6.003: Signal Processing

Discrete-Time Fourier Series

- Fourier series representations for discrete-time signals
- Comparison of Fourier series for CT and DT signals
- Properties of DT Fourier series
- Applications of Fourier analysis

September 23, 2021

Continuous-Time Fourier Series

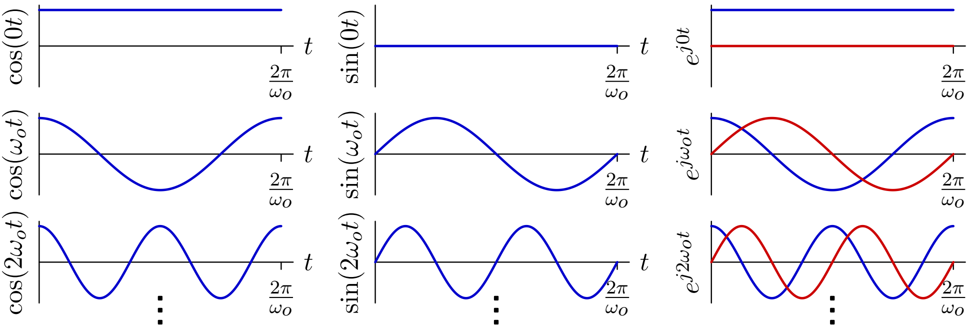
We previously explored the expansion of periodic CT functions as Fourier series using either trigonometric functions or complex exponentials

$$f(t) = f(t+T) \;=\; c_0 + \sum_{k=1}^{\infty} c_k \cos k\omega_o t + \sum_{k=1}^{\infty} d_k \sin k\omega_o t \;=\; \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

where $\omega_o = \frac{2\pi}{T}$ represents the fundamental frequency.

Real-valued basis functions

Complex basis functions



Continuous-Time Fourier Series

We found the Fourier series coefficients using two key insights.

1. Multiplying complex harmonics of ω_o yields a complex harmonic of ω_o:

$$e^{jk\omega_o t} \times e^{jl\omega_o t} = e^{j(k+l)\omega_o t}$$

2. Integrating a complex harmonic over a period T yields zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_o t} dt \equiv \int_T e^{jk\omega_o t} dt = \begin{cases} T & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$
$$= T\delta[k]$$

where $\delta[k]$ is the Kronecker delta function

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ Fourier components are **orthogonal**.

Continuous-Time Fourier Series

Use orthogonality to find the Fourier series coefficients.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Multiply $f(t)$ by the complex conjugate of the basis function of interest, and then integrate over T.

$$\begin{aligned}
\int_T f(t) e^{-jl\omega_o t} dt &= \int_T \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \right) e^{-jl\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j(k-l)\omega_o t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = a_l T
\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_o t} dt$$

Continuous-Time Fourier Series

Representing a periodic signal as a sum of harmonic sinusoids.

Synthesis Equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Analysis Equation

$$a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_o t}\, dt$$

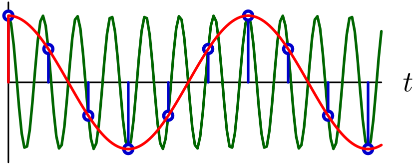
where $\omega_o = \dfrac{2\pi}{T}$

Today: Discrete-Time Fourier Series

Goal: Develop Fourier representation for discrete-time signals.

While the high-level goal is the same in DT as in CT, there are important differences between CT and DT signals.

As we saw last time the same set of samples can represent many different frequencies – a phenomenon called **aliasing**.



Samples (blue) of the original high-frequency signal (green) could just as easily have come from a much lower frequency signal (red).

Aliasing

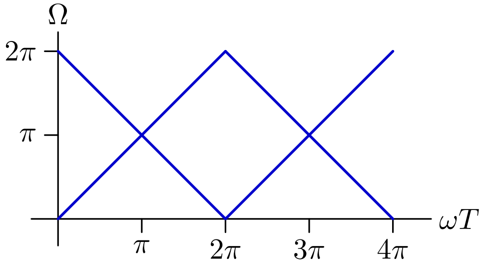
We saw that many CT frequencies ω **alias** to the same DT frequency Ω.

