6.003: Signal Processing

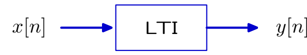
Frequency Response and Filtering

- Discrete-Time Frequency Response
- Continuous-Time Frequency Response

October 14, 2021

Representing Systems with Difference/Differential Equations

Discrete-time systems that can be described by linear **difference** equations with constant coefficients are linear and time-invariant.



$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

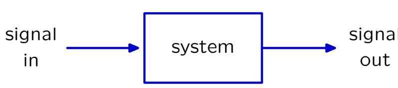
Continuous-time systems that can be described by linear **differential** equations with constant coefficients are linear and time-invariant.

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

– natural and compact representations of many systems

Context: The System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



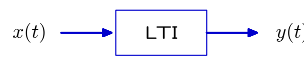
This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference/Differential Eq:** algebraic input/output **constraint** √
- **Convolution:** represent system by **unit-sample/impulse response** √
- Filter: represent a system by its frequency response

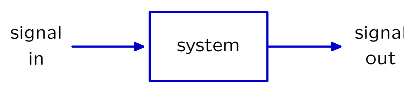
Unit-Sample Response and Impulse Response

Discrete-time systems that are linear and time-invariant can be completely specified by their response to a **unit-sample signal**.



If $\delta[n] \to h[n]$, then $x[n] \to y[n] = (x*h)[n] = \sum_m x[m]h[n-m]$

Continuous-time systems that are linear and time-invariant are completely specified by their response to a unit **impulse** function.

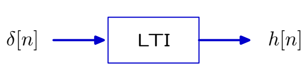


If $\delta(t) \to h(t)$, then $x(t) \to y(t) = (x*h)(t) = \int x(\tau)h(t-\tau)d\tau$

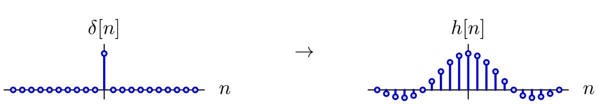
– an LTI system is completely characterized by **a single signal**

Unit-Sample Response

The unit-sample response is a **complete** description of a system.



This is a bit surprising since $\delta[n]$ is such a **simple signal.**
The unit-sample signal is the **shortest** possible non-trivial DT signal!



The response to this simple signal determines the response to **any** other input signal:

$$x[n] \to y[n] = (x*h)[n] = \sum_m x[m]h[n-m]$$

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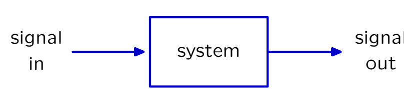
Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

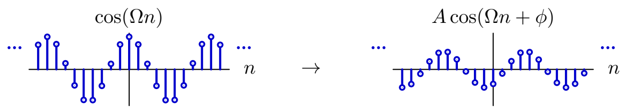
Three important representations for LTI systems:

- **Difference/Differential Eq:** algebraic input/output **constraint** √
- **Convolution:** represent system by **unit-sample/impulse response** √
- **Filter:** represent a system by its **frequency response**

Frequency Response

The **frequency response** is a third way to characterize a linear time-invariant system. This characterization is based on responses to **sinusoids.**

$\cos(\Omega n)$ → LTI → $A\cos(\Omega n+\phi)$



The idea is to characterize a system by the way A and ϕ vary with Ω.

Sinusoids differ from the unit-sample signal in important ways:

- **eternal** (longest possible signals) versus **transient** (shortest possible)
- comprises a **single** frequency versus a **sum** of all possible frequencies

Frequency Response

Using complex exponentials to characterize the frequency response.

$e^{j\Omega n}$ → LTI → $Ae^{j\Omega n}$

Notice that the complex valued A can represent both amplitude and phase. We can find A using convolution.

$$y[n]=(x*h)[n]=\sum_{m=-\infty}^{\infty}x[n-m]h[m]=\sum_{m=-\infty}^{\infty}e^{j\Omega(n-m)}h[m]$$

$$=e^{j\Omega n}\sum_{m=-\infty}^{\infty}h[m]e^{-j\Omega m}=H(\Omega)\,e^{j\Omega n}$$

The response to a complex exponential is a complex exponential with the **same frequency** but possibly **different amplitude and phase.**

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the unit-sample response.**

Frequency Response

The frequency response is a **complete** characterization of an LTI system.

