6.003: Signal Processing

Systems

- System Abstraction
- Linearity and Time Invariance

Homework 5 is posted

- shorter than usual
- no lab

Quiz 1 results are posted

- grades and grade definitions

October 7, 2021

Notes on Grading

We explicitly use the MIT definitions of grades.

- A **Exceptionally good performance** demonstrating a superior understanding of the subject matter, a foundation of extensive knowledge, and a skillful use of concepts and/or materials.
- B **Good performance** demonstrating capacity to use the appropriate concepts, a good understanding of the subject matter, and an ability to handle the problems and materials encountered in the subject.
- C **Adequate performance** demonstrating an adequate understanding of the subject matter, an ability to handle relatively simple problems, and adequate preparation for moving on to more advanced work in the field.
- D **Minimally acceptable performance** demonstrating at least partial familiarity with the subject matter and some capacity to deal with relatively simple problems, but also demonstrating deficiencies serious enough to make it inadvisable to proceed further in the field without additional work.
- F **Failed.**

Points, GPA Scale, and Letter Grade

Grading Procedure

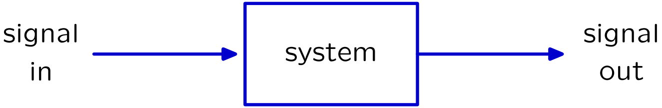
- We grade the exams on a **point** basis.
- We convert the 100 point score into a 5-point **GPA** scale using MIT's definitions of letter grades.
- Your final score in 6.003 will be a **weighted sum of your 5-point scores** for homeworks, labs, quizzes, and final exam.

total points	"GPA" scale	letter grade
100%	5	
		A
A/B boundary	4	
		B
B/C boundary	3	
		C
C/D boundary	2	
		D
D/F boundary	1	
		F
0%	0	

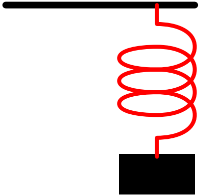
The goal of this scheme is to be transparent about your grade status.

The System Abstraction

Represent a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



Example: Mass and Spring



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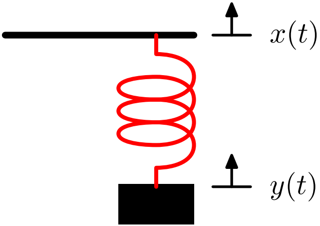

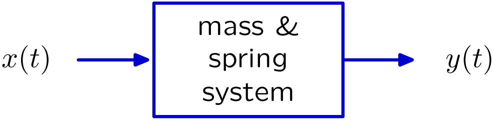

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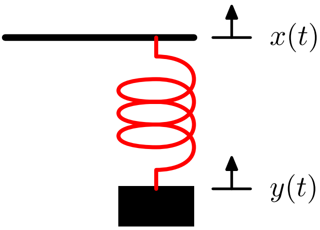
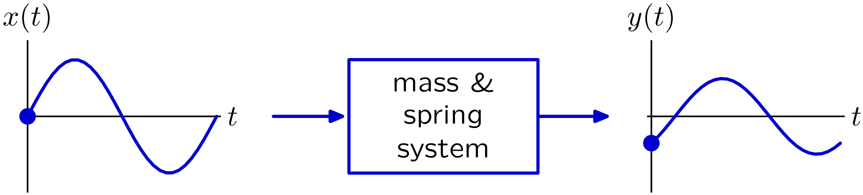
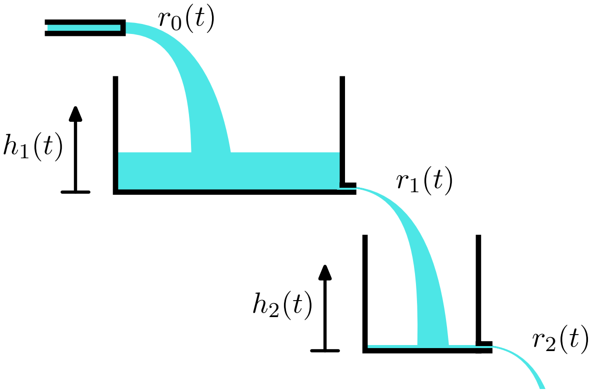
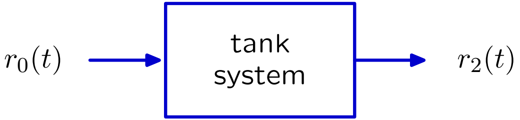




