6.003: Signal Processing

Impulse Response and Convolution

- Unit Sample Signal and Unit Sample Response
- Discrete-Time Convolution
- Impulse Function and Impulse Response
- Continuous-Time Convolution

Homework 6 is posted.
- Lab 6 check-in due Friday at 4pm.

October 12, 2021

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:
\[ \sum l \in \mathbb{R} c_l(y[n-l]) = \sum m \in \mathbb{R} d_m x[n-m] \]

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs
\[ \sum l \in \mathbb{R} c_l(y_1[n-l]) + y_2[n-l]) = \sum m \in \mathbb{R} d_m (x_1[n-m] + x_2[n-m]) \]

Homogeneity: scaling an input scales its output
\[ \sum l \in \mathbb{R} c_l(ax[n-l]) = \sum m \in \mathbb{R} d_m (ax[n-m]) \]

Time invariance: delaying an input delays its output
\[ \sum l \in \mathbb{R} c_l(y[(n-n_0)-l]) = \sum m \in \mathbb{R} d_m x[(n-n_0)-m] \]

Linear Differential Equations with Constant Coefficients

If a continuous-time system can be described by a linear differential equation with constant coefficients, then the system is linear and time-invariant.

General form:
\[ \sum l \in \mathbb{R} c_l \frac{d^{l}y(t)}{dt^l} = \sum m \in \mathbb{R} d_m \frac{d^{m}x(t)}{dt^m} \]

Such systems are easily shown to be linear and time-invariant.

Additivity: output of sum is sum of outputs
\[ \sum l \in \mathbb{R} c_l \frac{d^{l}y_1(t)}{dt^l} + \frac{d^{l}y_2(t)}{dt^l} = \sum m \in \mathbb{R} d_m \left( \frac{d^{m}x_1(t)}{dt^m} + \frac{d^{m}x_2(t)}{dt^m} \right) \]

Homogeneity: scaling an input scales its output
\[ \sum l \in \mathbb{R} c_l \left( \frac{d^{l}y(t)}{dt^l} \right) = \sum m \in \mathbb{R} d_m \left( \frac{d^{m}x(t)}{dt^m} \right) \]

Time invariance: delaying an input delays its output
\[ \sum l \in \mathbb{R} c_l \frac{d^{l}y(t-\tau)}{dt^l} = \sum m \in \mathbb{R} d_m \frac{d^{m}x(t-\tau)}{dt^m} \]

Today: Representing a System by its Unit-Sample Response

Describe a system (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.

This abstraction is particularly powerful for linear and time-invariant systems, which are both prevalent and mathematically tractable.

Three important representations for LTI systems:
- Difference Equation: algebraic constraint on samples
- Convolution: represent a system by its unit-sample response
- Filter: represent a system by its frequency response

Superposition

Break the input signal into additive parts and sum responses to the parts.
**Linearity**

A system is linear if its response to a weighted sum of inputs is equal to the weighted sum of its responses to each of the inputs.

Given

\[ x_1[n] \rightarrow \text{system} \rightarrow y_1[n] \]

and

\[ x_2[n] \rightarrow \text{system} \rightarrow y_2[n] \]

the system is linear if

\[ \alpha x_1[n] + \beta x_2[n] \rightarrow \text{system} \rightarrow \alpha y_1[n] + \beta y_2[n] \]

is true for all \( \alpha \) and \( \beta \) and all times \( n \).

**Time-Invariance**

A system is time-invariant if delaying the input to the system simply delays the output by the same amount of time.

Given

\[ x[n] \rightarrow \text{system} \rightarrow y[n] \]

the system is time invariant if

\[ x[n-n_0] \rightarrow \text{system} \rightarrow y[n-n_0] \]

is true for all \( n \) and all \( n_0 \).

**Unit-Sample Response**

If a system is linear and time-invariant (LTI), its input-output relation is completely specified by the system’s unit-sample response \( h[n] \).

1. One can always find the unit-sample response of a system.

\[ \delta[n] \rightarrow \text{LTI} \rightarrow h[n] \]

2. Time invariance implies that shifting the input simply shifts the output.

\[ \delta[n-k] \rightarrow \text{LTI} \rightarrow h[n-k] \]

3. Homogeneity implies that scaling the input simply scales the output.

\[ x[k] \delta[n-k] \rightarrow \text{LTI} \rightarrow x[k] h[n-k] \]

4. Additivity implies that the response to a sum is the sum of responses.

\[ x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \rightarrow \text{LTI} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \equiv (x \ast h)[n] \]

**Convolution**

The response of an LTI system to an arbitrary input \( x[n] \) can be found by convolving that input with the unit-sample response \( h[n] \) of the system.

\[ x[n] \rightarrow \text{LTI} \rightarrow y[n] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \equiv (x \ast h)[n] \]

**Notation**

Convolution is represented with an asterisk.

\[ \sum_{k=-\infty}^{\infty} x[k] h[n-k] \equiv (x \ast h)[n] \]

It is customary (but confusing) to abbreviate this notation:

\( (x \ast h)[n] = x[n] \ast h[n] \)

\( x[n] \ast h[n] \) looks like an operation of samples; but it is not!

