6.003: Signal Processing

Fourier Series – Complex Exponential Form

- complex numbers
- complex exponentials and their relation to sinusoids
- complex exponential form of Fourier series
- delay property of Fourier series

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Fourier Series

We have previously represented signals as weighted sums of sinusoids.

**Synthesis Equation**

\[ f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t) \quad \text{where} \quad \omega_o = \frac{2\pi}{T} \]

**Analysis Equations**

\[ c_0 = \frac{1}{T} \int_T^T f(t) \, dt \]

\[ c_k = \frac{2}{T} \int_T^T f(t) \cos(k\omega_o t) \, dt \]

\[ d_k = \frac{2}{T} \int_T^T f(t) \sin(k\omega_o t) \, dt \]

**Today:** Simplifying the math with complex numbers.
Simplifying Math By Using Complex Numbers

**Complex numbers** simplify thinking about roots of numbers / polynomials:
- all numbers have two square roots, three cube roots, etc.
- all polynomials of order $n$ have $n$ roots (some of which may be repeated).

→ much simpler than the rules that govern purely real-valued formulations. For example, a cubic equation with real-valued coefficients might have 1 or 3 real-valued roots; a quartic equation might have 0, 2, or 4.

**Complex exponentials** simplify thinking about trigonometric functions (Euler’s formula, Leonhard Euler, 1748):

$$e^{j\theta} = \cos \theta + j \sin \theta$$

where $j = \sqrt{-1}$.

This single equation virtually eliminates our need for trig tables. Richard Feynman called this ”the most remarkable formula in mathematics.”

Note that we will normally use $j$ (instead of $i$) to represent $\sqrt{-1}$. 

Euler showed the relation between complex exponentials and sinusoids by solving the following differential equation two ways.

\[ \frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0 \]

let \( f_1(\theta) = A \cos(\alpha \theta) + B \sin(\beta \theta) \)

\[ \frac{df_1(\theta)}{d\theta} = -\alpha A \sin(\alpha \theta) + \beta B \cos(\beta \theta) \]

\[ \frac{d^2 f_1(\theta)}{d\theta^2} = -\alpha^2 A \cos(\alpha \theta) - \beta^2 B \sin(\beta \theta) \]

\[ \alpha^2 = \beta^2 = 1 \]

\( f_1(\theta) = A \cos \theta + B' \sin \theta \)

Let \( f_2(\theta) = Ce^{\gamma \theta} \)

\[ \frac{df_2(\theta)}{d\theta} = \gamma Ce^{\gamma \theta} \]

\[ \frac{d^2 f_2(\theta)}{d\theta^2} = \gamma^2 Ce^{\gamma \theta} \]

\[ \gamma^2 = -1 \]

\( f_2(\theta) = Ce^{\pm j \theta} \)

If we arbitrarily take \( f_2(\theta) = e^{j \theta} \), \( f_2(0) = 1 \) and \( f_2'(0) = j \).

To make \( f_1(\theta) = f_2(\theta) \), we must set \( A = 1 \) and \( B' = j \).

It follows that

\[ e^{j \theta} = \cos \theta + j \sin \theta \]

This argument presumes the existance of a constant \( j \) whose square is \(-1\) and that can be manipulated as an ordinary algebraic constant.
Where Does Euler’s Formula Come From?

Euler’s formula also follows from Maclaurin expansion of the exponential function, assuming the $j$ behaves like any other algebraic constant.

Start with the expansion of the real-valued function:

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \cdots$$

Assume that the same expansion holds for complex-valued arguments:

$$e^{j\theta} = 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \frac{j^7\theta^7}{7!} + \cdots$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$\cos \theta \quad \sin \theta$$

Euler’s formula results by splitting the even and odd powers of $\theta$.

$$e^{j\theta} = \cos \theta + j \sin \theta$$
Geometric Interpretation

In 1799, Caspar Wessel was the first to describe complex numbers as points in the complex plane. Imaginary numbers had been in use since the 1500’s.

\[ c = a + jb \]
Algebraic Addition

Addition: the real part of a sum is the sum of the real parts, and the imaginary part of a sum is the sum of the imaginary parts.

Let $c_1$ and $c_2$ represent complex numbers:

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1+a_2) + j(b_1+b_2)$$
Geometric Addition

Rules for adding complex numbers are same as those for adding vectors.

