6.003: Signal Processing

Discrete-Time Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)

Announcements:

- Quiz 1: October 5, 2-4pm, 50-340 (Walker)
  - Coverage up to and including all of week 3, including HW3.
  - Closed book except for one page of notes (8.5"x11" both sides).
  - No electronic devices. (No headphones, cellphones, calculators, ...)
- No HW4
- A practice quiz has been posted.
  - Not turned in, not graded.
  - Solutions will be posted on Friday.
- If you have personal or medical difficulties, please contact S^3 and/or `6.003-instructors@mit.edu` for accommodations.

September 30, 2021

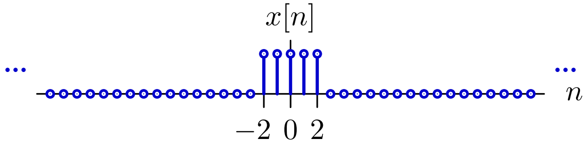
From Periodic to Aperiodic

Last time: representing arbitrary (aperiodic) CT signals as sums of sinusoidal components using the continuous-time Fourier transform.

Today: generalize the Fourier Transform idea to discrete-time signals.

Fourier Representions of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinusoids?



Fourier Representions of Aperiodic Signals

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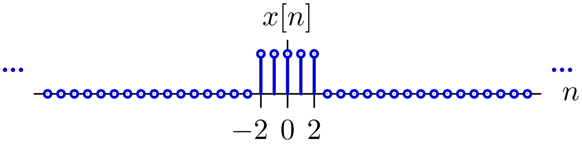
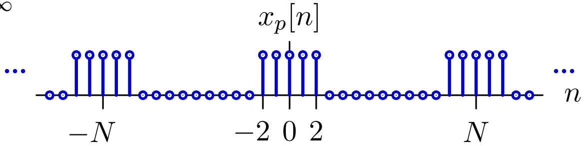


Strategy: make a periodic version of $x[n]$ by summing shifted copies:

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$



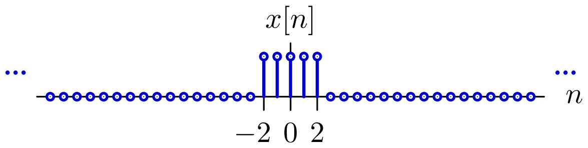
Since $x_p[n]$ is periodic, it has a Fourier series (which depends on N).

Find Fourier series coefficients $X_p[k]$ and take the limit of $X_p[k]$ as $N \to \infty$.

As $N \to \infty$, $x_p[n] \to x[n]$, and Fourier series will approach Fourier transform.

Fourier Representions of Aperiodic Signals

Example.



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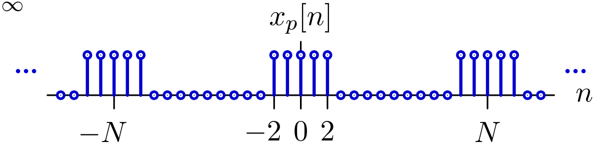


Calculate the Fourier series coefficients $X_p[k]$:

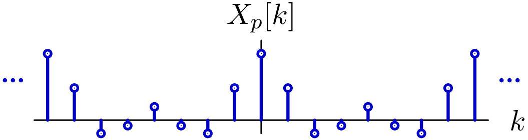
$$X_p[k] = \frac{1}{N} \sum_{n=-2}^{2} x_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

Fourier Representions of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x_p[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

Plot the resulting Fourier coefficients for $N{=}8$.



What happens if you double the period N?

Fourier Representions of Aperiodic Signals

Calculate the Fourier series coefficients $X_p[k]$:

$$X_p[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x_p[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{N} + \frac{2}{N}\cos\frac{2\pi k}{N} + \frac{2}{N}\cos\frac{4\pi k}{N}$$

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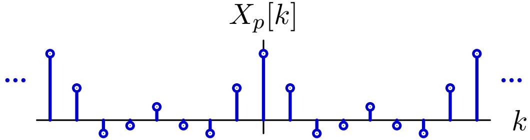
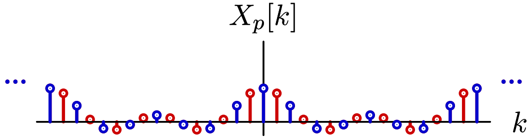


What happens if you double the period N? Make a plot for $N=16$.



There are now twice as many samples per period. (The red samples are at new intermediate frequencies.) The amplitude is halved.

