourier Series (DTFS)

October 19, 2021

Yet Another Fourier Representation

Why do we need another Fourier Representation?

Fourier series represent signals as sums of sinusoids. They provide insights that are not obvious from time representations, but Fourier series only defined for periodic signals.

$$X[k] = \sum_{n=\langle N\rangle} x[n]e^{-j2\pi kn/N} \qquad \text{(summed over a period)}$$

Fourier transforms have no periodicity constaint:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad \text{(summed over all samples } n\text{)}$$

but are functions of continuous domain (Ω).
$\rightarrow$ not convenient for numerical computations

Discrete Fourier Transform: discrete frequencies for aperiodic signals.

Discrete Fourier Transform

Definition and comparison to other Fourier representations.

	analysis	synthesis
DFT:	$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n}$
DTFS:	$X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi k}{N}n}$	$x[n] = \sum_{k=\langle N\rangle} X[k]e^{j\frac{2\pi k}{N}n}$
DTFT:	$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$

DTFS: $x[n]$ is presumed to be periodic in N
DTFT: $x[n]$ is arbitrary
DFT: only a portion of an arbitrary $x[n]$ is considered

Relation Between DFT and DTFS

If a signal is periodic in the DFT analysis period N, then the DFT coefficients are equal to the DTFS coefficients.

Let $x_1[n] = \cos\frac{2\pi n}{64}$. Then if N=64, the DFT coefficients are

$$X_1[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k-63]$$

as plotted below.

The coefficients are the same as the Fourier series coefficients.

Relation Between DFT and DTFS

If a signal is not periodic in the DFT analysis period N, then there are no Fourier series coefficients to compare.

Let $x_2[n] = \cos\frac{3\pi n}{64}$. Then if N=64, the DFT coefficients are

$$X_2[k] = \frac{1}{N}\sum_{n=0}^{N-1} x_2[n]e^{-j\frac{2\pi}{N}kn}$$

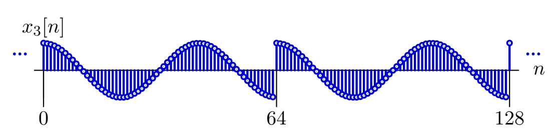
are plotted below.

Even though $x_2[n]$ contains a single frequency $\Omega = 3\pi/64$, there are large coefficients at many different frequencies k.

The reason is that $x_2[n]$ is not periodic in n with period $N = 64$.

Relation Between DFT and DTFS

Although $x_2[n] = \cos\frac{3\pi n}{64}$ is not periodic in N=64, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \le n < 64$ and that is periodic in N=64.



The DFT coefficients for this signal are the same as those for $x_2[n]$:

Furthermore, the DFT coefficients of $x_3[n]$ equal the DTFS coefficients of $x_3[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at $n = 64$.

Two Ways to Think About the DFT

Compare to DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in N.
$\rightarrow$ emphasizes the importance of **periodicity**.

Compare to DTFT:

2. The DFT is equal to samples of the **DTFT** of a windowed version of the original signal.
$\rightarrow$ emphasizes the importance of **spectral smear** in DFT representation.

Relation Between DFT and DTFT

The DFT can also be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $\frac{1}{N}$.

Let $x_w[n] = x[n] \times w[n]$ represent a **windowed** version of $x[n]$ where

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Then the Fourier transform of $x_w[n]$ is

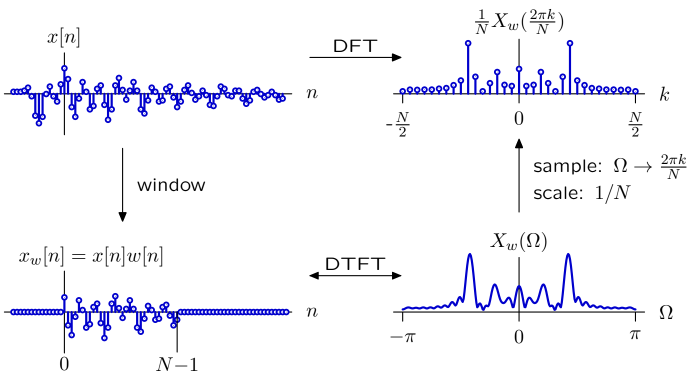
$$X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n] e^{-j\Omega n}$$

Scale both sides of this equation by $\frac{1}{N}$. Then **sample** the resulting function of Ω at $\Omega = \frac{2\pi k}{N}$ to obtain an expression for the DFT of $x[n]$:

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right)$$

Relation Between DFT and DTFT

Graphical depiction of relation between DFT and DTFT.



While sampling and scaling are important, it is the **windowing** that most affects frequency content.

Effect of Windowing on Fourier Representations

Example: characterize the effect of windowing on complex exponential signals, which are the basis functions for Fourier analysis.

