6.003: Signal Processing

Impulse Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

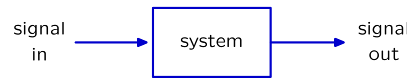
Homework 6 is posted.

- Lab 6 check-in due Friday at 4pm.

October 12, 2021

Last Time: The System Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference Equation:** algebraic **constraint** on samples √
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m]) \quad \surd$$

Homogeneity: scaling an input scales its output

$$\sum_l c_l (\alpha y[n-l]) = \sum_m d_m (\alpha x[n-m]) \quad \surd$$

Time invariance: delaying an input delays its output

$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m] \quad \surd$$

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l \frac{d^l y(t)}{dt^l} = \sum_m d_m \frac{d^m x(t)}{dt^m}$$

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs

$$\sum_l c_l \left(\frac{d^l y_1(t)}{dt^l} + \frac{d^l y_2(t)}{dt^l}\right) = \sum_m d_m \left(\frac{d^m x_1(t)}{dt^m} + \frac{d^m x_2(t)}{dt^m}\right) \quad \surd$$

Homogeneity: scaling an input scales its output

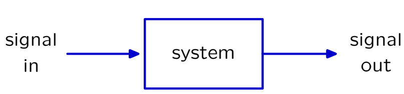
$$\sum_l c_l \left(\alpha \frac{d^l y(t)}{dt^l}\right) = \sum_m d_m \left(\alpha \frac{d^m x(t)}{dt^m}\right) \quad \surd$$

Time invariance: delaying an input delays its output

$$\sum_l c_l \frac{d^l y(t-\tau)}{dt^l} = \sum_m d_m \frac{d^m x(t-\tau)}{dt^m} \quad \surd$$

Today: Representing a System by its Unit-Sample Response

Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



This abstraction is particularly powerful for **linear and time-invariant** systems, which are both **prevalent** and **mathematically tractable**.

Three important representations for LTI systems:

- **Difference Equation:** algebraic **constraint** on samples √
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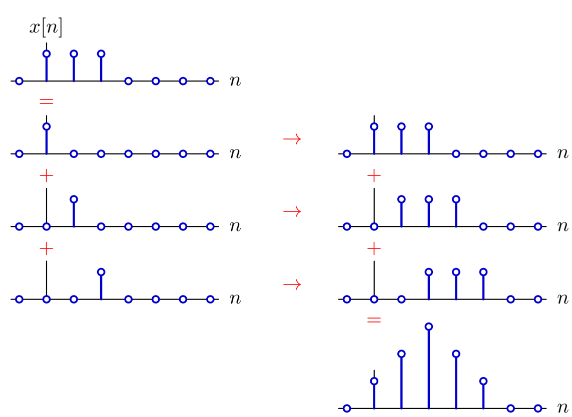
Superposition

Break the input signal into additive parts and sum responses to the parts.



Linearity

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

$x_1[n] \rightarrow$ system $\rightarrow y_1[n]$

and

$x_2[n] \rightarrow$ system $\rightarrow y_2[n]$

the system is linear if

$\alpha x_1[n] + \beta x_2[n] \rightarrow$ system $\rightarrow \alpha y_1[n] + \beta y_2[n]$

is true for all α and β and all times n.

Time-Invariance

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

$x[n] \rightarrow$ system $\rightarrow y[n]$

the system is time invariant if

$x[n-n_0] \rightarrow$ system $\rightarrow y[n-n_0]$

is true for all n and all n_0.

Unit-Sample Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's unit-sample response $h[n]$.

1. One can always find the unit-sample response of a system.

   $\delta[n] \rightarrow$ LTI $\rightarrow h[n]$

2. Time invariance implies that shifting the input simply shifts the output.

   $\delta[n-k] \rightarrow$ LTI $\rightarrow h[n-k]$

3. Homogeneity implies that scaling the input simply scales the output.

   $x[k]\delta[n-k] \rightarrow$ LTI $\rightarrow x[k]h[n-k]$

4. Additivity implies that the response to a sum is the sum of responses.

   $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow$ LTI $\rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$

Convolution

The response of an LTI system to an arbitrary input $x[n]$ can be found by **convolving** that input with the unit-sample response $h[n]$ of the system.

$x[n] \rightarrow$ LTI $\rightarrow y[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

Notation

Convolution is represented with an asterisk.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

It is customary (but confusing) to abbreviate this notation:

$(x*h)[n] = x[n]*h[n]$

$x[n]*h[n]$ looks like an operation of samples; but it is not!

$x[1]*h[1] \neq (x*h)[1]$

Convolution operates on signals not samples.