Fourier Representions of Aperiodic Signals

Define a new function $X(\Omega) = NX_p[k]$ where $\Omega = k\Omega_o = 2\pi k/N$.

$$NX_p[k] = 1 + 2\cos\frac{2\pi k}{N} + 2\cos\frac{4\pi k}{N} = 1 + 2\cos(\Omega) + 2\cos(2\Omega) = X(\Omega)$$

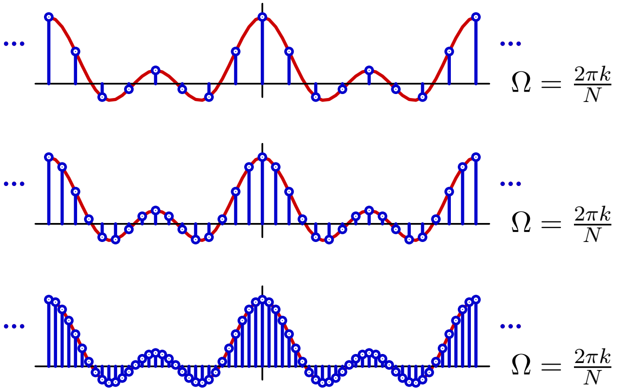
Then $NX_p[k]$ represents samples of $X(\Omega)$ with increasing resolution in Ω.

$$NX_p[k] = X(\Omega)$$

N=8:

N=16:

N=32:



The discrete function $NX_p[k]$ is a sampled version of the function $X(\Omega)$.

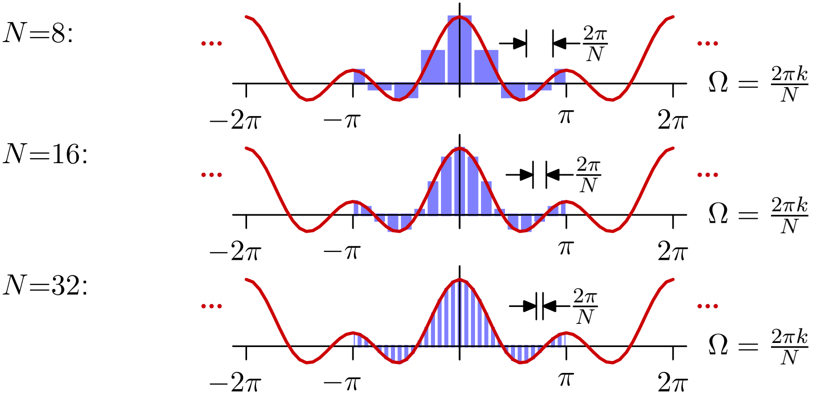
Fourier Representations of Aperiodic Signals

We can reconstruct $x[n]$ from $X(\Omega)$ using a Riemann sum.

$$x[n] = \lim_{N\to\infty} x_p[n] = \lim_{N\to\infty} \sum_{k=\langle N\rangle} X_p[k] e^{j\frac{2\pi}{N}kn}$$

$$= \lim_{N\to\infty} \left(\frac{1}{2\pi}\right) \sum_{k=\langle N\rangle} N X_p[k] e^{j\frac{2\pi}{N}kn} \left(\frac{2\pi}{N}\right) = \frac{1}{2\pi}\int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

$$N X_p[k] = X(\Omega)$$



Fourier Transform relation: $x[n] \overset{\text{FT}}{\Longleftrightarrow} X(\Omega)$

Fourier Series and Fourier Transform

Fourier series and transforms are similar:
both represent signals by their frequency content.

Discrete-Time Fourier Series

$$X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_o n}$$

analysis equation

$$x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

analysis equation

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n}\, d\Omega$$

synthesis equation

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period.

The information in an aperiodic signal can spread across all time.

Discrete-Time Fourier Series

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synthesis equation

Fourier Series and Fourier Transform

Periodic signals can be synthesized from a discrete set of k harmonics. Aperiodic signals generally require a continuous set of frequencies Ω.

Discrete-Time Fourier Series

$$X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_o n}$$ **analysis equation**

$$x[n] = x[n+N] = \sum_{k=\langle N \rangle} X[k] e^{jk\Omega_o n}$$ **synthesis equation**

where $\Omega_o = \frac{2\pi}{N}$

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ **analysis equation**

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Fourier Series and Fourier Transform

Harmonic frequencies $k\Omega_o$ are samples of continuous frequency Ω.

Discrete-Time Fourier Series

$$X[k] = X[k+N] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\Omega_o n}$$

analysis equation

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Discrete-Time Fourier Transform

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synthesis equation

CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.

