6.003: Signal Processing

**Frequency Response and Filtering**

- Discrete-Time Frequency Response
- Continuous-Time Frequency Response

*Context: The System Abstraction*

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

Three important representations for LTI systems:
- **Difference/Differential Eq**: algebraic input/output constraint
- **Convolution**: represent system by unit-sample/impulse response
- **Filter**: represent a system by its frequency response

**Unit-Sample Response**

The unit-sample response is a complete description of a system.

\[ \delta[n] \rightarrow \text{LTI} \rightarrow h[n] \]

This is a bit surprising since \( \delta[n] \) is such a simple signal. The unit-sample signal is the shortest possible non-trivial DT signal.

The response to this simple signal determines the response to any other input signal:

\[ x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \]

**Representing Systems with Difference/Differential Equations**

Discrete-time systems that can be described by linear difference equations with constant coefficients are linear and time-invariant.

\[ x[n] \rightarrow \text{LTI} \rightarrow y[n] \]

\[ \sum_l c_l y[n-l] = \sum_m d_m x[n-m] \]

Continuous-time systems that can be described by linear differential equations with constant coefficients are linear and time-invariant.

\[ x(t) \rightarrow \text{LTI} \rightarrow y(t) \]

\[ \sum_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m} \]

- natural and compact representations of many systems

**Unit-Sample Response and Impulse Response**

Discrete-time systems that are linear and time-invariant can be completely specified by their response to a unit-sample signal.

\[ x(t) \rightarrow \text{LTI} \rightarrow y(t) \]

If \( \delta[n] \rightarrow h[n] \), then \( x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m]h[n-m] \)

Continuous-time systems that are linear and time-invariant are completely specified by their response to a unit impulse function.

\[ x(t) \rightarrow h(t), \text{then } x(t) \rightarrow y(t) = (x * h)[t] = \int x(\tau)h(t-\tau)d\tau \]

- an LTI system is completely characterized by a single signal

**Context: The System Abstraction**

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

Three important representations for LTI systems:
- **Difference/Differential Eq**: algebraic input/output constraint
- **Convolution**: represent system by unit-sample/impulse response
- **Filter**: represent a system by its frequency response
**Frequency Response**

The frequency response is a **complete** characterization of an LTI system.

1. One can always find the frequency response of a system.

\[
e^{j\Omega n} \rightarrow H(\Omega) e^{j\Omega n}
\]

2. Scaling the input by a constant scales the output by the same constant.

\[
x(\Omega) e^{j\Omega n} \rightarrow X(\Omega) H(\Omega) e^{j\Omega n}
\]

3. Linearity implies that the response to a sum is the sum of the responses.

\[
x[n] = \sum_{m=-\infty}^{\infty} x[n-m] h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)} h[m]
\]

4. The Fourier transform of the output is \(X(\Omega) H(\Omega)\).

\[
X(\Omega) \rightarrow X(\Omega) H(\Omega)
\]

The transform of the output is \(H(\Omega)\) times the transform of the input.

**Frequency Response**

The frequency response can be an **insightful** description of a system.

Example:

A low-pass filter passes frequencies near 0 and rejects those near \(\pi\).

\[
H(\Omega)
\]

Very natural way to describe audio components:

- microphones
- loudspeakers
- audio equalizers

**Frequency Response**

Using complex exponentials to characterize the frequency response.

\[
e^{j\Omega n} \rightarrow LTI \rightarrow A_0 e^{j\Omega n}
\]

Notice that the complex valued \(A\) can represent both amplitude and phase. We can find \(A\) using convolution.

\[
y[n] = (x * h)[n] = \sum_{m=-\infty}^{\infty} x[n-m] h[m] = \sum_{m=-\infty}^{\infty} e^{j\Omega(n-m)} h[m]
\]

\[
= e^{j\Omega n} \sum_{m=-\infty}^{\infty} h[m] e^{-j\Omega m} = H(\Omega) e^{j\Omega n}
\]

The response to a complex exponential is a complex exponential with a different frequency. The map for how a system modifies the amplitude and phase of a complex exponential input is the Fourier transform of the unit-sample response.

**Unit-Sample Response and Frequency Response**

Two complete representations for linear, time-invariant systems.