Example: Tanks





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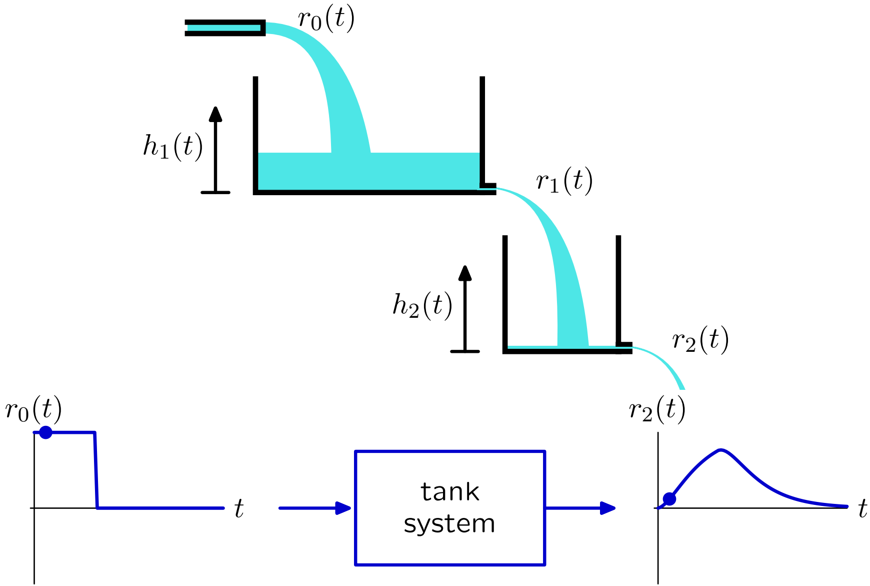
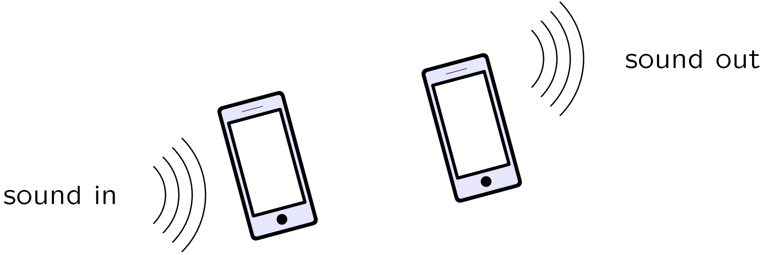
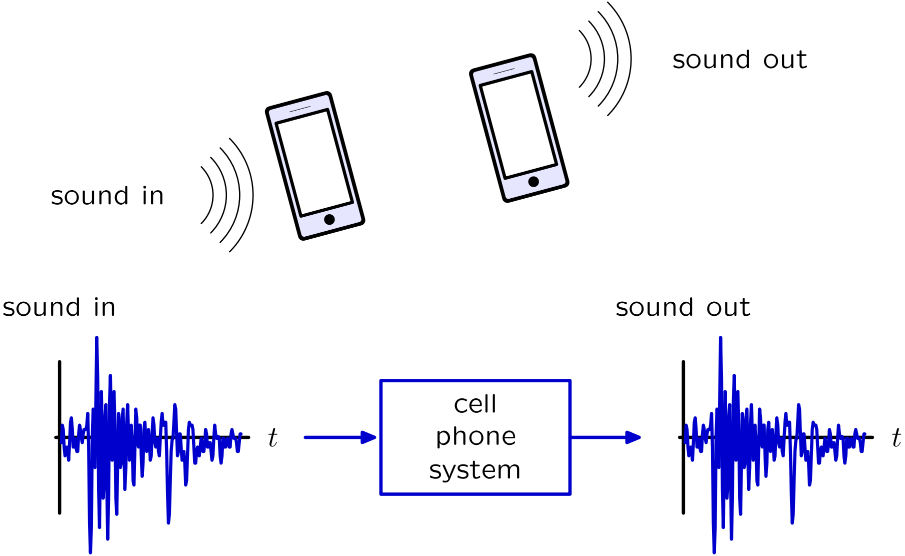