Let $c_1$ and $c_2$ represent complex numbers:

\[
\begin{align*}
  c_1 &= a_1 + jb_1 \\
  c_2 &= a_2 + jb_2
\end{align*}
\]

Then

\[
c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1 + a_2) + j(b_1 + b_2)
\]
Geometric Addition

Rules for adding complex numbers are same as those for adding vectors.

Let $c_1$ and $c_2$ represent complex numbers:

\[
c_1 = a_1 + jb_1
\]
\[
c_2 = a_2 + jb_2
\]

Then

\[
c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1+a_2) + j(b_1+b_2)
\]
Algebraic Multiplication

Multiplication is more complicated.

Let $c_1$ and $c_2$ represent complex numbers:

\[
c_1 = a_1 + jb_1 \\
c_2 = a_2 + jb_2
\]

Then

\[
c_1 \times c_2 = (a+jb) \times (c+jd) \\
= a \times c + a \times jd + jb \times c + jb \times jd \\
= (ac - bd) + j(ad + bc)
\]

Although the rules of algebra apply, the result is complicated:

- the real part of a product is NOT the product of the real parts, and
- the imaginary part is NOT the product of the imaginary parts.
The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a rotation.

Multiplying by $j$

- rotates 1 to $j$,
- rotates $j$ to $-1$,
- rotates $-1$ to $-j$, and
- rotates $-j$ to 1.
Geometric Multiplication

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a rotation.

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- rotates 1 to $j$,
- rotates $j$ to $-1$,
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Geometric Multiplication

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- rotates $-j$ to 1.
**Geometric Multiplication**

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a **rotation**.

Multiplying by $j$

- rotates 1 to $j$,
- rotates $j$ to $-1$,
- rotates $-1$ to $-j$, and
- rotates $-j$ to 1.

Multiplying by $j$ rotates a vector by $\pi/2$.

Multiplying by $j^2 = -1$ rotates a vector by $\pi$. 
Geometric Multiplication

Multiplying by $j$ rotates an arbitrary complex number by $\pi/2$.

\[ c = a + jb \]
\[ jc = ja - b \]
Geometric Approach: Polar Form

The magnitude of the product of complex numbers is the **product** of their magnitudes. The angle of a product is the **sum** of the angles.

\[
(r_1 \angle \theta_1) \times (r_2 \angle \theta_2) = r_1 \cos \theta_1 + j \sin \theta_1 \times r_2 \cos \theta_2 + j \sin \theta_2
\]

\[
= r_1 r_2 \cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)
\]

\[
= r_1 r_2 \angle (\theta_1 + \theta_2)
\]
Euler’s formula equates polar and rectangular descriptions of a unit vector at angle $\theta$.

$$e^{j\theta} = \cos \theta + j \sin \theta$$
Simplifying Math By Using Complex Numbers

Euler’s formula allows us to represent both sine and cosine basis functions with a single complex exponential:

\[
f(t) = \sum \left( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \right) = \sum a_k e^{jk\omega_o t}
\]

This halves the number of coefficients, but each is now complex-valued. More importantly, it replaces the trig functions with an exponential.
Converting From Trig Form To Complex Exponential Form

Assume that a function $f(t)$ can be written as a Fourier series in trig form.

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} \left( c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t) \right)$$

We can use Euler’s formula to convert sinusoids to complex exponentials.

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j \sin(k\omega_o t)$$

$$\cos(k\omega_o t) = \text{Re}\{e^{jk\omega_o t}\} = \frac{(e^{jk\omega_o t} + e^{-jk\omega_o t})}{2}$$

$$\sin(k\omega_o t) = \text{Im}\{e^{jk\omega_o t}\} = -j\frac{(e^{jk\omega_o t} - e^{-jk\omega_o t})}{2}$$

$$f(t) = c_0 + \frac{1}{2} \sum_{k=1}^{\infty} \left( c_k e^{jk\omega_o t} + c_k e^{-jk\omega_o t} - j d_k e^{jk\omega_o t} + j d_k e^{-jk\omega_o t} \right)$$

$$= c_0 + \frac{1}{2} \sum_{k=1}^{\infty} (c_k - j d_k) e^{jk\omega_o t} + \frac{1}{2} \sum_{k=1}^{\infty} (c_k + j d_k) e^{-jk\omega_o t}$$

$$= c_0 + \frac{1}{2} \sum_{k=1}^{\infty} (c_k - j d_k) e^{jk\omega_o t} + \frac{1}{2} \sum_{k=-\infty}^{-1} (c_{-k} + j d_{-k}) e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

where $a_k = \begin{cases} (c_k - j d_k)/2 & \text{if } k > 0 \\ c_0 & \text{if } k = 0 \\ (c_{-k} + j d_{-k})/2 & \text{if } k < 0 \end{cases}$
Negative Frequencies

The complex form of a Fourier series has both positive and negative $k$’s.