1. One can always find the frequency response of a system.

$e^{j\Omega n}$ → LTI → $H(\Omega)e^{j\Omega n}$

2. Scaling the input by a constant scales the output by the same constant.

$X(\Omega)e^{j\Omega n}$ → LTI → $X(\Omega)H(\Omega)e^{j\Omega n}$

3. Linearity implies that the response to a sum is the sum of the responses.

$x[n]=\frac{1}{2\pi}\int_{2\pi}X(\Omega)e^{j\Omega n}d\Omega$ → LTI → $y[n]=\frac{1}{2\pi}\int_{2\pi}X(\Omega)H(\Omega)e^{j\Omega n}d\Omega$

4. The Fourier transform of the output is $X(\Omega)H(\Omega)$.

$X(\Omega)$ → LTI → $X(\Omega)H(\Omega)$

The transform of the output is $H(\Omega)$ times the transform of the input.

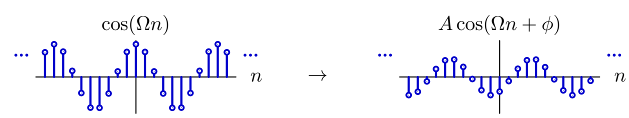
Frequency Response

The frequency response is a **complete** description of a system.

$e^{j\Omega n}$ → LTI → $H(\Omega)e^{j\Omega n}$

This is a bit surprising since $e^{j\Omega n}$ contains a **single** frequency.

→ can find the output of a system by breaking the input into its constituent frequencies and summing the responses to each frequency – one-at-a-time.



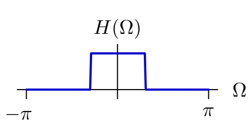
The frequency response can be used to find response to **any** input signal:

$$X(\Omega)\to Y(\Omega)=H(\Omega)X(\Omega)$$

Frequency Response

The frequency response can be an **insightful** description of a system.

Example:
A low-pass filter passes frequencies near 0 and rejects those near π.



Very natural way to describe audio components:

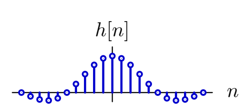
- microphones
- loudspeakers
- audio equalizers

Unit-Sample Response and Frequency Response

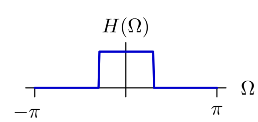
Two complete representations for linear, time-invariant systems.

$x[n]$ → LTI → $y[n]$

Unit-Sample Response: responses across time for a unit-sample input.



Frequency Response: responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **unit-sample response**!

Example

Find the frequency response of a system described by the following:

$$y[n] - \alpha y[n{-}1] = x[n]$$

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$$y[n] - \alpha y[n{-}1] = x[n]$$

Method 1:

Find the unit-sample response and take its Fourier transform.

$$h[n] - \alpha h[n-1] = \delta[n]$$

Solve the difference equation for $h[n]$.

$$h[n] = \delta[n] + \alpha h[n{-}1]$$

First order $\rightarrow$ need one initial condition: $h[-1] = 0$

$$\begin{aligned} h[0] &= \delta[0] + \alpha h[-1] = 1 \\ h[1] &= \delta[1] + \alpha h[0] = \alpha \\ h[2] &= \delta[2] + \alpha h[1] = \alpha^2 \\ h[3] &= \delta[3] + \alpha h[2] = \alpha^3 \end{aligned}$$

$$h[n] = \alpha^n u[n]$$

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\Omega}\right)^n = \frac{1}{1{-}\alpha e^{-j\Omega}}$$

Example

Find the frequency response of a system described by the following:

$$y[n] - \alpha y[n{-}1] = x[n]$$

Method 2:

Find the response to $e^{j\Omega n}$ directly.

$$x[n] = e^{j\Omega n}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y[n] = H(\Omega)e^{j\Omega n}$$

$$y[n{-}1] = H(\Omega)e^{j\Omega(n-1)} = H(\Omega)e^{-j\Omega}e^{j\Omega n}$$

Substitute into the difference equation.

$$H(\Omega)e^{j\Omega n} - \alpha H(\Omega)e^{-j\Omega}e^{j\Omega n} = H(\Omega)(1{-}\alpha e^{-j\Omega})e^{j\Omega n} = e^{j\Omega n}$$

Since $e^{j\Omega n}$ is never 0, we can divide it out.

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as method 1.