Sample a CT signal

$$f(t) = \cos(\omega t)$$

at integer multiples of the sampling period T to generate a DT signal:

$$f[n] = f(t)|_{t=nT} = f(nT) = \cos(\omega nT) = \cos(\Omega n) \quad \text{where} \quad \Omega = \omega T$$



Each point on the blue lines in this figure indicates a pair of frequencies (ω, Ω) for which $\cos(\omega T n) = \cos(\Omega n)$.

Aliasing

We saw that many CT frequencies ω **alias** to the same DT frequency Ω.

Sample a CT signal

$$f(t) = \cos(\omega t)$$

at integer multiples of the sampling period T to generate a DT signal:

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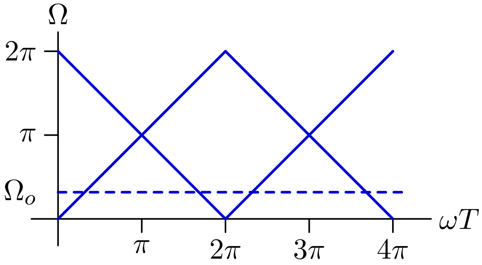


Many continuous frequencies ω produce the same samples:

$$f[n] = \cos(\Omega_o n) = \cos(\omega T n) \quad \text{for} \quad \omega = \frac{\Omega_o}{T}, \frac{2\pi - \Omega_o}{T}, \frac{2\pi + \Omega_o}{T}, \frac{4\pi - \Omega_o}{T}, \ldots$$

Aliasing

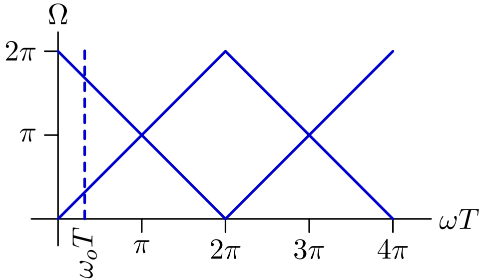
We saw that many CT frequencies ω **alias** to the same DT frequency Ω.

Sample a CT signal

$$f(t) = \cos(\omega t)$$

at integer multiples of the sampling period T to generate a DT signal:

$$f[n] = f(t)|_{t=nT} = f(nT) = \cos(\omega nT) = \cos(\Omega n) \quad \text{where} \quad \Omega = \omega T$$

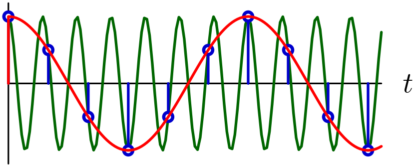


Many discrete frequencies Ω could represent the samples produced by ω_o:

$$f[n] = \cos(\omega_o Tn) = \cos(\Omega n) \;\; \text{for } \Omega = \omega_o T,\ 2\pi{-}\Omega_o T,\ 2\pi{+}\Omega_o T,\ 4\pi{-}\Omega_o T \ldots$$

Aliasing

Aliasing is an intrinsic property of DT sinusoids.



And there are other unique properties of DT sinsuoids that are important for representing DT signals as Fourier series.

Check Yourself

What is the fundamental (shortest) period of each of the following signals?

1. $f_1[n] = \cos\left(\frac{\pi n}{12}\right)$
2. $f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right)$
3. $f_3[n] = \cos(n)$

Check Yourself

If

$$f_1[n] = \cos\left(\frac{\pi n}{12}\right) = f_1[n+N]$$

then

$$\frac{\pi N}{12} = 2\pi m$$

where both N and m are integers. Solving, we find

$$N = \frac{24\pi m}{\pi}$$

which is 24 if $m = 1$. Therefore $N = 24$.

Check Yourself

Similarly if

$$f_2[n] = \cos\left(\frac{\pi n}{12}\right) + 3\cos\left(\frac{\pi n}{15}\right) = f_2[n+N]$$

then

$$\frac{\pi N}{12} = 2\pi m_1 \qquad \to N = \frac{24\pi m_1}{\pi}$$

and

$$\frac{\pi N}{15} = 2\pi m_2 \qquad \to N = \frac{30\pi m_2}{\pi}$$

for integers m_1 and m_2.