Continuous-Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$ **analysis equation**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$ **synthesis equation**

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega+2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$ **analysis equation**

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CT and DT Fourier Transforms

DT frequencies alias because adding 2π to Ω does not change $e^{-j\Omega n}$.
Because $X(\Omega)$ is periodic in 2π, we need only integrate $d\Omega$ over a 2π interval.

Continuous-Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ **analysis equation**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$ **synthesis equation**

Discrete-Time Fourier Transform

$$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$ **analysis equation**

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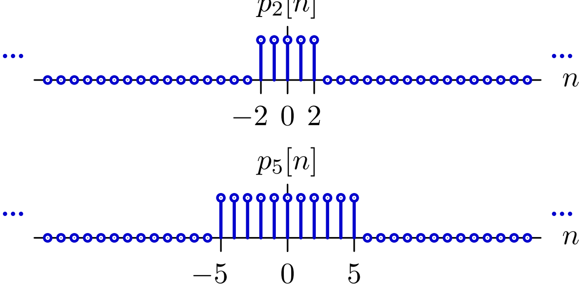
Examples of Fourier Transforms

Find the Fourier Transform (FT) of a rectangular pulse of width $2S+1$:

$$p_S[n] = \begin{cases} 1 & -S \le N \le S \\ 0 & \text{otherwise} \end{cases}$$



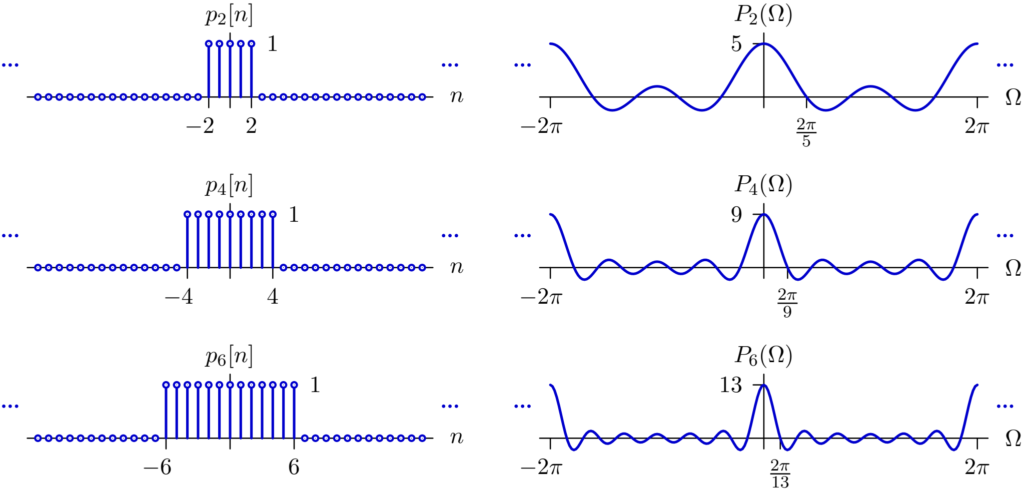
$$P_S(\Omega) = \sum_{n=-\infty}^{\infty} p_S[n] e^{-j\Omega n} = \sum_{n=-S}^{S} e^{-j\Omega n}$$

$$= e^{j\Omega S} + e^{j\Omega(S-1)} + \cdots + 1 + \cdots + e^{-j\Omega(S-1)} + e^{-j\Omega S}$$

$$= 1 + 2\cos(\Omega) + 2\cos(2\Omega) + \cdots + 2\cos(S\Omega)$$

Examples of Fourier Transforms

Compare Fourier transforms of pulses with different widths.

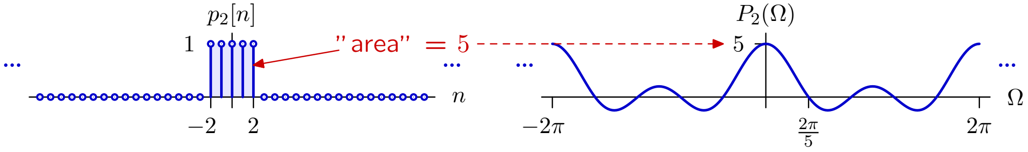


As the function widens in n the Fourier transform narrows in Ω.

Moments

Similar to CT, the value of $X(\Omega)$ at $\Omega = 0$ is the sum of $x[n]$ over time t.

