6.003: Signal Processing

Discrete-Time Fourier Transform

- Definition
- Examples
- Properties
- Relations between Fourier series and transforms (DT and CT)

Announcements:
- Quiz 1: October 5, 2-4pm, 50-340 (Walker)
  - Coverage up to and including all of week 3, including HW3.
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September 30, 2021

Fourier Representations of Aperiodic Signals

How can we represent an aperiodic signal as a sum of sinuosids?

\[ x[n] \]

\[ \vdots \]

\[ -2 \quad 0 \quad 2 \quad n \]

Strategy: make a periodic version of \( x[n] \) by summing shifted copies:

\[ x_p[n] = \sum_{m=-\infty}^{\infty} x[n-mN] \]

Since \( x_p[n] \) is periodic, it has a Fourier series (which depends on \( N \)).

Find Fourier series coefficients \( X_p[k] \) and take the limit of \( X_p[k] \) as \( N \to \infty \).

As \( N \to \infty \), \( x_p[n] \to x[n] \), and Fourier series will approach Fourier transform.

Fourier Representations of Aperiodic Signals

Define a new function \( X(\Omega) = NX_p[k] \) where \( \Omega = k\Omega_0 = 2\pi k/n \).

\[ NX_p[k] = 1 + 2\cos \frac{2\pi k}{N} + 2\cos \frac{4\pi k}{N} \]

Then \( NX_p[k] \) represents samples of \( X(\Omega) \) with increasing resolution in \( \Omega \).

The discrete function \( NX_p[k] \) is a sampled version of the function \( X(\Omega) \).
Fourier Representations of Aperiodic Signals

We can reconstruct $x[n]$ from $X(\Omega)$ using a Riemann sum.

$$ x[n] = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_k(\Omega) e^{j\Omega n} d\Omega $$

N=8:

$$ X_k(\Omega) = \begin{cases} 1 & |\Omega| = \frac{2\pi k}{8} \\ 0 & \text{otherwise} \end{cases} $$

N=16:

$$ X_k(\Omega) = \begin{cases} 1 & |\Omega| = \frac{2\pi k}{16} \\ 0 & \text{otherwise} \end{cases} $$

N=32:

$$ X_k(\Omega) = \begin{cases} 1 & |\Omega| = \frac{2\pi k}{32} \\ 0 & \text{otherwise} \end{cases} $$

Fourier Transform relation: $x[n] \leftrightarrow X(\Omega)$

Fourier Series and Fourier Transform

All of the information in a periodic signal is contained in one period. The information in an aperiodic signal can spread across all time.

Discrete-Time Fourier Series

$$ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega kn} $$

$$ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\Omega kn} $$

Discrete-Time Fourier Transform

$$ X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x[n] e^{-j\Omega n} d\Omega $$

$$ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega $$

Continuous-Time Fourier Transform

$$ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt $$

$$ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega $$

CT and DT Fourier Transforms

DT frequencies alias because adding $2\pi$ to $\Omega$ does not change $e^{-j\Omega n}$. 

Continuous-Time Fourier Transform

$$ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt $$

$$ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega $$

Discrete-Time Fourier Transform

$$ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega kn} $$

$$ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\Omega kn} $$

$$ X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x[n] e^{-j\Omega n} d\Omega $$

$$ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega $$

Fourier Series and Fourier Transform

Fourier series and transforms are similar: both represent signals by their frequency content.

Discrete-Time Fourier Series

$$ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\Omega kn} $$

$$ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\Omega kn} $$

Discrete-Time Fourier Transform

$$ X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x[n] e^{-j\Omega n} d\Omega $$

$$ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega $$
**Continuous-Time Fourier Transform**

\[ X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \]  

*analysis equation*

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \]  

*synthesis equation*

**Discrete-Time Fourier Transform**

\[ X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \]  

*analysis equation*

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega \]  

*synthesis equation*

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**Examples of Fourier Transforms**

Compare Fourier transforms of pulses with different widths.

- **p₂[n]**
  - 1
  - Ω
  - P₂(Ω)

- **p₄[n]**
  - 1
  - Ω
  - P₄(Ω)

- **p₆[n]**
  - 1
  - Ω
  - P₆(Ω)

As the function widens in \(n\) the Fourier transform narrows in \(\Omega\).