We seek the smallest possible value of N so

$$N = 24m_1 = 2 \times 2 \times 2 \times 3 \times m_1 = 30m_2 = 2 \times 3 \times 5 \times m_2$$

and the smallest possible N is $2 \times 2 \times 2 \times 3 \times 5 = 120$.

Check Yourself

If

$$f_3[n] = \cos(n) = f_3[n+N]$$

then

$$N = 2\pi m$$

where both N and m are integers.

Since the is not possible, $f_3[n]$ is not periodic.

Check Yourself

What is the fundamental (shortest) period of each of the following signals?

1. $x_1[n] = \cos\frac{\pi n}{12}$ 24
2. $x_2[n] = \cos\frac{\pi n}{12} + 3\cos\frac{\pi n}{15}$ 120
3. $x_3[n] = \cos n + \cos 2n + \cos 3n$ ∞

Check Yourself

What is the fundamental (shortest) period of each of the following signals?

1. $x_1[n] = \cos\frac{\pi n}{12}$ 24
2. $x_2[n] = \cos\frac{\pi n}{12} + 3\cos\frac{\pi n}{15}$ 120
3. $x_3[n] = \cos n + \cos 2n + \cos 3n$ ∞

The period of a periodic DT signal **must be an integer**. Therefore the fundamental frequency $\Omega_o = \frac{2\pi}{N}$ must be an **integer submultiple** of 2π.

No such constraints on fundamental frequencies in CT. In CT, the fundamental frequency $\omega = \frac{2\pi}{T}$ can be any real number.

$\rightarrow$ This is an intrinsic difference between CT and DT signals.

Discrete-Time Sinusoids

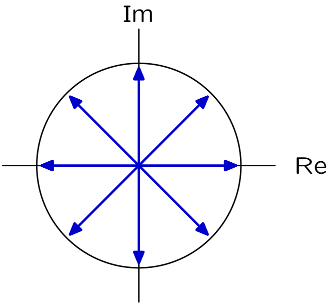
If Ω_o is a submultiple of 2π, and the harmonic frequencies alias, then there are (only) N distinct complex exponentials with period N.
(There were an infinite number in CT!)

If $f[n] = e^{j\Omega n}$ is periodic in N then

$$f[n] = e^{j\Omega n} = f[n+N] = e^{j\Omega(n+N)} = e^{j\Omega n}e^{j\Omega N}$$

and $e^{j\Omega N}$ must be 1. Therefore $e^{j\Omega}$ must be one of the N^{th} roots of 1.

Example: $N = 8$

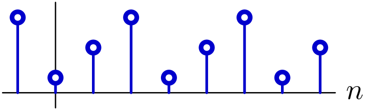


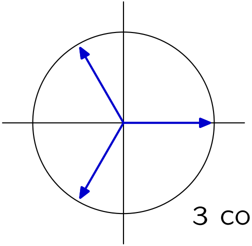
There are only 8 unique harmonics of Ω_o: $0,\ \frac{1\pi}{4},\ \frac{2\pi}{4},\ \frac{3\pi}{4},\ \frac{4\pi}{4},\ \frac{5\pi}{4},\ \frac{6\pi}{4},\ \frac{7\pi}{4}.$

Discrete-Time Sinusoids

There are N distinct complex exponentials with period N.