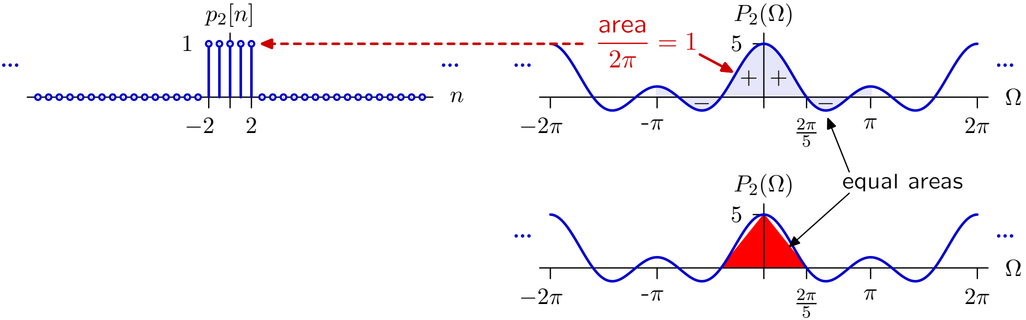
$$X(0) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n]$$



Moments

The value of $x[0]$ is $\frac{1}{2\pi}$ times the integral of $X(\Omega)$ over $\Omega = [-\pi, \pi]$.

$$x[0] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} X(\Omega)\, d\Omega$$

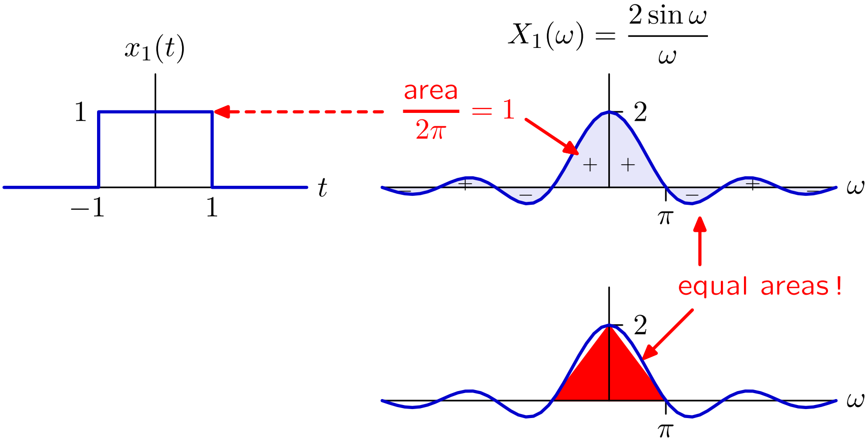


Very similar to analogous relations for CT Fourier transforms.

Moments

The value of $x(0)$ is the integral of $X(\omega)$ divided by 2π.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$



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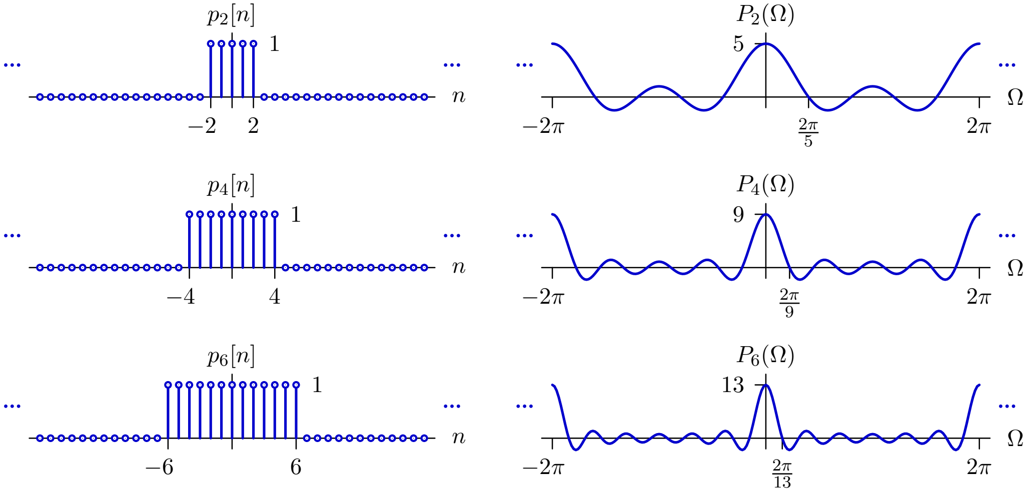
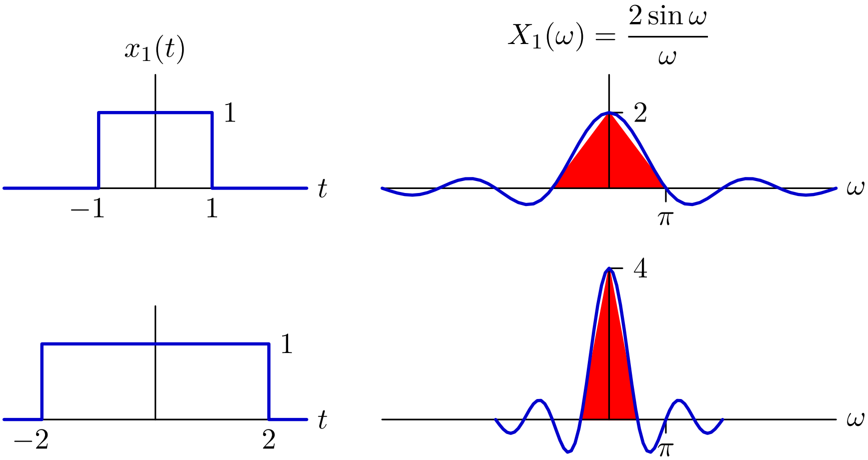