Example

Find the frequency response of a system described by the following:

$$y[n] - \alpha y[n{-}1] = x[n]$$

Method 3:

Take the Fourier transform of the difference equation.

$$Y(\Omega) - \alpha e^{-j\Omega}Y(\Omega) = X(\Omega)$$

Solve for $Y(\Omega)$.

$$Y(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}X(\Omega)$$

Since $Y(\Omega) = H(\Omega)X(\Omega)$,

$$H(\Omega) = \frac{1}{1 - \alpha e^{-j\Omega}}$$

Same answer as methods 1 and 2.

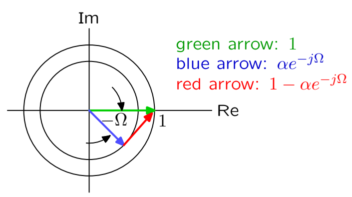
Example

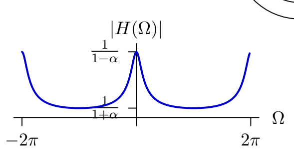
Plot the frequency response.

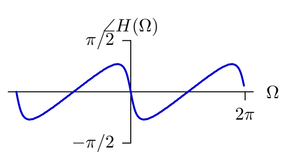
$$H(\Omega) = \frac{1}{1{-}\alpha e^{-j\Omega}}$$

Note that denominator is the difference of 2 complex numbers.

If $0 < \alpha < 1$:







Amplifies at low frequencies and attenuates high frequencies. Adds delay.

The System Abstraction

The system abstraction applies equally well for continuous-time signals.

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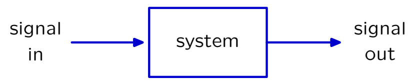

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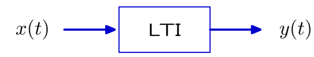

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

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Unit-Sample Response and Impulse Response

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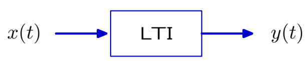


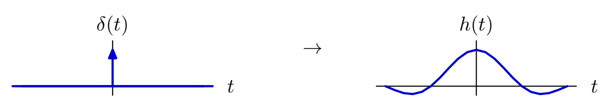
If $\delta(t) \to h(t)$, then $x(t) \to y(t) = (x*h)(t) = \int x(\tau)h(t-\tau)d\tau$

– an LTI system is completely characterized by a single signal

Impulse Response

The impulse response is a **complete** description of a system.

This is a bit surprising since $\delta(t)$ is zero almost everywhere.
The impulse function is the **shortest** possible non-trivial CT signal!



The response to this signal determines the response to **any** other input.

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Frequency Response

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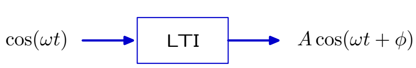

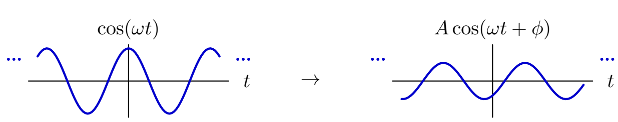

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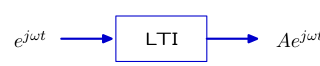
Frequency Response

Using complex exponentials to characterize the frequency response.



Notice that the complex valued A can represent both amplitude and phase.
We can find A using convolution.

$$y(t) = (x*h)(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(\omega)\, e^{j\omega t}$$

The response to a complex exponential is a complex exponential with the **same frequency** but possibly **different amplitude and phase.**

The map for how a system modifies the amplitude and phase of a complex exponential input is the **Fourier transform of the impulse response.**

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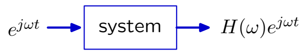
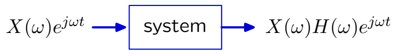
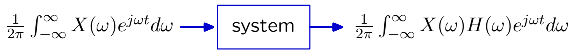


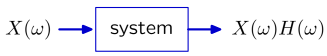
2. Scaling the input by a constant scales the output by the same constant.



3. Linearity implies that the response to a sum is the sum of the responses.



4. The Fourier transform of the output is $X(\omega)H(\omega)$.



The transform of the output is $H(\omega)$ times the transform of the input.

Frequency Response

The frequency response is a **complete** description of a system.

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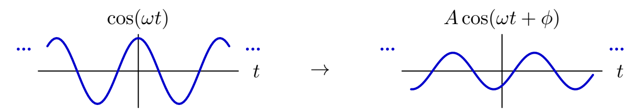
$\rightarrow$ can find the output of a system by breaking the input into its constituent frequencies and summing the responses to each frequency – one-at-a time.