Example: periodic in N=3

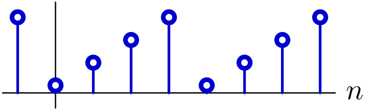


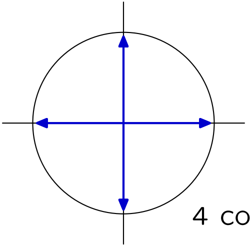


3 samples repeated in time

3 complex exponentials

Example: periodic in N=4





4 samples repeated in time

4 complex exponentials

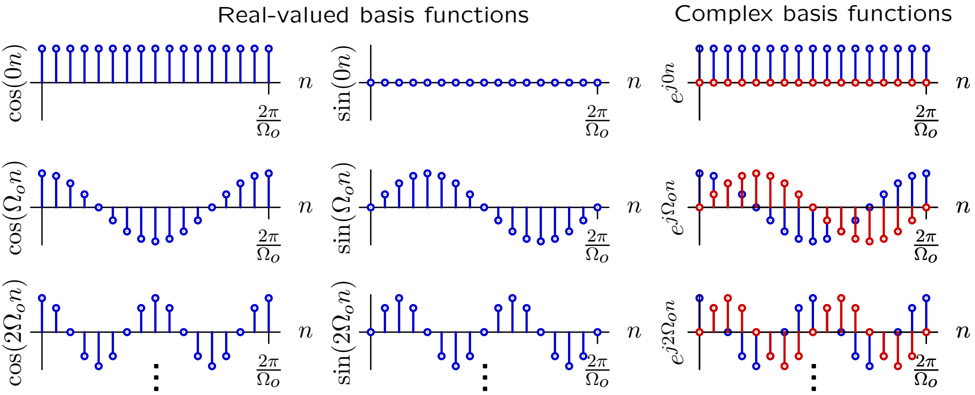
If a DT signal is periodic with period N,
then its Fourier series will contain just N terms.

Discrete Time Fourier Series

A DT Fourier Series has just N harmonic frequencies $k\Omega_o$.

$$f[n] = c_0 + \sum_{k=\langle N \rangle} c_k \cos(k\Omega_o n) + \sum_{k=\langle N \rangle} d_k \sin(k\Omega_o n)$$

where Ω_o represents the fundamental frequency **(radians/sample)**.
Otherwise, DT Fourier series are similar to CT Fourier series.



Discrete-Time Fourier Series

The same two key insights apply to both CT and DT analysis.

1. Multiplying complex **DT** harmonics of Ω_o yields a new harmonic of Ω_o:

$$e^{jk\Omega_o n} \times e^{jl\Omega_o n} = e^{j(k+l)\Omega_o n}$$

2. **Summing** a complex harmonic over a period N is zero unless the harmonic is at DC:

$$\sum_{n=n_0}^{n_0+N} e^{jk\Omega_o n} \equiv \sum_{n=\langle N\rangle} e^{jk\Omega_o n} = \begin{cases} N & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

$$= N\delta[k]$$

$\rightarrow$ **DT** Fourier components are **orthogonal**.

Discrete-Time Fourier Series

Using orthogonality to find the DT Fourier series coefficients.

$$f[n] = f[n+N] = \sum_{k=\langle N \rangle} a_k e^{jk\Omega_o n}$$

Multiply $f[n]$ by the complex conjugate of the basis function of interest, and then sum over N.

$$\begin{aligned}
\sum_{n=\langle N \rangle} f[n] e^{-jl\Omega_o n} &= \sum_{n=\langle N \rangle} \left(\sum_{k=\langle N \rangle} a_k e^{jk\Omega_o n} \right) e^{-jl\Omega_o n} \\
&= \sum_{k=\langle N \rangle} a_k \sum_{n=\langle N \rangle} e^{j(k-l)\Omega_o n} \\
&= \sum_{k=\langle N \rangle} a_k N \delta[k-l] = a_l N
\end{aligned}$$

Solving for a_l and then substituting k for l yields

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\Omega_o n}$$

Discrete-Time Fourier Series

Both $f[n]$ and a_k are periodic in N.

The DT signal is periodic by construction:

$$f[n] = f[n+N]$$

Its Fourier series coefficients are also periodic.

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\frac{2\pi}{N}n}$$

$$a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-j(k+N)\frac{2\pi}{N}n}$$

$$a_{k+N} = \underbrace{\left(\frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\frac{2\pi}{N}n}\right)}_{a_k}\underbrace{\left(e^{-jN\frac{2\pi}{N}n}\right)}_{1}$$

$$a_{k+N} = a_k$$

Discrete-Time Fourier Series

Representing a periodic DT signal as a sum of harmonic sinusoids.

Synthesis Equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

Analysis Equation

$$a_k = a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} f[n] e^{-jk\Omega_o n}$$

where $\Omega_o = \dfrac{2\pi}{N}$

Fourier Series Summary

CT and DT Fourier Series are similar, but DT Fourier Series have just N coefficients while CT Fourier Series have an infinite number.