Only positive values of $k$ are used in the trig form:

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

but both positive and negative values of $k$ are used in the exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

If we only included positive $k$ in the previous sum, the result would always have an imaginary component (unless $a_k = 0$ for all $k$).

If $f(t)$ is real-valued (as it must be for the trig form), then the complex coefficients $a_k$ are **conjugate symmetric**:

$$a_{-k} = a_k^*$$

where the $*$ denotes the complex conjugate.

$k : \ldots \, -3 \, -2 \, -1 \, 0 \, 1 \, 2 \, 3 \, \ldots$

$a_k : \ldots \, \frac{c_3+jd_3}{2} \, \frac{c_2+jd_2}{2} \, \frac{c_1+jd_1}{2} \, c_0 \, \frac{c_1-jd_1}{2} \, \frac{c_2-jd_2}{2} \, \frac{c_3-jd_3}{2} \, \ldots$
Fourier Series Directly From Complex Exponential Form

Assume that $f(t)$ is periodic in $T$ and is composed of a weighted sum of harmonically related complex exponentials.

$$f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

We can “sift” out the component at $l\omega_0$ by multiplying both sides by $e^{-jl\omega_0 t}$ and integrating over a period.

$$\int_T f(t) e^{-j\omega_0 l t} dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t} e^{-j\omega_0 l t} dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 (k-l) t} dt$$

$$= \begin{cases} Ta_l & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

Solving for $a_l$ provides an explicit formula for the coefficients:

$$a_k = \frac{1}{T} \int_T f(t) e^{-j\omega_0 k t} dt ; \quad \text{where } \omega_0 = \frac{2\pi}{T} .$$

This formulation works even if $f(t)$ has complex values.
Orthogonality and Projection

Fourier components are separable because they are **orthogonal**. Similar to separating a vector $\vec{r}$ into $x$ and $y$ components. Since $\hat{x}$ and $\hat{y}$ are orthogonal, we can separate the $x$ and $y$ components of $\vec{r}$ by **projection**:

\[
a = \vec{r} \cdot \hat{x} \\
b = \vec{r} \cdot \hat{y}
\]

Then

\[
\vec{r} = a\hat{x} + b\hat{y}
\]
Orthogonal Decompositions

**Vector representation:** let $\vec{r}$ represent a vector with components $a$ and $b$ in the $\hat{x}$ and $\hat{y}$ directions, respectively.

\[
a = \vec{r} \cdot \hat{x} \\
b = \vec{r} \cdot \hat{y}
\]

\[
\vec{r} = a\hat{x} + b\hat{y}
\]

(“analysis” equations)

(“synthesis” equation)
Orthogonal Decompositions

**Vector representation:** let $\vec{r}$ represent a vector with components $a$ and $b$ in the $\hat{x}$ and $\hat{y}$ directions, respectively.

\[
a = \vec{r} \cdot \hat{x} \\
b = \vec{r} \cdot \hat{y}
\]

(“analysis” equations)

\[
\vec{r} = a\hat{x} + b\hat{y}
\]

(“synthesis” equation)

**Fourier series:** let $f(t)$ represent a signal with harmonic components $a_0, a_1, \ldots, a_k$ for harmonics $e^{j0t}, e^{j\frac{2\pi}{T}t}, \ldots, e^{j\frac{2\pi}{T}kt}$ respectively.

\[
a_k = \frac{1}{T} \int_T f(t)e^{-j\frac{2\pi}{T}kt} \, dt
\]

(“analysis” equation)

\[
f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}
\]

