6.003: Signal Processing

Fourier Series – Complex Exponential Form

- complex numbers
- complex exponentials and their relation to sinusoids
- complex exponential form of Fourier series
- delay property of Fourier series

September 16, 2021

Fourier Series

We have previously represented signals as weighted sums of sinusoids.

Synthesis Equation

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k \sin(k\omega_o t) \quad \text{where } \omega_o = \frac{2\pi}{T}$$

Analysis Equations

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt$$

$$d_k = \frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt$$

Today: Simplifying the math with complex numbers.

Simplifying Math By Using Complex Numbers

Complex numbers simplify thinking about roots of numbers / polynomials:

- all numbers have two square roots, three cube roots, etc.
- all polynomials of order n have n roots (some of which may be repeated).

→ much simpler than the rules that govern purely real-valued formulations. For example, a cubic equation with real-valued coefficients might have 1 or 3 real-valued roots; a quartic equation might have 0, 2, or 4.

Complex exponentials simplify thinking about trigonometric functions (Euler's formula, Leonhard Euler, 1748):

$$e^{j\theta} = \cos\theta + j\sin\theta$$

where $j = \sqrt{-1}$.

This single equation virtually eliminates our need for trig tables. Richard Feynman called this "the most remarkable formula in mathematics."

Note that we will normally use j (instead of i) to represent $\sqrt{-1}$.

Where Does Euler's Formula Come From?

Euler showed the relation between complex exponentials and sinusoids by solving the following differential equation two ways.

$$\frac{d^2 f(\theta)}{d\theta^2} + f(\theta) = 0$$

let $f_1(\theta) = A\cos(\alpha\theta) + B\sin(\beta\theta)$

$$\frac{df_1(\theta)}{d\theta} = -\alpha A\sin(\alpha\theta) + \beta B\cos(\beta\theta)$$

$$\frac{d^2 f_1(\theta)}{d\theta^2} = -\alpha^2 A\cos(\alpha\theta) - \beta^2 B\sin(\beta\theta)$$

$$\alpha^2 = \beta^2 = 1$$

$$f_1(\theta) = A\cos\theta + B'\sin\theta$$

Let $f_2(\theta) = Ce^{\gamma\theta}$

$$\frac{df_2(\theta)}{d\theta} = \gamma Ce^{\gamma\theta}$$

$$\frac{d^2 f_2(\theta)}{d\theta^2} = \gamma^2 Ce^{\gamma\theta}$$

$$\gamma^2 = -1$$

$$f_2(\theta) = Ce^{\pm j\theta}$$

If we arbitrarily take $f_2(\theta) = e^{j\theta}$, $f_2(0) = 1$ and $f_2'(0) = j$.
To make $f_1(\theta) = f_2(\theta)$, we must set $A = 1$ and $B' = j$.
It follows that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This argument presumes the existance of a constant j whose square is -1 and that can be manipulated as an ordinary algebraic constant.

Where Does Euler's Formula Come From?

Euler's formula also follows from Maclaurin expansion of the exponential function, assuming the j behaves like any other algebraic constant.

Start with the expansion of the real-valued function:

$$e^{\theta} = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} + \frac{\theta^6}{6!} + \frac{\theta^7}{7!} + \cdots$$

Assume that the same expansion holds for complex-valued arguments:

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \frac{j^5\theta^5}{5!} + \frac{j^6\theta^6}{6!} + \frac{j^7\theta^7}{7!} + \cdots \\ &= 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{j\theta^7}{7!} + \cdots \\ &= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta} \end{aligned}$$

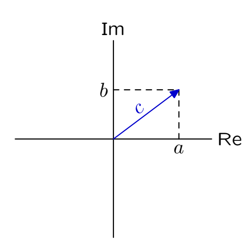
Euler's formula results by splitting the even and odd powers of θ.

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Geometric Interpretation

In 1799, Caspar Wessel was the first to describe complex numbers as points in the complex plane. Imaginary numbers had been in use since the 1500's.

$$c = a + jb$$



Algebraic Addition

Addition: the real part of a sum is the sum of the real parts, and the imaginary part of a sum is the sum of the imaginary parts.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1+a_2) + j(b_1+b_2)$$

Geometric Addition

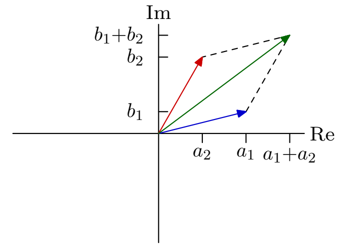
Rules for adding complex numbers are same as those for adding vectors.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then

$$c_1 + c_2 = (a_1 + jb_1) + (a_2 + jb_2) = (a_1+a_2) + j(b_1+b_2)$$



Algebraic Multiplication

Multiplication is more complicated.