Unambiguous notation:

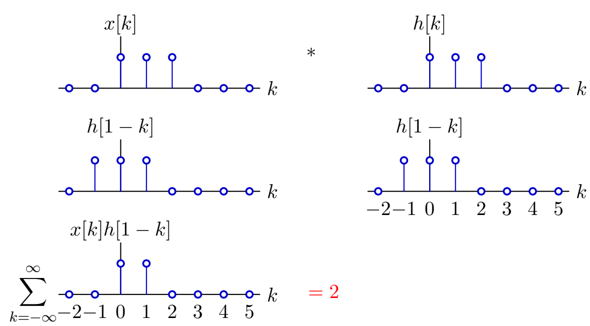
$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x*h)[n]$$

The symbols x and h represent DT signals.
Convolving x with h generates a new DT signal $x*h$.

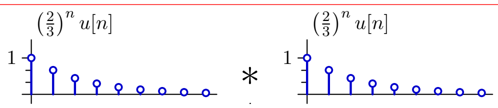
Structure of Convolution

Focus on computing the n^{th} output sample.

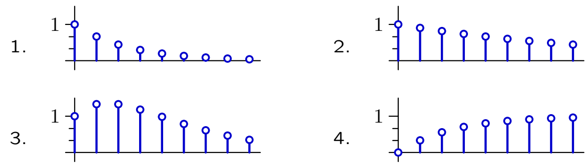
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



Check Yourself



Which plot shows the result of the convolution above?



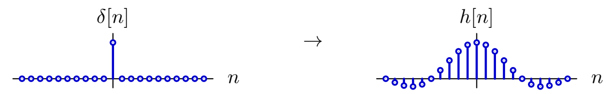
5. none of the above

Unit-Sample Response

The unit-sample response is a **complete** description of a system.

$\delta[n]$ → LTI → $h[n]$

It can be used to determine the response to **any** other input.



Given $h[n]$ one can compute the response to any arbitrary input signal.

$$y[n] = (x*h)[n] \equiv \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

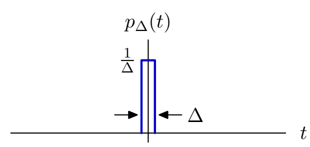
Continuous-Time Systems

Superposition and convolution are of equal importance for CT systems.

Impulse Response

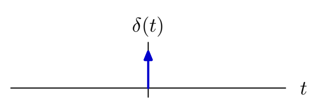
A CT system is completely characterized by its **impulse response**, much as a DT system is completely characterized by its unit-sample response.

We have worked with the impulse (Dirac delta) function $\delta(t)$ previously. It's defined in a limit as follows.
Let $p_\Delta(t)$ represent a pulse of width Δ and height $\frac{1}{\Delta}$ so that its area is 1.



Then

$$\delta(t) = \lim_{\Delta\to 0} p_\Delta(t)$$

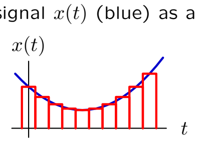


The impulse function can be used to break an arbitrary input $x(t)$ into time-based components, much as $\delta[k]$ is used for discrete-time signals.

Impulse Response

An arbitrary CT signal can be represented by an infinite sum of infinitesimal impulses (which define an integral).

Approximate an arbitrary signal $x(t)$ (blue) as a sum of pulses $p_\Delta(t)$ (red).



$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta$$

and the limit of $x_\Delta(t)$ as $\Delta \to 0$ will approximate $x(t)$.

$$\lim_{\Delta\to 0} x_\Delta(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)p_\Delta(t-k\Delta)\Delta \to \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

The result in CT is much like the result for DT:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \qquad x[n] = \sum_{m=-\infty}^{\infty} x[m]\delta(n-m)$$

Impulse Response

If a system is linear and time-invariant (LTI), its input-output relation is **completely specified** by the system's impulse response $h(t)$.

1. One can always find the impulse response of a system.

   $\delta(t)$ → system → $h(t)$

2. Time invariance implies that shifting the input simply shifts the output.

   $\delta(t-\tau)$ → system → $h(t-\tau)$

3. Homogeneity implies that scaling the input simply scales the output.

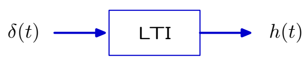
   $x(\tau)\delta(t-\tau)$ → system → $x(\tau)h(t-\tau)$

4. Additivity implies that the response to a sum is the sum of responses.

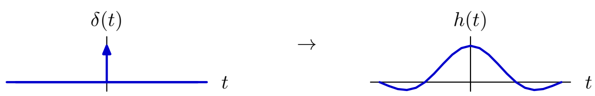
   $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$ → system → $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x*h)(t)$

Impulse Response

The impulse response is a **complete** description of a system.