Example: Cell Phone System



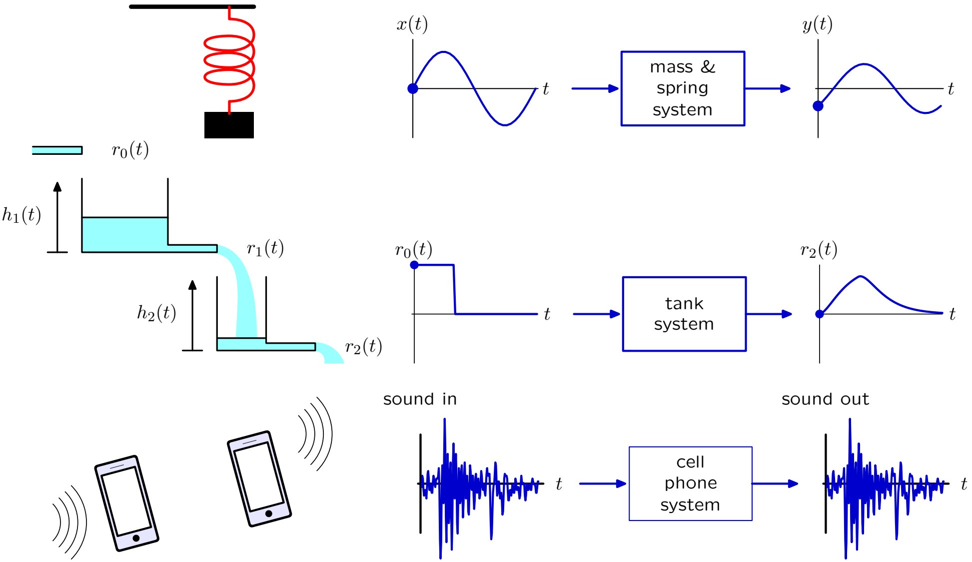
sound in → $\boxed{\text{cell phone system}}$ → sound out

Example: Cell Phone System



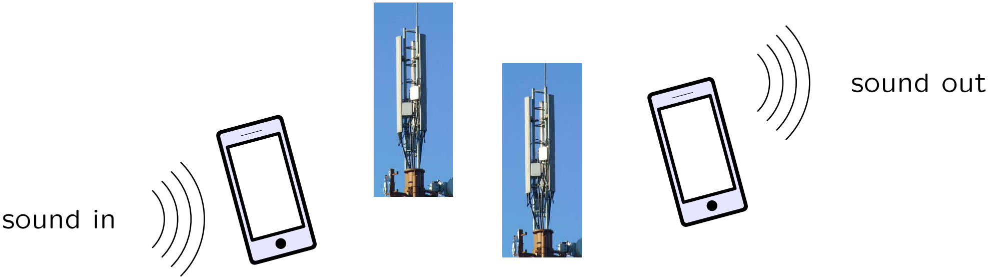
Signals and Systems: Widely Applicable

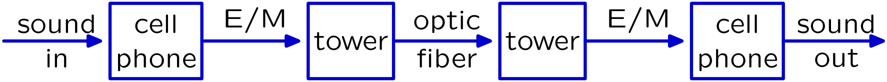
The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...