(“synthesis” equation)
Orthogonal Decompositions

Integrating over a period sifts out the $k^{th}$ component of the series. Sifting as a dot product:

$$x = \bar{r} \cdot \hat{x} \equiv |\bar{r}| |\hat{x}| \cos \theta$$

Sifting as an inner product:

$$a_k = e^{j \frac{2\pi}{T} kt} \cdot f(t) \equiv \frac{1}{T} \int_T f(t) e^{-j \frac{2\pi}{T} kt} dt$$

where

$$a(t) \cdot b(t) = \frac{1}{T} \int_T a^*(t) b(t) dt.$$ 

The complex conjugate (*) makes the inner product of the $k^{th}$ and $m^{th}$ components equal to 1 iff $k = m$:

$$\frac{1}{T} \int_T \left( e^{j \frac{2\pi}{T} kt} \right)^* \left( e^{j \frac{2\pi}{T} mt} \right) dt = \frac{1}{T} \int_T e^{-j \frac{2\pi}{T} kt} e^{j \frac{2\pi}{T} mt} dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{otherwise} \end{cases}$$
Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?
Check Yourself

How many of the following are orthogonal \((\perp)\) in \(T = 3\)?

\[ \cos 2\pi t \perp \sin 2\pi t \]
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \sin 2\pi t$?

$\cos 2\pi t \perp \sin 2\pi t$

$\cos 2\pi t$

$\sin 2\pi t$

$\cos 2\pi t \sin 2\pi t = \frac{1}{2} \sin 4\pi t$

$\int_{0}^{3} dt = 0$ therefore YES
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$?
Check Yourself

How many of the following are orthogonal ($\perp$) in $T = 3$?

$\cos 2\pi t \perp \cos 4\pi t$?

1. $\cos 2\pi t$
2. $\cos 4\pi t$
3. $\cos 2\pi t \cos 4\pi t$

$\int_0^3 dt = 0$ therefore YES
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp \sin \pi t$?

$\cos 2\pi t \perp \sin \pi t$

$\int_0^3 dt \neq 0$ therefore NO
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp e^{j2\pi t}$?
Check Yourself

How many of the following are orthogonal (⊥) in $T = 3$?

$\cos 2\pi t \perp e^{j2\pi t}$?

$e^{2\pi t} = \cos 2\pi t + j\sin 2\pi t$

$\cos 2\pi t \perp \sin 2\pi t$ but not $\cos 2\pi t$

Therefore NO
How many of the following pairs of functions are orthogonal ($\perp$) in $T = 3$? 2

1. $\cos 2\pi t \perp \sin 2\pi t$? ✓
2. $\cos 2\pi t \perp \cos 4\pi t$? ✓
3. $\cos 2\pi t \perp \sin \pi t$? ×
4. $\cos 2\pi t \perp e^{j2\pi t}$? ×
Fourier Series

Comparison of trigonometric and complex exponential forms.

**Complex Exponential Form**

\[ f(t) = f(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \]

\[ a_k = \frac{1}{T} \int_T f(t) e^{-jk\omega_0 t} \, dt \]

**Trigonometric Form**

\[ f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t) \]

\[ c_0 = \frac{1}{T} \int_T f(t) \, dt \]

\[ c_k = \frac{2}{T} \int_T \cos(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots \]

\[ d_k = \frac{2}{T} \int_T \sin(k\omega_0 t) \, dt; \quad k = 1, 2, 3, \ldots \]
It seems as though it takes more numbers to characterize the complex exponential form:

• Each harmonic frequency in the complex exponential form depends on two complex-valued numbers: $a_k$ and $a_{-k}$.

• Each harmonic frequency in the trig form depends on two real-valued numbers: $c_k$ and $d_k$.

Q: What is going on?
A: The complex exponential form allows $f(t)$ to have complex values. The trigonometric form requires that $f(t)$ be real-valued.

Q: Isn’t it twice the work to compute both $a_k$ and $a_{-k}$?
A: Only if $f(t)$ is complex-valued.

If $f(t)$ is real-valued, then $a_{-k}$ is the complex conjugate of $a_k$. 

Is the Complex Exponential Form Actually Easier?

Last time, we determined the effect of a half-period shift on the Fourier coefficients of the trig form. The result was a bit complicated.