It can be used to determine the response to **any** other input.



Given $h(t)$ one can compute the response to any arbitrary input signal.

$$y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

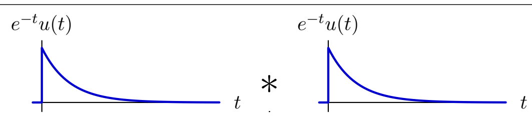
Comparison of CT and DT Convolution

Convolution of CT signals is analogous to convolution of DT signals.

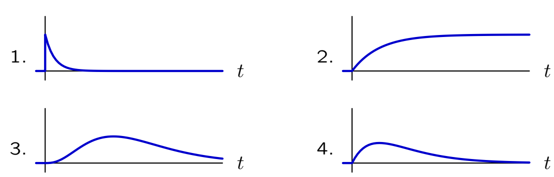
DT: $y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

CT: $y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

Check Yourself



Which plot shows the result of the convolution above?



5. none of the above

Properties of Convolution

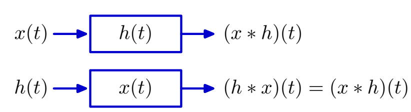
Commutivity:

$$(x * y)(t) = (y * x)(t)$$

$$(x * y)(t) \equiv \int_{-\infty}^{\infty} x(t-\tau)y(\tau)\,d\tau$$

let $\lambda = t - \tau$

$$(x * y)(t) = \int_{\infty}^{-\infty} x(\lambda)y(t-\lambda)(-d\lambda)$$
$$= \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)\,d\lambda$$
$$= (y * x)(t)$$



Properties of Convolution

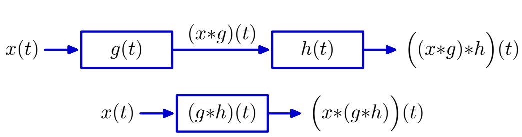
Associativity.

$$\Big((x * y) * z\Big)(t) = \Big(x * (y * z)\Big)(t)$$

$$\Big((x * y) * z\Big)(t) \equiv \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\lambda-\tau)y(\tau)\,d\tau\right) z(\lambda)\,d\lambda$$

let $\mu = \lambda + \tau$

$$\Big((x * y) * z\Big)(t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t-\mu)y(\mu-\lambda)\,d\mu\right) z(\lambda)\,d\lambda$$
$$= \int_{-\infty}^{\infty} x(t-\mu) \left(\int_{-\infty}^{\infty} y(\mu-\lambda)z(\lambda)\,d\lambda\right) d\mu$$
$$= \Big(x * (y * z)\Big)$$

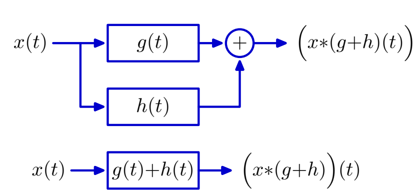


Properties of Convolution

Distributivity over addition.

$$\Big(x * (g + h)\Big)(t) = (x * g)(t) + (x * h)(t)$$

$$\Big(x * (g + h)\Big) = \int_{-\infty}^{\infty} x(t-\tau)\Big(g(\tau) + h(\tau)\Big)\,d\tau$$
$$= \int_{-\infty}^{\infty} x(t-\tau)g(\tau)d\tau + \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$
$$= (x * g)(t) + (x * h)(t)$$



Check Yourself

Match expressions on the left with functions on the right where

$f(t) = e^{-t}\,u(t)$

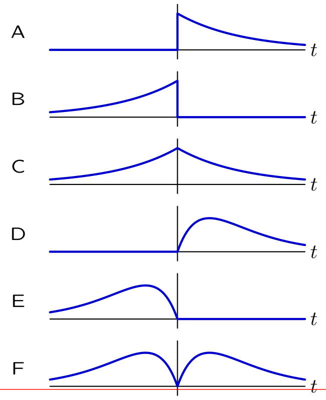
$g(t) = e^{t}\,u(-t)$

$(f * f)(t)$ ☐

$(g * g)(t)$ ☐

$(f * g)(t)$ ☐

$(g * f)(t)$ ☐



Summary

The response of a discrete-time, LTI system to an input $x[n]$ can be computed by convolving the input with the system's **unit sample response**.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n]$$

The response of a continuous-time, LTI system to an input $x(t)$ can be computed by convolving the input with the system's **impulse response**.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t)$$

Convolution allows us to represent a system by a **single signal!**