Continuous-Time Fourier Series

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

analysis equation

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

synthesis equation

where $\omega_o = \frac{2\pi}{T}$

Discrete-Time Fourier Series

$$a_k = a_{k+N} = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-jk\Omega_o n}$$

analysis equation

$$f[n] = f[n+N] = \sum_{k=\langle N\rangle} a_k e^{jk\Omega_o n}$$

synthesis equation

where $\Omega_o = \frac{2\pi}{N}$

Properties of Discrete-Time Fourier Series

Operations on the time representation of a signal can often be interpreted as equivalent (but easier) operations on the series coefficients.

Here we will discuss four (of many) properites of Fourier series.

- linearity
- time shift
- time reversal
- conjugate symmetry

Linearity

The Fourier series coefficients of a linear combination of two signals is the linear combination of their Fourier series coefficients.

Let

$$f[n] = af_1[n] + bf_2[n] \quad \text{where} \quad f_1[n] = f_1[n+N] \text{ and } f_2[n] = f_2[n+N]$$

then the Fourier series coefficents for $f[n]$ are given by

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} (af_1[n] + bf_2[n]) e^{-jk\frac{2\pi}{N}n}$$

$$= a \; \frac{1}{N} \sum_{n=\langle N \rangle} f_1[n] e^{-jk\frac{2\pi}{N}n} + b \; \frac{1}{N} \sum_{n=\langle N \rangle} f_2[n] e^{-jk\frac{2\pi}{N}n}$$

$$= aF_1[k] + bF_2[k]$$

where $F_1[k]$ and $F_2[k]$ are Fourier series coefficients for $f_1[n]$ and $f_2[n]$.

Time Shift

Shifting time changes the phases of a signal's Fourier coefficients.

Let

$$g[n] = f[n - n_0] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

then

$$G[k] = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[n - n_0] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{-jk\frac{2\pi}{N}(m+n_0)} \quad \text{where} \quad m = n - n_0$$

$$= e^{-jk\frac{2\pi}{N}n_0} \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{-jk\frac{2\pi}{N}m} = e^{-jk\frac{2\pi}{N}n_0} F[k]$$

Time Reversal

Reversing time reverses frequency.

Let

$$g[n] = f[-n] \quad \text{where} \quad f[n] = f[n+N]$$

If

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

then

$$\begin{aligned} G[k] &= \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=\langle N \rangle} f[-n] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{m=\langle N \rangle} f[m] e^{jk\frac{2\pi}{N}m} \quad \text{where} \quad m = -n \\ &= F[-k] \end{aligned}$$

Conjugate Symmetry

If $f[n]$ is real-valued, then its Fourier coefficients have conjugate symmetry.

If $f[n]$ is real-valued, then $f[n] = f^*[n]$.

$$F[k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{-jk\frac{2\pi}{N}n}$$

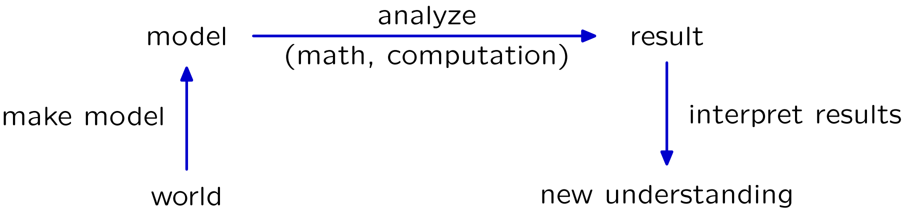
$$F[-k] = \frac{1}{N} \sum_{n=\langle N \rangle} f[n] e^{jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \left(f[n] e^{-jk\frac{2\pi}{N}n} \right)^* = F^*[k]$$

Applications of Fourier Series

Signal processing is **widely used** in science and engineering to . . .

- **model** some aspect of the world,
- **analyze** the model, and
- **interpret** results to gain a new or better understanding.



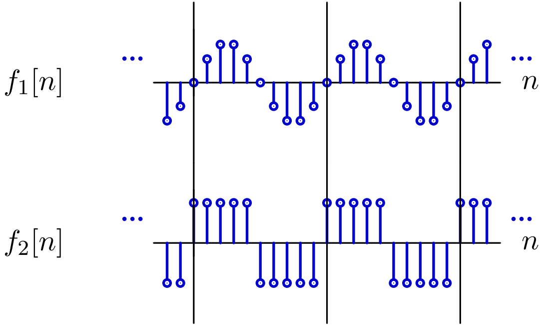
We previously touched on applications in physics, including the wave equation and how it leads directly to Fourier analysis.