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**Moments**

The value of \(x(0)\) is the integral of \(X(\omega)\) divided by \(2\pi\).

\[ x(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \frac{\sin \omega}{\omega} d\omega \]

**Examples of Fourier Transforms**

Find the Fourier Transform (FT) of a rectangular pulse of width \(2S+1\):

\[ p_S[n] = \begin{cases} 1 - S & \text{if } S \leq n \leq S \\ 0 & \text{otherwise} \end{cases} \]

\[ p_S[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_S(\Omega)e^{j\Omega n}d\Omega \]

**Moments**

Similar to CT, the value of \(X(\Omega)\) at \(\Omega = 0\) is the sum of \(x[n]\) over time \(t\).

\[ X(0) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n] \]

**Examples of Fourier Transforms**

Compare Fourier transforms of pulses with different widths.

- **p₂[n]**
  - 1
  - Ω
  - P₂(Ω)

- **p₄[n]**
  - 1
  - Ω
  - P₄(Ω)

- **p₆[n]**
  - 1
  - Ω
  - P₆(Ω)

As the function widens in \(n\) the Fourier transform narrows in \(\Omega\).
Stretching Time
Stretching time compresses frequency and increases amplitude (preserving area).

Math With Impulses
Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 1: Find the Fourier transform of a unit impulse function.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$$

Since $\delta(t)$ is zero except near $t=0$, only values of $e^{-j\omega t}$ near $t=0$ are important. Because $e^{-j\omega t}$ is a smooth function of $t$, $e^{-j\omega t}$ can be replaced by $e^{-j\omega 0}$.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0}dt = 1$$

$$\delta(t) \quad \text{CTFT} \quad \frac{1}{\omega}$$

This matches our previous result which was based explicitly on a limit. Here the limit is implicit.

Compressing Time to the Limit
Alternatively, we could compress time while keeping area $= 1$.

Example 2: Find the function whose Fourier transform is a shifted impulse.

$$x(t) = 1$$

$$X(\omega) = \sin \omega / 2 \omega / 2$$

In the limit, the pulse has zero width but area $1$!

We represent this limit with the delta function: $\delta(t)$.

Math With Impulses
Although physically unrealizable, the impulse (a.k.a. Dirac delta) function is useful as a mathematically tractable approximation to a very brief signal.

Example 3: Find the function whose Fourier transform is a shifted impulse.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \quad \text{CTFT} \quad 2\pi \delta(\omega - \omega_0)$$

Use this result to relate Fourier series to Fourier transforms.

Relation Between Fourier Series and Fourier Transforms
If a periodic signal $x(t) = x(t+T)$ has a Fourier series representation, then it can also be represented by an equivalent Fourier transform.

$$e^{j\omega_0 t} \quad \text{CTFS} \quad 2\pi \delta(\omega - \omega_0)$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k t} \quad \text{CTFS} \quad X[k]$$

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k t} \quad \text{CTFT} \quad \sum_{k=-\infty}^{\infty} 2\pi X[k] \delta(\omega - 2\pi k)$$

Each term in the Fourier series is replaced by an impulse in the Fourier transform.
Relation between Fourier Transform and Fourier Series
Each Fourier series term is replaced by an impulse in the Fourier transform.

\[ x(t) = \sum_{k=-\infty}^{\infty} x_p(t-kT) \]

Math With Impulses
Delta functions are similarly useful in discrete-time transforms.

Let

\[ X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) \]

where the sum results because DT Fourier Transforms are periodic in 2\pi.

Then

\[ x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\Omega)d\Omega = \frac{1}{2\pi} \]

Thus if \( x[n] = 1 \) for all \( n \), the transform is a delta function in frequency.

1. \[ \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - 2\pi m) \]

We previously showed a similar result for CT:

1. \[ \frac{1}{2\pi} 2\pi \delta(\omega) \]

Math With Impulses
We can similarly find the transform of a DT complex exponential.

Let

\[ X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_o - 2\pi m) \]

where the sum results because DT Fourier Transforms are periodic in 2\pi.

Then

\[ x[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\Omega)e^{j(\Omega + \Omega_o) n}d\Omega = \frac{1}{2\pi} e^{j\Omega_o n} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_o - 2\pi m) \]

Thus if \( x[n] = e^{j\Omega_o n} \) for all \( n \), the transform is a shifted delta function in frequency.

\[ e^{j\Omega_o n} \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_o - 2\pi m) \]

We previously showed a similar result for CT:

\[ e^{j\omega n} \sum_{m=-\infty}^{\infty} 2\pi \delta(\omega - \omega_o) \]

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