Let c_1 and c_2 represent complex numbers:

$$c_1 = a_1 + jb_1$$
$$c_2 = a_2 + jb_2$$

Then

$$\begin{aligned} c_1 \times c_2 &= (a+jb) \times (c+jd) \\ &= a\times c + a\times jd + jb\times c + jb\times jd \\ &= (ac - bd) + j(ad + bc) \end{aligned}$$

Although the rules of algebra apply, the result is complicated:

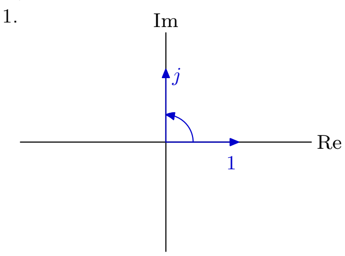
- the real part of a product is NOT the product of the real parts, and
- the imaginary part is NOT the product of the imaginary parts.

Geometric Multiplication

The two-dimensional view of complex numbers allows us to think about multiplication by an imaginary number as a **rotation**.

Multiplying by j

- **rotates 1 to j**,
- rotates j to -1,
- rotates -1 to $-j$, and
- rotates $-j$ to 1.

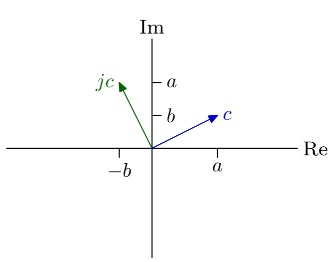


Multiplying by j rotates a vector by $\pi/2$.
Multiplying by $j^2 = -1$ rotates a vector by π.

Geometric Multiplication

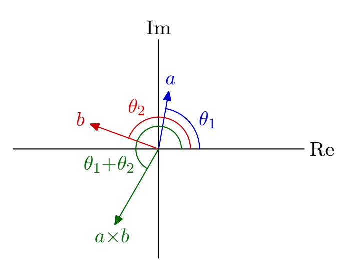
Multiplying by j rotates an arbitrary complex number by $\pi/2$.

$$c = a + jb$$
$$jc = ja - b$$



Geometric Approach: Polar Form

The magnitude of the product of complex numbers is the **product** of their magnitudes. The angle of a product is the **sum** of the angles.

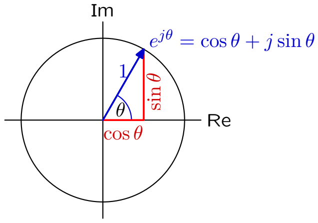


$$\begin{aligned} (r_1\angle\theta_1) \times (r_2\angle\theta_2) &= r_1(\cos\theta_1 + j\sin\theta_1) \times r_2(\cos\theta_2 + j\sin\theta_2) \\ &= r_1 r_2(\underbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{j\cos\theta_1\sin\theta_2 + j\sin\theta_1\cos\theta_2}_{j\sin(\theta_1+\theta_2)}) \\ &= r_1 r_2\angle(\theta_1 + \theta_2) \end{aligned}$$

Geometric Interpretation of Euler's Formula

Euler's formula equates polar and rectangular descriptions of a unit vector at angle θ.

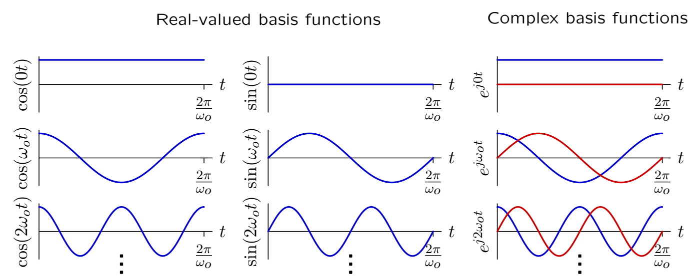
$$e^{j\theta} = \cos\theta + j\sin\theta$$



Simplifying Math By Using Complex Numbers

Euler's formula allows us to represent both sine and cosine basis functions with a single complex exponential:

$$f(t) = \sum \Big(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\Big) = \sum a_k e^{jk\omega_o t}$$



This halves the number of coefficients, but each is now complex-valued. More importantly, it replaces the trig functions with an exponential.