**Unit-Sample Response**: responses across time for a unit-sample input.

\[
x[n] \rightarrow LTI \rightarrow y[n]
\]

**Frequency Response**: responses across frequencies for sinusoidal inputs.

\[
H(\Omega)
\]

The frequency response is Fourier transform of unit-sample response!
Find the frequency response of a system described by the following:
\[ y[n] - \alpha y[n-1] = x[n] \]

**Method 1:**
Find the unit-sample response and take its Fourier transform.
\[ h[n] - \alpha h[n-1] = \delta[n] \]
Solve the difference equation for \( h[n] \).
\[ h[0] = \delta[0] + \alpha h[-1] = 1 \]
\[ h[1] = \delta[1] + \alpha h[0] = \alpha \]
\[ h[2] = \delta[2] + \alpha h[1] = \alpha^2 \]
\[ h[3] = \delta[3] + \alpha h[2] = \alpha^3 \]
\[ h[n] = \alpha^n \delta[n] \]
\[ H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j \Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j \Omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j \Omega}\right)^n = \frac{1}{1-\alpha e^{-j \Omega}} \]

**Method 2:**
Find the response to \( e^{j \Omega n} \) directly.
\[ x[n] = e^{j \Omega n} \]
Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.
\[ y[n] = H(\Omega) e^{j \Omega n} \]
\[ y[n-1] = H(\Omega) e^{-j \Omega (n-1)} = H(\Omega) e^{-j \Omega n} e^{j \Omega} = H(\Omega) e^{j \Omega n} \]
Substitute into the difference equation.
\[ H(\Omega) e^{j \Omega n} - \alpha H(\Omega) e^{-j \Omega n} e^{j \Omega} = H(\Omega) (1 - \alpha e^{-j \Omega}) e^{j \Omega n} = e^{j \Omega n} \]
Since \( e^{j \Omega n} \) is never 0, we can divide it out.
\[ H(\Omega) = \frac{1}{1-\alpha e^{-j \Omega}} \]
Same answer as method 1.

**Example**
Find the frequency response of a system described by the following:
\[ y[n] - \alpha y[n-1] = x[n] \]

**Method 3:**
Take the Fourier transform of the difference equation.
\[ Y(\Omega) - \alpha e^{-j \Omega} Y(\Omega) = X(\Omega) \]
Solve for \( Y(\Omega) \).
\[ Y(\Omega) = \frac{1}{1-\alpha e^{-j \Omega}} X(\Omega) \]
Since \( Y(\Omega) = H(\Omega) X(\Omega) \),
\[ H(\Omega) = \frac{1}{1-\alpha e^{-j \Omega}} \]
Same answer as methods 1 and 2.

**Example**
Find the frequency response of a system described by the following:
\[ y[n] - \alpha y[n-1] = x[n] \]

**Example**
Find the frequency response of a system described by the following:
\[ y[n] - \alpha y[n-1] = x[n] \]

**The System Abstraction**
The system abstraction applies equally well for continuous-time signals.
**The System Abstraction**
Describe a system (physical, mathematical, or computational) by the way
it transforms an input signal into an output signal.

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

Three important representations for LTI systems:
- Differential Equation: algebraic constraint on derivatives ✓
- Convolution: represent a system by its impulse response ✓
- Filter: represent a system by its frequency response

**Unit-Sample Response and Impulse Response**
Discrete-time systems that are linear and time-invariant can be completely specified by their response to a unit-sample signal.

\[
x[n] \rightarrow \text{LTI} \rightarrow y[n]
\]

If \( \delta[n] \rightarrow h[n] \), then \( x[n] \rightarrow y[n] = (x * h)[n] = \sum_m x[m] h[n-m] \)

Continuous-time systems that are linear and time-invariant are completely specified by their response to a unit impulse function.

\[
x(t) \rightarrow \text{LTI} \rightarrow y(t)
\]

If \( \delta(t) \rightarrow h(t) \), then \( x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau) h(t-\tau) d\tau \)

- an LTI system is completely characterized by a single signal

**Frequency Response**
The frequency response is a third way to characterize a linear time-invariant system. This characterization is based on responses to sinusoids.

\[
\cos(\omega t) \rightarrow \text{LTI} \rightarrow A \cos(\omega t + \phi)
\]

The idea is to characterize a system by the way \( A \) and \( \phi \) vary with \( \omega \).

Sinusoids differ from the unit-sample signal in important ways:
- eternal (longest possible signals) versus transient (shortest possible)
- comprises a single frequency versus a sum of all possible frequencies

**Representing Systems with Difference/Differential Equations**
Discrete-time systems that can be described by linear difference equations with constant coefficients are linear and time-invariant.

\[
x[n] \rightarrow \text{LTI} \rightarrow y[n]
\]

\[
\sum_l c_l y[n-l] = \sum_m d_m x[n-m]
\]

Continuous-time systems that can be described by linear differential equations with constant coefficients are linear and time-invariant.

\[
x(t) \rightarrow \text{LTI} \rightarrow y(t)
\]

\[
\sum_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}
\]

- natural and compact representations of many systems

**Impulse Response**
The impulse response is a complete description of a system.

\[
\delta(t) \rightarrow \text{LTI} \rightarrow h(t)
\]

This is a bit surprising since \( \delta(t) \) is zero almost everywhere.
The impulse function is the shortest possible non-trivial CT signal!

\[
x(t) \rightarrow y(t) = (x * h)(t) = \int x(\tau) h(t-\tau) d\tau
\]

The response to this signal determines the response to any other input.