Assume that \( f(t) \) is periodic in time with period \( T \):

\[
f(t) = f(t + T).
\]

Let \( g(t) \) represent a version of \( f(t) \) shifted by half a period:

\[
g(t) = f(t - T/2).
\]

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients \( c_k \) are negated \( \times \)
- sine coefficients \( d_k \) are negated \( \times \)
- odd-numbered coefficients \( c_1, d_1, c_3, d_3, \ldots \) are negated \( \sqrt{ } \)
- sine and cosine coefficients are swapped: \( c_k \rightarrow d_k \) and \( d_k \rightarrow c_k \) \( \times \)
Halp-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t - T/2) = \sum_{k=0}^{\infty} c_k \cos(k\omega_o(t - T/2)) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o(t - T/2))$$

- **cosine basis functions**
  - $\cos(0t)$
  - $\cos(\omega_o t)$
  - $\cos(2\omega_o t)$
  - $\cdots$

- **delayed half a period**
  - $\cos(0t)$
  - $\cos(\omega_o t)$
  - $\cos(2\omega_o t)$
  - $\cdots$

Half-period shift inverts $c_k$ terms if $k$ is odd. It has no effect if $k$ is even.
Half-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

\[
f(t - T/2) = \sum_{k=0}^{\infty} c_k \cos(k\omega_0(t - T/2)) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0(t - T/2))
\]

Half-period shift inverts $d_k$ terms if and only if $k$ is odd.
Quarter-Period Shift

Shifting by $T/4$ is **even more complicated**.

\[ f(t - T/4) = \sum_{k=0}^{\infty} c_k \cos(k\omega_o(t - T/4)) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o(t - T/4)) \]

- **Cosine basis functions**
- **Delayed one fourth period**

\[
\begin{align*}
\cos(0t) & \rightarrow \sin(\omega_o t) \\
\cos(\omega_o t) & \rightarrow -\cos(2\omega_o t) \\
\cos(2\omega_o t) & \rightarrow -\sin(3\omega_o t) \\
\cos(3\omega_o t) & \rightarrow \sin(\omega_o t)
\end{align*}
\]
Check Yourself: Alternative (more intuitive) Approach

Shifting $f(t)$ shifts the underlying basis functions of its Fourier expansion.

$$f(t - T/4) = \sum_{k=0}^{\infty} c_k \cos(k\omega_0(t - T/4)) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0(t - T/4))$$

sine basis functions

$\sin(0t)$

$\sin(\omega_0 t)$

$\sin(2\omega_0 t)$

$\sin(3\omega_0 t)$

delayed 1/4 period

$\sin(0t)$

$\sin(\omega_0 t)$

$\sin(2\omega_0 t)$

$\sin(3\omega_0 t)$

$\sin(\omega_0 t) \rightarrow -\cos(\omega_0 t); \quad \sin(2\omega_0 t) \rightarrow -\sin(2\omega_0 t); \quad \sin(3\omega_0 t) \rightarrow \cos(3\omega_0 t)$
Summary of Shift Results

Let $c_k$ and $d_k$ represent the Fourier series coefficients for $f(t)$

\[
 f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)
\]

and $c'_k$ and $d'_k$ represent those for a half-period delay.

\[
 g(t) = f(t - T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_0 t)
\]

Then $c'_k = (-1)^k c_k$ and $d'_k = (-1)^k d_k$.

Let $c''_k$ and $d''_k$ represent those for a quarter-period delay.

\[
 g(t) = f(t - T/2) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_0 t)
\]

Then

\[
 c''_k = \begin{cases} 
 c_k & \text{if } k = 0, 4, 8, 12, \ldots \\
 d_k & \text{if } k = 1, 5, 9, 13, \ldots \\
 -c_k & \text{if } k = 2, 6, 10, 14, \ldots \\
 -d_k & \text{if } k = 3, 7, 11, 15, \ldots 
\end{cases} \quad \text{and} \quad
 d''_k = \begin{cases} 
 d_k & \text{if } k = 0, 4, 8, 12, \ldots \\
 -c_k & \text{if } k = 1, 5, 9, 13, \ldots \\
 -d_k & \text{if } k = 2, 6, 10, 14, \ldots \\
 c_k & \text{if } k = 3, 7, 11, 15, \ldots 
\end{cases}
\]
Other Shifts Yield Even More Complicated Results

Let $c_k$ and $d_k$ represent the Fourier series coefficients for $f(t)$

$$f(t) = f(t + T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0 t)$$

and $c_k'''$ and $d_k'''$ represent those for n eighth-period delay.