Applications of Fourier analysis in **hearing**.

Applications of Fourier Analysis in Hearing

What determines the pitch of a sound? This seemingly simple question has evoked debate (sometimes fierce) for more than 150 years.

Compare two periodic signals with the same period, each played with 4000 samples per second

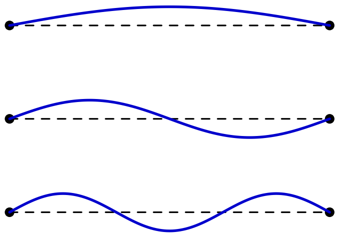


Different sounds, same pitch. We would like to understand why.

Pitch Experiments

Early experiments were based on stringed instruments and tubes, which were known to produce not just a fundamental but also harmonic overtones.

dominant frequency	dominant mode
fundamental	
2nd harmonic	
3rd harmonic	



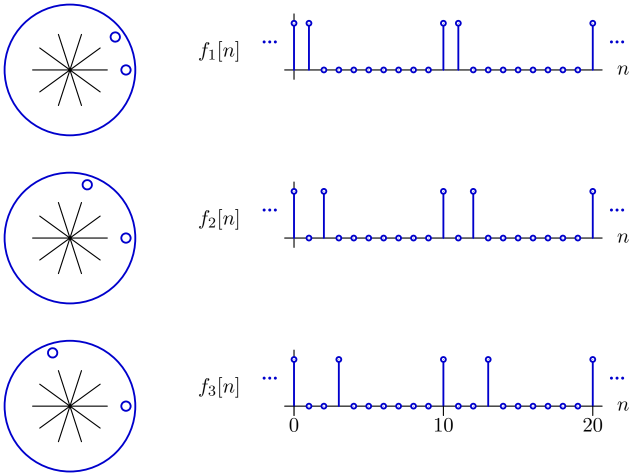
Although different sources produced different mixtures of harmonics, it was difficult to separate effects of one harmonic from those of others.

A breakthrough occured with the work of Seebeck who used sirens to generate more complicated sounds.

Very clever experiment, but very controversial interpretations.

Sirens

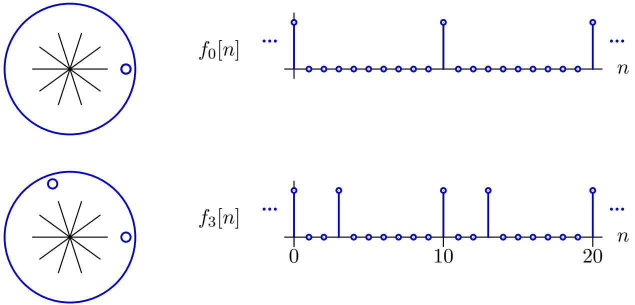
Seebeck used a siren to generate more complicated sounds (circa 1841) by passing a jet of compressed air through holes in a spinning disk.



The pattern of holes determined the pattern of pulses in each period. The speed of spinning controlled the number of periods per second.

Sirens

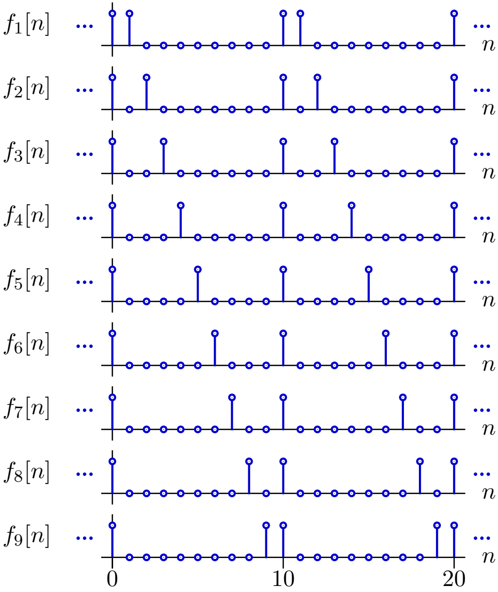
Strangely, adding a second hole per period didn't seem to affect the pitch.