As the function widens in n the Fourier transform narrows in Ω.

Stretching Time

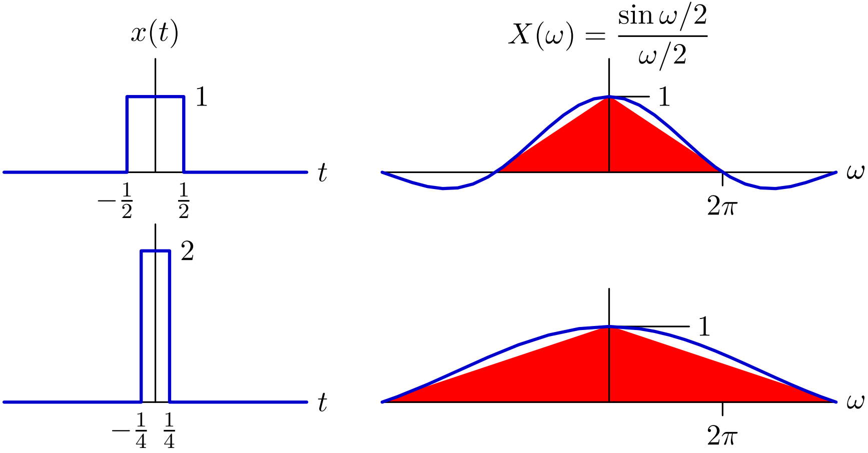
Stretching time compresses frequency and increases amplitude (preserving area).



Very similar in CT and DT.

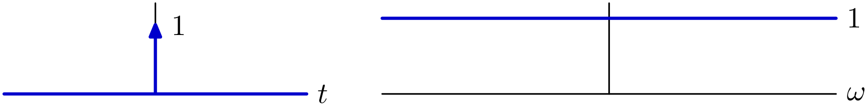
Compressing Time to the Limit

Alternatively, we could compress time while keeping area = 1.



In the limit, the pulse has zero width but area 1!

We represent this limit with the delta function: $\delta(t)$.



Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

Since $\delta(t)$ is zero except near $t{=}0$, only values of $e^{-j\omega t}$ near $t{=}0$ are important. Because $e^{-j\omega t}$ is a smooth function of t, $e^{-j\omega t}$ can be replaced by $e^{-j\omega 0}$:

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega 0} dt = 1$$

$$\delta(t) \quad \stackrel{\text{CTFT}}{\Longleftrightarrow} \quad 1$$

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.

Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 2: Find the function whose Fourier transform is a unit impulse.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j0t} d\omega = \frac{1}{2\pi}$$

$$1 \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 2\pi\delta(\omega)$$

Notice the similarity to the previous result:

$$\delta(t) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 1$$

These relations are **duals** of each other.

- A constant in time consists of a single frequency at $\omega = 0$.
- An impulse in time contains components at all frequencies.

Math With Impulses

Although physically unrealizeable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 3: Find the function whose Fourier transform is a shifted impulse.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_o) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega') e^{j(\omega'+\omega_o)t} d\omega'$$
$$= \frac{1}{2\pi} e^{j\omega_o t}\int_{-\infty}^{\infty} \delta(\omega') e^{j\omega' t} d\omega' = \frac{1}{2\pi} e^{j\omega_o t}$$

$$e^{j\omega_o t} \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 2\pi\delta(\omega-\omega_o)$$

Use this result to relate Fourier series to Fourier transforms.