Converting From Trig Form To Complex Exponential Form

Assume that a function $f(t)$ can be written as a Fourier series in trig form.

$$f(t) = f(t+T) = c_0 + \sum_{k=1}^{\infty}\Big(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\Big)$$

We can use Euler's formula to convert sinusoids to complex exponentials.

$$e^{jk\omega_o t} = \cos(k\omega_o t) + j\sin(k\omega_o t)$$
$$\cos(k\omega_o t) = \mathrm{Re}\{e^{jk\omega_o t}\} = (e^{jk\omega_o t} + e^{-jk\omega_o t})/2$$
$$\sin(k\omega_o t) = \mathrm{Im}\{e^{jk\omega_o t}\} = -j(e^{jk\omega_o t} - e^{-jk\omega_o t})/2$$

$$\begin{aligned} f(t) &= c_0 + \frac{1}{2}\sum_{k=1}^{\infty}\Big(c_k e^{jk\omega_o t} + c_k e^{-jk\omega_o t} - jd_k e^{jk\omega_o t} + jd_k e^{-jk\omega_o t}\Big) \\ &= c_0 + \frac{1}{2}\sum_{k=1}^{\infty}(c_k - jd_k)e^{jk\omega_o t} + \frac{1}{2}\sum_{k=1}^{\infty}(c_k + jd_k)e^{-jk\omega_o t} \\ &= c_0 + \frac{1}{2}\sum_{k=1}^{\infty}(c_k - jd_k)e^{jk\omega_o t} + \frac{1}{2}\sum_{k=-1}^{-\infty}(c_{-k} + jd_{-k})e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} \end{aligned}$$

where $a_k = \begin{cases} (c_k - jd_k)/2 & \text{if } k > 0 \\ c_0 & \text{if } k = 0 \\ (c_{-k} + jd_{-k})/2 & \text{if } k < 0 \end{cases}$

Negative Frequencies

The complex form of a Fourier series has both positive and negative k's.

Only positive values of k are used in the trig form:

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k\cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k\sin(k\omega_o t)$$

but both positive and negative values of k are used in the exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

If we only included positive k in the previous sum, the result would always have an imaginary component (unless $a_k = 0$ for all k).

If $f(t)$ is real-valued (as it must be for the trig form), then the complex coefficients a_k are **conjugate symmetric**:

$$a_{-k} = a_k^*$$

where the $*$ denotes the complex conjugate.

k :	...	-3	-2	-1	0	1	2	3	...
a_k :	...	$\frac{c_3+jd_3}{2}$	$\frac{c_2+jd_2}{2}$	$\frac{c_1+jd_1}{2}$	c_0	$\frac{c_1-jd_1}{2}$	$\frac{c_2-jd_2}{2}$	$\frac{c_3-jd_3}{2}$	...

Fourier Series Directly From Complex Exponential Form

Assume that $f(t)$ is periodic in T and is composed of a weighted sum of harmonically related complex exponentials.

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}$$

We can "sift" out the component at $l\omega_o$ by multiplying both sides by $e^{-jl\omega_o t}$ and integrating over a period.

$$\int_T f(t)e^{-j\omega_o lt}dt = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt}e^{-j\omega_o lt}dt = \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_o(k-l)t}dt$$
$$= \begin{cases} Ta_l & \text{if } l = k \\ 0 & \text{otherwise} \end{cases}$$

Solving for a_l provides an explicit formula for the coefficients:

$$a_k = \frac{1}{T}\int_T f(t)e^{-j\omega_o kt}dt; \quad \text{where } \omega_o = \frac{2\pi}{T}.$$

This formulation works even if $f(t)$ has complex values.

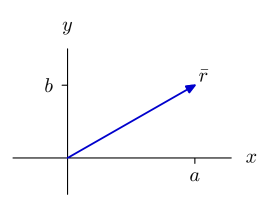
Orthogonality and Projection

Fourier components are separable because they are **orthogonal**.
Similar to separating a vector $\bar{r}$ into x and y components.
Since $\hat{x}$ and $\hat{y}$ are orthogonal, we can separate the x and y components of $\bar{r}$ by **projection**:

$$a = \bar{r}\cdot\hat{x}$$
$$b = \bar{r}\cdot\hat{y}$$

Then

$$\bar{r} = a\hat{x} + b\hat{y}$$



Orthogonal Decompositions

Vector representation: let $\bar{r}$ represent a vector with components a and b in the $\hat{x}$ and $\hat{y}$ directions, respectively.

$$a = \bar{r}\cdot\hat{x} \qquad b = \bar{r}\cdot\hat{y} \qquad \text{("analysis" equations)}$$

$$\bar{r} = a\hat{x} + b\hat{y} \qquad \text{("synthesis" equation)}$$

Fourier series: let $f(t)$ represent a signal with harmonic components $a_0, a_1, \ldots, a_k$ for harmonics $e^{j0t}, e^{j\frac{2\pi}{T}t}, \ldots, e^{j\frac{2\pi}{T}kt}$ respectively.

$$a_k = \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi}{T}kt}dt \qquad \text{("analysis" equation)}$$

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \qquad \text{("synthesis" equation)}$$

Orthogonal Decompositions

Integrating over a period **sifts** out the k^{th} component of the series.