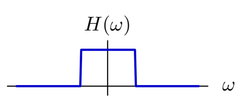
The frequency response can be used to find response to **any** input signal:

$$X(\omega) \rightarrow Y(\omega) = H(\omega)X(\omega)$$

Frequency Response

The frequency response can be an **insightful** description of a system.

Example:
A low-pass filter passes frequencies near 0 and rejects those far from 0.



Very natural way to describe audio enhancements:

- microphones
- loudspeakers
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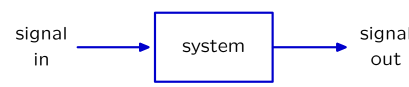
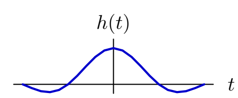
System Abstraction

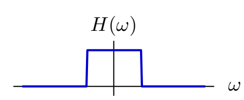
Two complete representations for linear, time-invariant systems.



Impulse Response: responses across time for a inpulse input.



Frequency Response: responses across frequencies for sinusoidal inputs.



The **frequency response** is Fourier transform of **impulse response**!

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha\frac{dy(t)}{dt} = x(t)$$

Example

Find the frequency response of a system described by the following:

$$y(t) + \alpha\frac{dy(t)}{dt} = x(t)$$

Method 1:

Find the response to $e^{j\omega t}$ directly.

$$x(t) = e^{j\omega t}$$

Because the system is linear and time-invariant, the output will have the same frequency as the input, but possibly different amplitude and phase.

$$y(t) = H(\omega)e^{j\omega t}$$

$$\frac{dy(t)}{dt} = j\omega H(\omega)e^{j\omega t}$$

Substitute into the differential equation.

$$H(\omega)e^{j\omega t} + j\omega\alpha H(\omega)e^{j\omega t} = (1 + j\omega\alpha)H(\omega)e^{j\omega t} = e^{j\omega t}$$

Since $e^{j\omega t}$ is never 0, we can divide it out.

$$H(\omega) = \frac{1}{1 + j\omega\alpha}$$

Example

Find the frequency response of a system described by the following:

$$y(t)+\alpha\frac{dy(t)}{dt}=x(t)$$

Method 2:

Take the Fourier transform of the differential equation.

$$Y(\omega)+j\omega\alpha Y(\omega)=X(\omega)$$

Solve for $Y(\omega)$.

$$Y(\omega)=\frac{1}{1+j\omega\alpha}X(\omega)$$

Since $Y(\omega)=H(\omega)X(\omega)$,

$$H(\omega)=\frac{1}{1+j\omega\alpha}$$

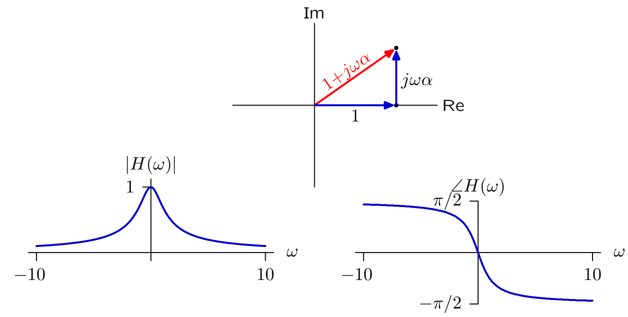
Same answer as method 1.

Example

Plot the frequency response.

$$H(\omega)=\frac{1}{1+j\omega\alpha}$$

Note that denominator is sum of 2 complex numbers.



Amplifies low frequencies, attenuates high frequencies, adds phase delay.

Check Yourself

Find the frequency response of a rectangular box filter:

$$y(t)=\frac{1}{2}\int_{t-1}^{t+1}x(\tau)d\tau$$

(This CT filter is analogous to the three-point averager in DT.)

Summary

The Fourier transform of the response of a DT LTI system is the product of the Fourier transform of the input times the system's **frequency response.**

$X(\Omega)$ → LTI → $Y(\Omega)$

$$Y(\Omega)=H(\Omega)X(\Omega)$$

The frequency response $H(\Omega)$ is the Fourier transform of the unit-sample response $h[n]$.

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$X(\omega)$ → LTI → $Y(\omega)$

$$Y(\omega)=H(\omega)X(\omega)$$

The frequency response $H(\omega)$ is the Fourier transform of the impulse response $h(t)$.