Signals and Systems: Modular

The representation does not depend upon the physical substrate.



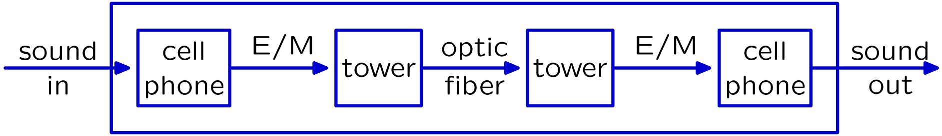


focuses on the flow of **information**, abstracts away everything else

Signals and Systems: Hierarchical

Representations of component systems are easily combined.

Example: cascade of component systems

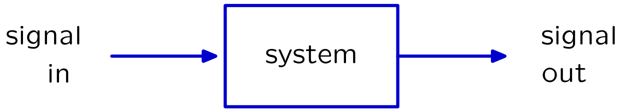


Composite system

Component and composite systems have the same form, and are analyzed with same methods.

System Abstraction

The system abstraction builds on and extends our work with signals.



We will use this approach to process a variety of signals:

- **audio:** equalization, noise reduction, reverberation reduction, echo cancellation, pitch shift (auto-tune)
- **image:** smoothing, edge enhancement, unsharp masking, feature detection
- **video:** image stabilization, motion magnification

System Abstraction

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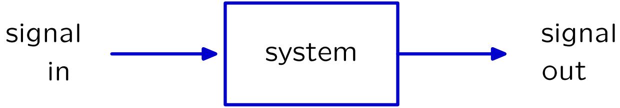


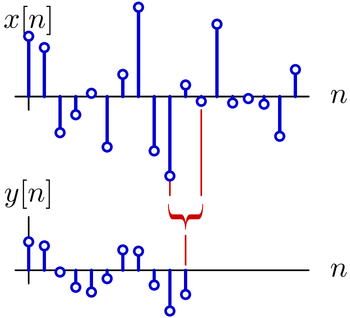
We will look at three different representations for systems:

- **Difference Equation:** algebraic **constraint** on samples
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**

Example: Three-Point Averaging

The output at time n is average of inputs at times $n-1$, n, and $n+1$.

$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



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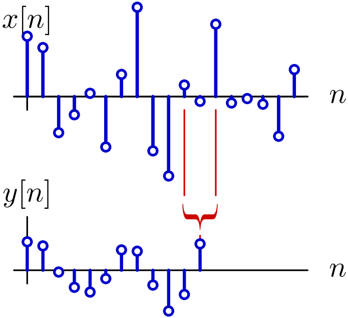

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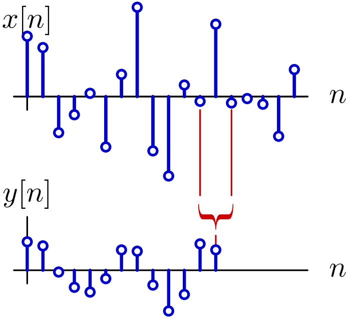

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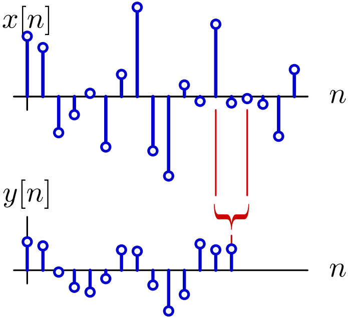

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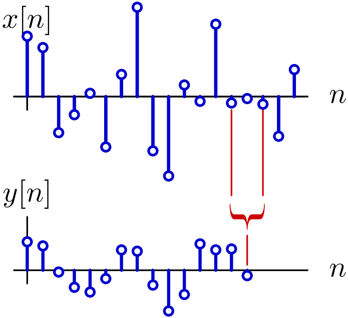