$$g(t) = f(t - T/8) = c_0 + \sum_{k=1}^{\infty} c'_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} d'_k \sin(k\omega_0 t)$$

$$c_k''' = \begin{cases} 
  c_k & \text{if } k = 0, 8, 16, 24, \ldots \\
  \frac{\sqrt{2}}{2}(c_k + d_k) & \text{if } k = 1, 9, 17, 25, \ldots \\
  d_k & \text{if } k = 2, 10, 18, 26, \ldots \\
  \frac{\sqrt{2}}{2}(-c_k + d_k) & \text{if } k = 3, 11, 19, 27, \ldots \\
  -c_k & \text{if } k = 4, 12, 20, 28, \ldots \\
  \frac{\sqrt{2}}{2}(-c_k - d_k) & \text{if } k = 5, 13, 21, 29, \ldots \\
  -d_k & \text{if } k = 6, 14, 22, 30, \ldots \\
  \frac{\sqrt{2}}{2}(c_k - d_k) & \text{if } k = 7, 15, 23, 31, \ldots 
\end{cases}$$

$$d_k''' = \ldots$$
Effects of Time Shifts on Complex Exponential Series

Delaying time by \( \tau \) multiplies the complex exponential coefficients of a Fourier series by a constant \( e^{-jk\omega_0 \tau} \).

Let \( a_k \) represent the complex exponential series coefficients of \( f(t) \) and \( a'_k \) represent the complex exponential series coefficients of \( g(t) = f(t - \tau) \).

\[
\begin{align*}
    a'_k &= \frac{1}{T} \int_T g(t)e^{-jk\omega_0 t} dt \\
         &= \frac{1}{T} \int_T f(t - \tau)e^{-jk\omega_0 t} dt \\
         &= \frac{1}{T} \int_T f(s)e^{-jk\omega_0 (s+\tau)} ds \\
         &= e^{-jk\omega_0 \tau} \frac{1}{T} \int_T f(s)e^{-jk\omega_0 s} ds \\
         &= e^{-jk\omega_0 \tau} a_k
\end{align*}
\]

Each coefficient \( a'_k \) in the series for \( g(t) \) is a constant \( e^{-jk\omega_0 t} \) times the corresponding coefficient \( a_k \) in the series for \( f(t) \).
Summary

We introduced the complex exponential form of Fourier series.

• complex numbers
• complex exponentials and their relation to sinusoids
• analysis and synthesis with complex exponentials
• delay property: much simpler with complex exponentials
Trig Table

\[
\begin{align*}
\sin(a+b) &= \sin(a) \cos(b) + \cos(a) \sin(b) \\
\sin(a-b) &= \sin(a) \cos(b) - \cos(a) \sin(b) \\
\cos(a+b) &= \cos(a) \cos(b) - \sin(a) \sin(b) \\
\cos(a-b) &= \cos(a) \cos(b) + \sin(a) \sin(b) \\
\tan(a+b) &= (\tan(a)+\tan(b))/(1-\tan(a) \tan(b)) \\
\tan(a-b) &= (\tan(a)-\tan(b))/(1+\tan(a) \tan(b)) \\
\sin(A) + \sin(B) &= 2 \sin((A+B)/2) \cos((A-B)/2) \\
\sin(A) - \sin(B) &= 2 \cos((A+B)/2) \sin((A-B)/2) \\
\cos(A) + \cos(B) &= 2 \cos((A+B)/2) \cos((A-B)/2) \\
\cos(A) - \cos(B) &= -2 \sin((A+B)/2) \sin((A-B)/2) \\
\sin(a+b) + \sin(a-b) &= 2 \sin(a) \cos(b) \\
\sin(a+b) - \sin(a-b) &= 2 \cos(a) \sin(b) \\
\cos(a+b) + \cos(a-b) &= 2 \cos(a) \cos(b) \\
\cos(a+b) - \cos(a-b) &= -2 \sin(a) \sin(b) \\
2 \cos(A) \cos(B) &= \cos(A-B) + \cos(A+B) \\
2 \sin(A) \sin(B) &= \cos(A-B) - \cos(A+B) \\
2 \sin(A) \cos(B) &= \sin(A+B) + \sin(A-B) \\
2 \cos(A) \sin(B) &= \sin(A+B) - \sin(A-B) 
\end{align*}
\]