Relation Between Fourier Series and Fourier Transforms

If a periodic signal $x(t) = x(t+T)$ has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_o t} \overset{\text{FT}}{\Longleftrightarrow} 2\pi\delta(\omega - \omega_o)$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFS}}{\longleftrightarrow} \quad X[k]$$

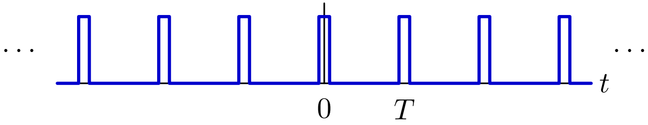
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j\frac{2\pi}{T}kt} \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta\left(\omega - \frac{2\pi}{T}k\right)$$

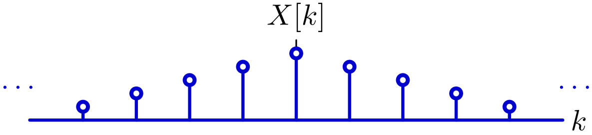
Each term in the Fourier series is replaced by an impulse in the Fourier transform.

Relation between Fourier Transform and Fourier Series

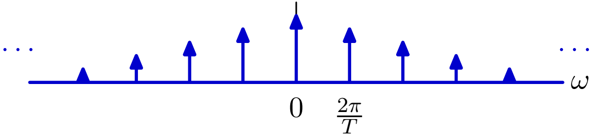
Each Fourier series term is replaced by an impulse in the Fourier transform.

$$x(t) = \sum_{k=-\infty}^{\infty} x_p(t - kT)$$





$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, X[k]\, \delta(\omega - k\frac{2\pi}{T})$$



Math With Impulses

Delta functions are similarly useful in discrete-time transforms.

Let

$$X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$$

where the sum results because DT Fourier Transforms are periodic in 2π. Then

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} \delta(\Omega)e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} \delta(\Omega)\, d\Omega = \frac{1}{2\pi}$$

Thus if $x[n] = 1$ for all n, the transform is a delta function in frequency.

$$1 \quad \overset{\text{DTFT}}{\Longleftrightarrow} \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi m)$$

We previously showed a similar result for CT:

$$1 \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 2\pi\delta(\omega)$$

Math With Impulses

We can similarly find the transform of a DT complex exponential.

Let

$$X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi m)$$

where the sum results because DT Fourier Transforms are periodic in 2π. Then

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \delta(\Omega - \Omega_o) e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{2\pi} \delta(\Omega') e^{j(\Omega' + \Omega_o)n}\, d\Omega' = \frac{1}{2\pi} e^{j\Omega_o n}$$

Thus if $x[n] = e^{j\Omega_o n}$ for all n, the transform is a shifted delta function in frequency.

$$e^{j\Omega_o n} \quad \overset{\text{DTFT}}{\Longleftrightarrow} \quad \sum_{m=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_o - 2\pi m)$$

We previously showed a similar result for CT:

$$e^{j\omega_o t} \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad 2\pi\delta(\omega - \omega_o)$$

Relations Between Fourier Series and Fourier Transforms

Continuous Time:

$$e^{j\frac{2\pi k}{T}t} \overset{\text{CTFT}}{\Longleftrightarrow} 2\pi\delta\left(\omega - \frac{2\pi k}{T}\right)$$

$$x(t) = x(t+T) \overset{\text{CTFS}}{\Longleftrightarrow} X[k]$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k]e^{j\frac{2\pi}{T}kt} \overset{\text{CTFT}}{\Longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi X[k]\delta\left(\omega - \frac{2\pi}{T}k\right)$$

Discrete Time:

$$e^{j\frac{2\pi k}{N}n} \overset{\text{DTFT}}{\Longleftrightarrow} \sum_{m=-\infty}^{\infty} 2\pi\delta\left(\Omega - \frac{2\pi k}{N} - 2\pi m\right)$$

$$x[n] = x[n+N] \overset{\text{DTFS}}{\Longleftrightarrow} X[k]$$

$$x[n] = x[n+N] = \sum_{k=\langle N\rangle} X[k]e^{j\frac{2\pi}{N}kn} \overset{\text{DTFT}}{\Longleftrightarrow} \sum_{k=\langle N\rangle} \sum_{m=-\infty}^{\infty} 2\pi X[k]\delta\left(\Omega - \frac{2\pi}{N}k - 2\pi m\right)$$

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