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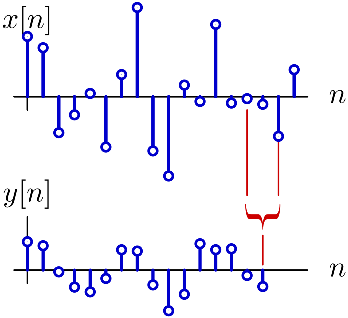

Example: Three-Point Averaging

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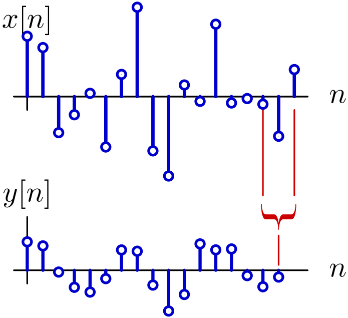
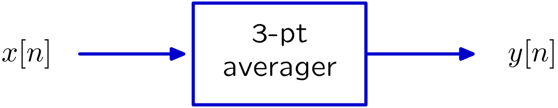
$$y[n] = \frac{1}{3}\Big(x[n-1] + x[n] + x[n+1]\Big)$$



Think of this process as a system with input $x[n]$ and output $y[n]$.



Properties of Systems

We will focus primarily on systems that have two important properties:

- **linearity**
- **time invariance**

Such systems are both useful and mathematically tractable.

Additivity

A system is additive if its response to a **sum of signals** is equal to the **sum of the responses** to each signal taken one at a time.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is additive** if

$$x_1[n] + x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n] + y_2[n]$$

is true for all possible inputs and all times n.

Example: the three-point averager is additive:
The three-point average of the sum of two signals is equal to the sum of the three-point averages of the individual signals.

Homogeneity

A system is homogeneous if multiplying its input signal by a constant multiplies the output signal by the same constant.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

the **system is homogeneous** if

$$\alpha x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n]$$

is true for all α and all possible inputs and all times n.

Example: the three-point averager is homogeneous.
Doubling an input signal doubles its three-point average.

Linearity

A system is linear if its response to a **weighted sum of input signals** is equal to the **weighted sum of its responses** to each of the input signals.

Given

$$x_1[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_1[n]$$

and

$$x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow y_2[n]$$

the **system is linear** if

$$\alpha x_1[n] + \beta x_2[n] \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y_1[n] + \beta y_2[n]$$

is true for all α and β and all possible inputs and all times n.

A system is linear if it is both additive and homogeneous.

Time-Invariance

A system is time-invariant if delaying the input signal simply delays the output signal by the same amount of time.

Given

$$x[n] \longrightarrow \boxed{\text{system}} \longrightarrow y[n]$$

the **system is time invariant** if

$$x[n-n_0] \longrightarrow \boxed{\text{system}} \longrightarrow y[n-n_0]$$

is true for all n_0 and for all possible inputs and all times n.

Example: The three-point averager is time-invariant.
Shifting the input to a 3-pt averager simply shifts the output by that same amount.

Representing Systems with Difference Equations

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1]$$

for all n.

Is this system **linear**?

Representing Systems with Difference Equations

Consider a system represented by the following difference equation:

$$y[n] = x[n] + x[n-1]$$

for all n.

Is this system **linear**?

Assume that $x_1[n] \rightarrow y_1[n]$. Then $y_1[n] = x_1[n] + x_1[n-1]$.

Assume that $x_2[n] \rightarrow y_2[n]$. Then $y_2[n] = x_2[n] + x_2[n-1]$.

Multiply α times equation 1 and add β times equation 2:

$$\alpha y_1[n] + \beta y_2[n] = \alpha x_1[n] + \beta x_2[n] + \alpha x_1[n-1] + \beta x_2[n-1]$$

This equation shows that $\alpha x_1[n] + \beta x_2[n] \rightarrow \alpha y_1[n] + \beta y_2[n]$.