Step 1: Find $X(\Omega)$, the DTFT of a complex exponential signal: $x[n] = e^{j\Omega_o n}$

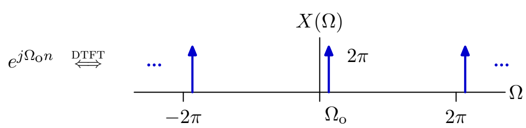
Step 2: Find $X_w(\Omega)$, the DTFT of a windowed version of $x[n]$: $x_w[n] = x[n]w[n]$

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

Effect of Windowing on Fourier Representations

Step 1: Find the DTFT of a complex exponential $x[n] = e^{j\Omega_o n}$.

The DTFT of a complex exponential is a train of impulses.



This is easy to verify using the DTFT synthesis equation.

$$\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_o) e^{j\Omega n}\, d\Omega \\ &= \int_{-\pi}^{\pi} \delta(\Omega - \Omega_o) e^{j\Omega n}\, d\Omega \\ &= e^{j\Omega_o n} \end{aligned}$$

Effect of Windowing on Fourier Representations

Step 2: Find the DTFT of $x_w[n]$, a windowed version of $x[n]$.

Let $x_w[n]$ represent a windowed version of $x[n]$.

$$x_w[n] = x[n]w[n] = e^{j\Omega_o n} w[n]$$

Then

$$\begin{aligned} X_w(\Omega) &= \sum_{n=-\infty}^{\infty} e^{j\Omega_o n} w[n] e^{-j\Omega n} && \text{DTFT analysis equation} \\ &= \sum_{n=-\infty}^{\infty} w[n] e^{-j(\Omega - \Omega_o)n} && \text{combine exponential terms} \\ &= W(\Omega - \Omega_o) && \text{DTFT of } w[n]\text{, shifted in frequency} \end{aligned}$$

The DTFT of a windowed version of a complex exponential signal is a shifted version of the DTFT of the window signal.

$$e^{j\Omega_o n} w[n] \overset{\text{DTFT}}{\Longleftrightarrow} W(\Omega - \Omega_o)$$

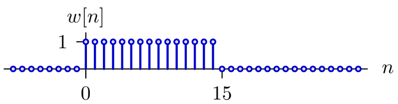
$\rightarrow$ Need to know $W(\Omega)$.

Effect of Windowing on Fourier Representations

Simplest window is rectangular, with width of N (length of DFT analysis)

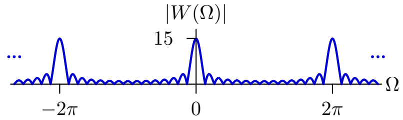
$$w[n]=\begin{cases}1 & 0\le n<N\\ 0 & \text{otherwise}\end{cases}$$

as shown below for $N=15$.



The DTFT of $w[n]$ is

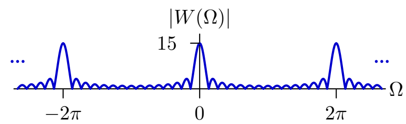
$$W(\Omega)=\sum_{n=-\infty}^{\infty}w[n]e^{-j\Omega n}=\sum_{n=0}^{N-1}e^{-j\Omega n}=\frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}}=\frac{\sin\frac{N\Omega}{2}}{\sin\frac{\Omega}{2}}e^{-j\Omega\frac{N-1}{2}}$$

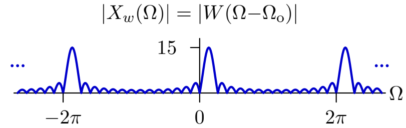


Effect of Windowing on Fourier Representations

The DTFT of a windowed version of a complex exponential signal is a shifted version of the DTFT of the window signal.

$$x_w[n]=e^{j\Omega_o n}w[n] \overset{\text{DTFT}}{\Longleftrightarrow} X_w(\Omega)=W(\Omega-\Omega_o)$$

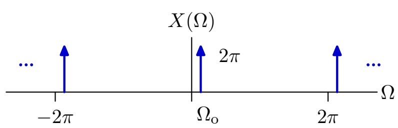




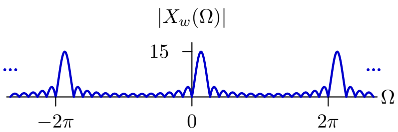
Effect of Windowing on Fourier Representations

Step 3: Compare $X_w(\Omega)$ to $X(\Omega)$.

$x[n]=e^{j\Omega_o n} \overset{\text{DTFT}}{\Longleftrightarrow}$



$x_w[n]=x[n]w[n] \overset{\text{DTFT}}{\Longleftrightarrow}$

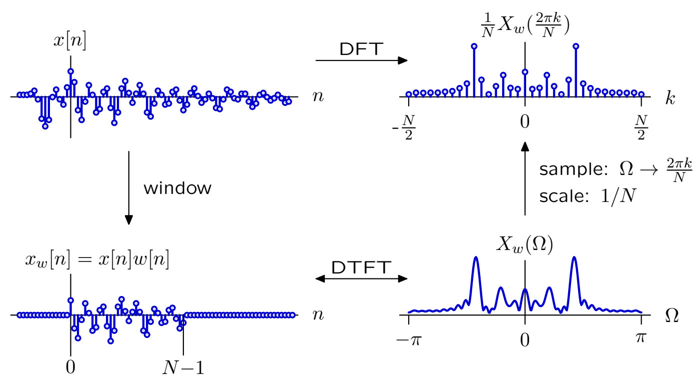


The frequency content of $X(\Omega)$ is at discrete frequencies $\Omega=\Omega_o+2\pi m$.