\( x[1] \ast h[1] \neq (x \ast h)[1] \)

Convolution operates on signals not samples.

Unambiguous notation:

\[ \sum_{k=-\infty}^{\infty} x[k] h[n-k] \equiv (x \ast h)[n] \]

The symbols \( x \) and \( h \) represent DT signals.

Convolving \( x \) with \( h \) generates a new DT signal \( x \ast h \).

**Structure of Convolution**

Focus on computing the \( n^{th} \) output sample.

\[ y[1] = \sum_{k=-\infty}^{\infty} x[k] h[1-k] \]

\[ \begin{array}{c|c|c}
  x[k] & \ast & h[k] \\
  \hline
  -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
  \hline
  x[k]h[1-k] & = & 2
\end{array} \]
Check Yourself

Which plot shows the result of the convolution above?

1.  
2.  
3.  
4.  
5. none of the above

Continuous-Time Systems
Superposition and convolution are of equal importance for CT systems.

Unit-Sample Response
The unit-sample response is a complete description of a system.

Impulse Response
If a system is linear and time-invariant (LTI), its input-output relation is completely specified by the system’s impulse response $h(t)$.

1. One can always find the impulse response of a system.
2. Time invariance implies that shifting the input simply shifts the output.
3. Homogeneity implies that scaling the input simply scales the output.
4. Additivity implies that the response to a sum is the sum of responses.
Impulse Response

The impulse response is a complete description of a system.

\[ \delta(t) \xrightarrow{\text{LTI}} h(t) \]

It can be used to determine the response to any other input.

\[ \delta(t) \rightarrow h(t) \]

Given \( h(t) \) one can compute the response to any arbitrary input signal.

\[ y(t) = (x * h)(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]

Comparison of CT and DT Convolution

Convolusion of CT signals is analogous to convolution of DT signals.

**DT:** \[ y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \]

**CT:** \[ y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \]

Check Yourself

Which plot shows the result of the convolution above?

1. 2. 3. 4. 5. none of the above

Properties of Convolution

**Commutivity:**

\[ (x * y)(t) = (y * x)(t) \]

\[ x * (y + z)(t) = (x * y)(t) + (x * z)(t) \]

\[ (x * (g + h))(t) = (x * g)(t) + (x * h)(t) \]

\[ x(t) \xrightarrow{g(t)} \xrightarrow{(x * g)(t)} \xrightarrow{h(t)} \xrightarrow{(x * g)(t)} \xrightarrow{z(t)} \xrightarrow{(x * g)(t) + h(t)} \xrightarrow{(x * (g + h))(t)} \]

Properties of Convolution

**Associativity.**

\[ (x * (y * z))(t) = (x * (y * z))(t) \]

\[ x * (y * z)(t) = \int_{-\infty}^{\infty} x(t-\mu)h(\mu) \cdot z(\mu) d\mu \]

\[ = \int_{-\infty}^{\infty} x(t-\mu)\int_{-\infty}^{\infty} h(\mu)z(\mu) d\mu \]

\[ = (x * (y * z)) \]

\[ x(t) \xrightarrow{g(t)} \xrightarrow{(x * g)(t)} \xrightarrow{h(t)} \xrightarrow{(x * g)(t) + h(t)} \xrightarrow{(x * (g + h))(t)} \]

Properties of Convolution

**Distributivity over addition.**

\[ (x * (g + h))(t) = (x * g)(t) + (x * h)(t) \]

\[ (x * (g + h))(t) = \int_{-\infty}^{\infty} x(t-\tau)g(\tau) h(t-\tau) d\tau \]

\[ = \int_{-\infty}^{\infty} x(t-\tau)g(\tau) d\tau + \int_{-\infty}^{\infty} x(t-\tau)h(\tau) d\tau \]

\[ = (x * g)(t) + (x * h)(t) \]

\[ x(t) \xrightarrow{g(t)} \xrightarrow{h(t)} \xrightarrow{(x * g)(t)} \xrightarrow{(x * h)(t)} \xrightarrow{(x * (g + h))(t)} \]
Check Yourself

Match expressions on the left with functions on the right where

\[ f(t) = e^{-t}u(t) \]
\[ g(t) = e^{t}u(-t) \]

\[ (f * f)(t) \]
\[ (g * g)(t) \]
\[ (f * g)(t) \]
\[ (g * f)(t) \]

\[ (f * f)(t) \]
\[ (g * g)(t) \]
\[ (f * g)(t) \]
\[ (g * f)(t) \]

Summary

The response of a discrete-time, LTI system to an input \( x[n] \) can be computed by convolving the input with the system’s unit sample response.

\[ x[n] \xrightarrow{\text{LTI}} y[n] \]

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \equiv (x * h)[n] \]

The response of a continuous-time, LTI system to an input \( x(t) \) can be computed by convolving the input with the system’s impulse response.

\[ x(t) \xrightarrow{\text{LTI}} y(t) \]

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \equiv (x * h)(t) \]

Convolution allows us to represent a system by a single signal!