Pitch should be different if it is determined by the intervals between pulses.

Sirens

There was one very interesting exception.



But hearing this exception required precise alignment of the siren's holes.

Sirens and Controversy

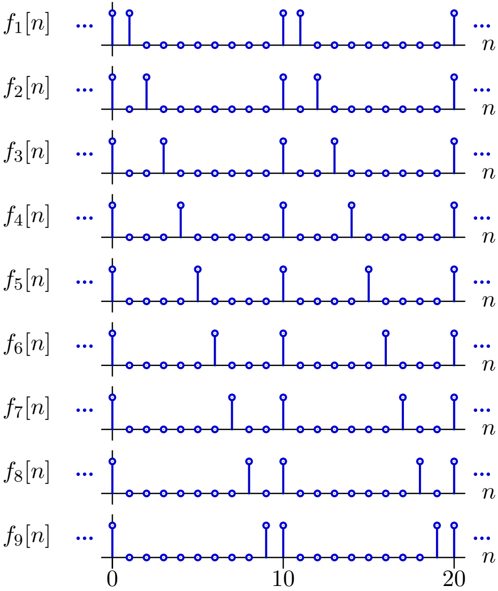
Seebeck interpreted his results in terms of the intervals between the holes. He held that pitch results from **timing** with some intervals being more important than others. As the lengths of the two intervals in his experiment converged, the pitch favored what had been the second harmonic and that frequency increasingly dominated.

Georg Ohm (already known for his work on electrical conduction) interpreted Seebeck's results using Fourier's recently described series. He held that the pulses generated by a siren contained a **fundamental** and **harmonics** that were physically present just as much as they are in a stringed instrument.

A bitter controversy ensued.

Fourier Interpretation

To understand Ohm's argument, compute the Fourier series for the siren's sound.



Fourier Interpretation

Find the k^{th} coefficient of the i^{th} signal.

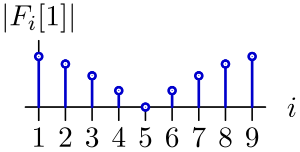
$$F_i[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f_i[n]\,e^{-j\frac{2\pi k}{N}n} = \frac{1}{10}\sum_{n=0}^{9} f_i[n]\,e^{-j\frac{2\pi k}{10}n} = \frac{1}{10}\left(1+e^{-j\frac{2\pi k}{10}i}\right)$$

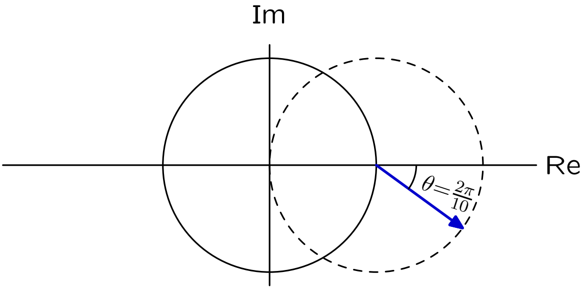
DC: $k=0$ term

$$F_i[0] = \frac{1}{10}\left(1+e^{-j\frac{2\pi 0}{10}i}\right) = \frac{2}{10}$$

Fundamental: $k=1$ term

$$F_i[1] = \frac{1}{10}\left(1+e^{-j\frac{2\pi}{10}i}\right)$$





Notice that $f_5[n]$ has no fundamental component!