**Frequency Response**
Using complex exponentials to characterize the frequency response.

\[
e^{j\omega t} \rightarrow \text{LTI} \rightarrow A e^{j\omega t}
\]

Notice that the complex valued \( A \) can represent both amplitude and phase.

We can find \( A \) using convolution.

\[
y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)} h(\tau) d\tau
\]

\[
= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau = H(\omega) e^{j\omega t}
\]

The response to a complex exponential is a complex exponential with the same frequency but possibly different amplitude and phase.

The map for how a system modifies the amplitude and phase of a complex exponential input is the Fourier transform of the impulse response.
**Frequency Response**

The frequency response is a complete characterization of an LTI system.

1. One can always find the frequency response of a system.
   \[
   e^{j\omega t} \quad \text{system} \implies H(\omega)e^{j\omega t}
   \]

2. Scaling the input by a constant scales the output by the same constant.
   \[
   X(\omega)e^{j\omega t} \quad \text{system} \implies X(\omega)H(\omega)e^{j\omega t}
   \]

3. Linearity implies that the response to a sum is the sum of the responses.
   \[
   \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad \text{system} \implies \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}d\omega
   \]

4. The Fourier transform of the output is \(X(\omega)H(\omega)\).
   \[
   X(\omega) \quad \text{system} \implies X(\omega)H(\omega)
   \]

The transform of the output is \(H(\omega)\) times the transform of the input.

**Example**

Find the frequency response of a system described by the following:

\[
y(t) + \alpha \frac{dy(t)}{dt} = x(t)
\]

**System Abstraction**

Two complete representations for linear, time-invariant systems.

**Impulse Response:** responses across time for a impulse input.

**Frequency Response:** responses across frequencies for sinusoidal inputs.

The frequency response is Fourier transform of impulse response!

**Frequency Response**

The frequency response is a complete description of a system.

\[
e^{j\omega t} \rightarrow \text{LTI} \rightarrow H(\omega)e^{j\omega t}
\]

This is a bit surprising since \(e^{j\omega t}\) contains a single frequency.

\[\rightarrow\] can find the output of a system by breaking the input into its constituent frequencies and summing the responses to each frequency — one-at-a-time.

\[
\cos(\omega t) \rightarrow \ldots \rightarrow A\cos(\omega t + \phi) \ldots
\]

The frequency response can be used to find response to any input signal:

\[
X(\omega) \rightarrow Y(\omega) = H(\omega)X(\omega)
\]

**Example**

Find the frequency response of a system described by the following:

\[
y(t) + \alpha \frac{dy(t)}{dt} = x(t)
\]

**Method 1:**

Find the response to \(e^{j\omega t}\) directly.

\[
x(t) = e^{j\omega t}
\]

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

\[
y(t) = H(\omega)e^{j\omega t}
\]

\[
\frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t}
\]

Substitute into the differential equation.

\[
H(\omega)e^{j\omega t} + j\omega \alpha H(\omega)e^{j\omega t} = (1 + j\omega \alpha)H(\omega)e^{j\omega t} = e^{j\omega t}
\]

Since \(e^{j\omega t}\) is never 0, we can divide it out.

\[
H(\omega) = \frac{1}{1 + j\omega \alpha}
\]
Example
Find the frequency response of a system described by the following:
\[ y(t) + \alpha \frac{dy(t)}{dt} = x(t) \]

**Method 2:**
Take the Fourier transform of the differential equation.
\[ Y(\omega) + j\omega \alpha Y(\omega) = X(\omega) \]
Solve for \( Y(\omega) \).
\[ Y(\omega) = \frac{1}{1 + j\omega \alpha} X(\omega) \]
Since \( Y(\omega) = H(\omega)X(\omega) \),
\[ H(\omega) = \frac{1}{1 + j\omega \alpha} \]
Same answer as method 1.

Check Yourself
Find the frequency response of a rectangular box filter:
\[ y(t) = \frac{1}{2} \int_{t-1}^{t+1} x(\tau)d\tau \]
(This CT filter is analogous to the three-point averager in DT.)

Example
Plot the frequency response.
\[ H(\omega) = \frac{1}{1 + j\omega \alpha} \]
Note that denominator is sum of 2 complex numbers. Amplifies low frequencies, attenuates high frequencies, adds phase delay.

Summary
The Fourier transform of the response of a DT LTI system is the product of the Fourier transform of the input times the system’s **frequency response**.
\[ X(\Omega) \xrightarrow{\text{LTI}} Y(\Omega) \]
\[ Y(\Omega) = H(\Omega)X(\Omega) \]

The frequency response \( H(\Omega) \) is the Fourier transform of the unit-sample response \( h[n] \).

The Fourier transform of the response of a CT LTI system is the product of the Fourier transform of the input times the system’s **frequency response**.
\[ X(\omega) \xrightarrow{\text{LTI}} Y(\omega) \]
\[ Y(\omega) = H(\omega)X(\omega) \]

The frequency response \( H(\omega) \) is the Fourier transform of the impulse response \( h(t) \).