The frequency content of $X_w(\Omega)$ is most dense at these same frequencies, but is spread out over almost all other frequencies as well.

Relation Between DFT and DTFT

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ **scaled** by $\frac{1}{N}$.



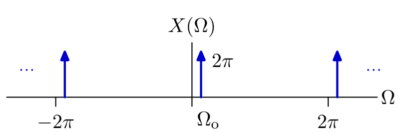
The remaining steps are to sample and scale.

Effect of Windowing on Fourier Representations

The DFT can be thought of as **samples** of the DTFT of a **windowed** version of $x[n]$ scaled by $1/N$. Here $\Omega_o=\frac{2\pi}{15}$.

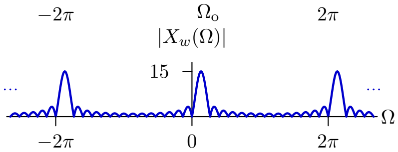
original signal

$x[n]=e^{j\Omega_o n} \overset{\text{DTFT}}{\Longleftrightarrow}$



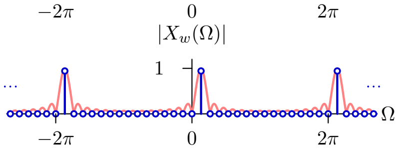
windowed

$x_w[n]=x[n]w[n] \overset{\text{DTFT}}{\Longleftrightarrow}$



sampled and scaled

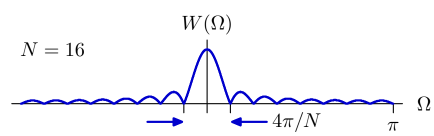
$x_w[n]=x[n]w[n] \overset{\text{DFT}}{\Longleftrightarrow}$

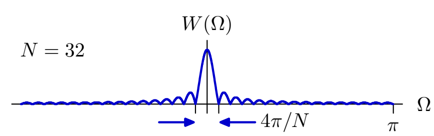


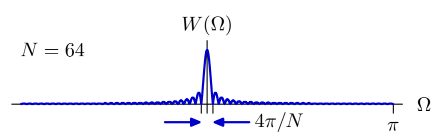
One sample is taken at the peak, and the others fall on zeros.

Spectral Blurring introduces a Time/Frequency Tradeoff

Longer windows provide finer frequency resolution.





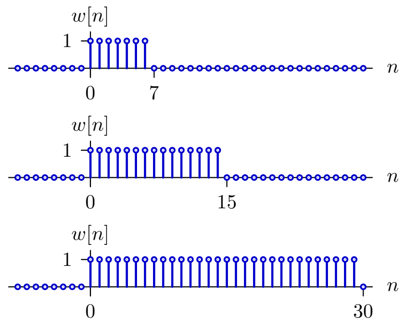


The width of the central lobe is inversely related to window length.

Spectral Blurring introduces a Time/Frequency Tradeoff

However, longer windows provide less temporal resolution.

$$w[n] = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

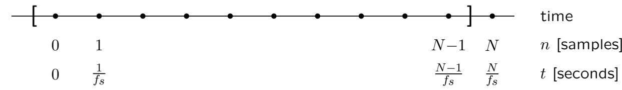


→ fundamental tradeoff between resolution in frequency and time.

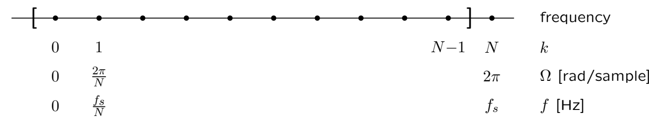
Time and Frequency Resolution

We can think about the tradeoff between time and frequency by examining resolution in time and resolution in frequency.
The DFT analysis period N determines both.

The time window is divided into N samples numbered $n = 0$ to $N-1$.



Discrete frequencies are similarly numbered as $k = 0$ to $N-1$.



Two Ways to Think About the DFT

Compare to DTFS:

1. The DFT of a signal $x[n]$ is equal to the **DTFS** of a version of $x[n]$ that is periodically extended so that it is periodic in N.
→ emphasizes the importance of **periodicity**.

Compare to DTFT:

2. The DFT is equal to samples of the **DTFT** of a windowed version of the original signal.
→ emphasizes the importance of **spectral smear** in DFT representation.

These **views are equivalent** – but they highlight different phenomena.

Summary

Today we introduced a new Fourier representation for DT signals: the Discrete Fourier Transform (DFT).

The DFT has a number of features that make it particular convenient.
- It is not limited to periodic signals.
- It has discrete domain (k instead of Ω) and finite length: convenient for numerical computation.

A central issue is using the DFT is understanding the tradeoff between time and frequency.
- Long analysis windows N provide high resolution in frequency but poor resolution in time.
- Short analysis windows N provide high resolution in time but poor resolution in frequency.

More about these issues in recitation and next lecture.