Fourier Series

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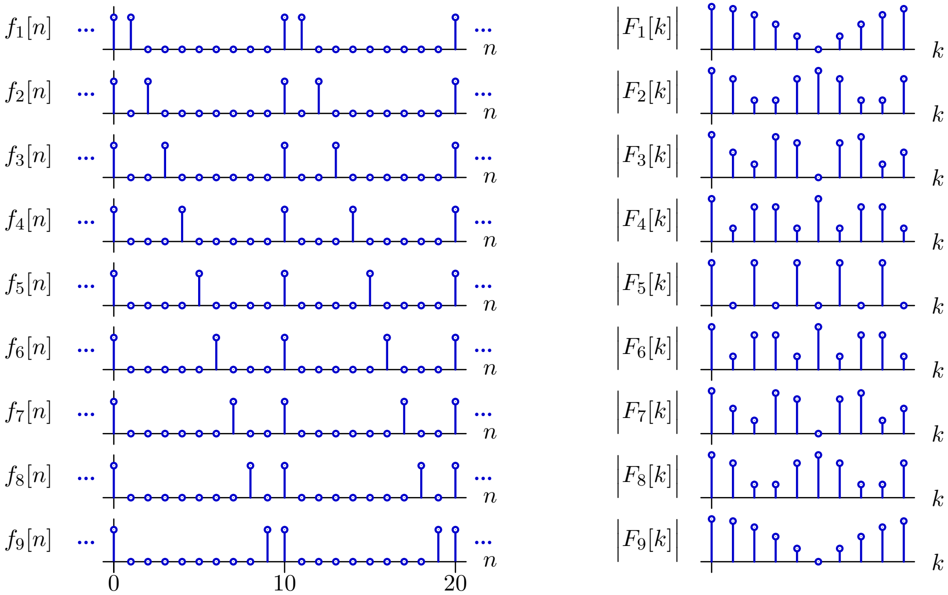
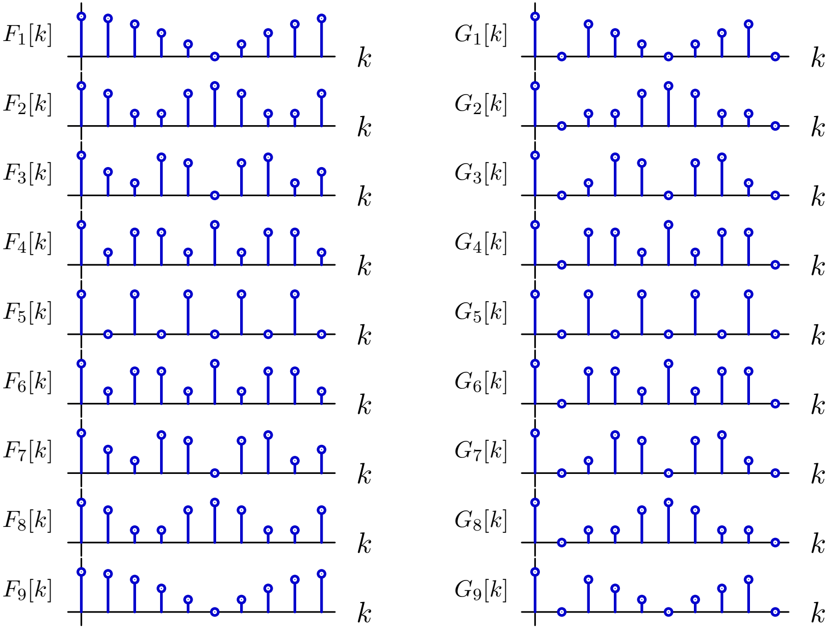


Fourier Series With and Without the Fundamental

Resynthesize each waveform without its fundamental component.



Although perception of the fundamental is weakened, it is not gone!

Summary

Seebeck designed an extremely clever **experiment** to test pitch perception.

Ohm analyzed an important **theory** (from Fourier) and argued that harmonics are present even in the pulsatile sounds generated by a siren.

Neither Seebeck nor Ohm could convincingly account for experimental results that demonstrated the dominance of the fundamental, even when it was weak or missing.

Progress in understanding the "missing fundamental" awaited Helmholtz, who demonstrated the importance of "combination tones" in the ear.

$$
\begin{tikzcd}
\text{model} \arrow[r, "\text{analyze}", "\text{(math, computation)}"'] & \text{result} \arrow[d, "\text{interpret results}"] \\
\text{world} \arrow[u, "\text{make model}"] & \text{new understanding}
\end{tikzcd}
$$

Summary

Today we focused on discrete-time Fourier analysis.

- We developed Fourier series for discrete-time signals.
- We compared Fourier series for CT and DT signals.
- We looked at four (of many) properties of DT Fourier series.
- We looked briefly at applications of Fourier analysis in hearing.

Next week: Fourier analysis of aperiodic signals.