Therefore the system is **linear.**

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 2.

$$y[n] = x[n] \times x[n-1]$$

for all n.

Is this system **linear**?

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 2.

$$y[n] = x[n] \times x[n-1]$$

for all n.

Is this system **linear**?

Assume that $x_1[n] \to y_1[n]$. Then $y_1[n] = x_1[n] \times x_1[n-1]$.

Find the response $y_2[n]$ when $x_2[n] = \alpha x_1[n]$:

$$\begin{aligned} y_2[n] &= x_2[n] \times x_2[n-1] \\ &= \alpha x_1[n] \times \alpha x_1[n-1] \\ &= \alpha^2 x_1[n] \times x_1[n-1] \\ &= \alpha^2 y_1[n] \end{aligned}$$

Multiplying input $x_1[n]$ by α does **not** multiply the output $y_1[n]$ by α. It multiplies $y_1[n]$ by α^2 !

Therefore the system is **neither homogeneous not linear**.

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 3:

$$y[n] = nx[n]$$

for all n.

Is the system **linear**?

Representing Systems with Difference Equations

Determining linearity from a difference equation representation.

Example 3:

$$y[n] = nx[n]$$

for all n.

Is the system **linear**?

Let $x[n] = \alpha x_1[n] + \beta x_2[n]$.

Then

$$\begin{aligned} y[n] &= n(\alpha x_1[n] + \beta x_2[n]) \\ &= \alpha n x_1[n] + \beta n x_2[n] \\ &= \alpha y_1[n] + \beta y_2[n] \end{aligned}$$

Therefore the system is **linear.**

Representing Systems with Difference Equations

Determining time invariance from a difference equation.

Example 3.

$$y[n] = nx[n]$$

for all n.

Is the system **time-invariant**?

Representing Systems with Difference Equations

Determining time invariance from a difference equation.

Example 3.

$$y[n] = nx[n]$$

for all n.

Is the system **time-invariant**?

If time-invariant, delaying input by 1 should delay output by 1.
Let $x_1[n]$ represent a delayed version of the input.

$$x_1[n] = x[n-1]$$

The corresponding output $y_1[n]$ is given by

$$y_1[n] = nx_1[n] = nx[n-1]$$

This is not the same as delaying the original output:

$$y[n-1] = (n-1)x[n-1]$$

Since $y_1[n] \neq y[n-1]$, the system is **not time-invariant.**

Linear Difference Equations with Constant Coefficients

If a discrete-time system can be described by a linear difference equation with constant coefficients, then the system is linear and time-invariant.

General form:

$$\sum_l c_l y[n-l] = \sum_m d_m x[n-m]$$

Additivity: output of sum is sum of outputs

$$\sum_l c_l (y_1[n-l] + y_2[n-l]) = \sum_m d_m (x_1[n-m] + x_2[n-m]) \qquad \surd$$

Homogeneity: scaling an input scales its output

$$\sum_l \alpha c_l y[n-l] = \sum_m \alpha d_m x[n-m] \qquad \surd$$

Time invariance: delaying an input delays its output

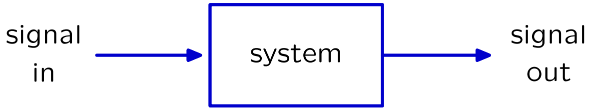
$$\sum_l c_l y[(n-n_0)-l] = \sum_m d_m x[(n-n_0)-m] \qquad \surd$$

Notice that $y[n] = x[n] + 1$ does **not** represent a linear system. Why?

Summary: System Abstraction

The system abstraction builds on and extends our work with signals.

Goal: characterize a **system** to better understand the relation between two signals.



Next Time: Three representations for systems:

- **Difference Equation:** algebraic **constraint** on samples
- **Convolution:** represent a system by its **unit-sample response**
- **Filter:** represent a system by its **frequency response**