Sifting as a dot product:

$$x = \bar{r}\cdot\hat{x} \equiv |\bar{r}||\hat{x}|\cos\theta$$

Sifting as an inner product:

$$a_k = e^{j\frac{2\pi}{T}kt}\cdot f(t) \equiv \frac{1}{T}\int_T f(t)e^{-j\frac{2\pi}{T}kt}dt$$

where

$$a(t)\cdot b(t) = \frac{1}{T}\int_T a^*(t)b(t)dt\,.$$

The complex conjugate $(^*)$ makes the inner product of the k^{th} and m^{th} components equal to 1 iff $k=m$:

$$\frac{1}{T}\int_T \left(e^{j\frac{2\pi}{T}kt}\right)^*\left(e^{j\frac{2\pi}{T}mt}\right)dt = \frac{1}{T}\int_T e^{-j\frac{2\pi}{T}kt}e^{j\frac{2\pi}{T}mt}dt = \begin{cases}1 & \text{if } k=m\\ 0 & \text{otherwise}\end{cases}$$

Check Yourself

How many of the following pairs of functions are orthogonal ($\perp$) in $T=3$?

1. $\cos 2\pi t \perp \sin 2\pi t$?
2. $\cos 2\pi t \perp \cos 4\pi t$?
3. $\cos 2\pi t \perp \sin \pi t$?
4. $\cos 2\pi t \perp e^{j2\pi t}$?

Fourier Series

Comparison of trigonometric and complex exponential forms.

Complex Exponential Form

$$f(t) = f(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T}\int_T f(t)e^{-jk\omega_o t}dt$$

Trigonometric Form

$$f(t) = f(t+T) = c_o + \sum_{k=1}^{\infty} c_k\cos(k\omega_o t) + \sum_{k=1}^{\infty} d_k\sin(k\omega_o t)$$

$$c_0 = \frac{1}{T}\int_T f(t)\,dt$$

$$c_k = \frac{2}{T}\int_T \cos(k\omega_o t)\,dt; \quad k = 1,2,3,\ldots$$

$$d_k = \frac{2}{T}\int_T \sin(k\omega_o t)\,dt; \quad k = 1,2,3,\ldots$$

Comparison of Trigonometric and Complex Exponential Forms

It seems as though it takes more numbers to characterize the complex exponential form:

- Each harmonic frequency in the complex exponential form depends on two complex-valued numbers: a_k and a_{-k}.
- Each harmonic frequency in the trig form depends on two real-valued numbers: c_k and d_k.

Q: What is going on?
A: The complex exponential form allows $f(t)$ to have complex values. The trigonometric form requires that $f(t)$ be real-valued.

Q: Isn't it twice the work to compute both a_k and a_{-k}?
A: Only if $f(t)$ is complex-valued. If $f(t)$ is real-valued, then a_{-k} is the complex conjugate of a_k.

Is the Complex Exponential Form Actually Easier?

Last time, we determined the effect of a half-period shift on the Fourier coefficients of the trig form. The result was a bit complicated.

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T)\,.$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t-T/2)\,.$$

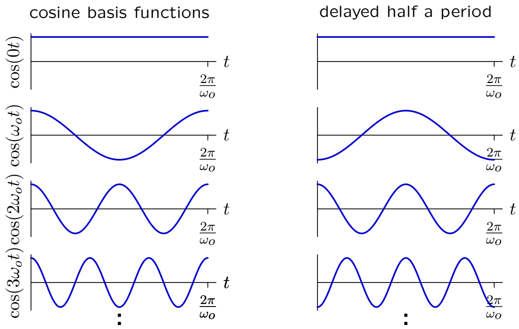
How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated ✗
- sine coefficients d_k are negated ✗
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated ✓
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$ ✗

Half-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2)=\sum_{k=0}^{\infty}c_k\cos\left(k\omega_o(t-T/2)\right)+\sum_{k=1}^{\infty}d_k\sin\left(k\omega_o(t-T/2)\right)$$

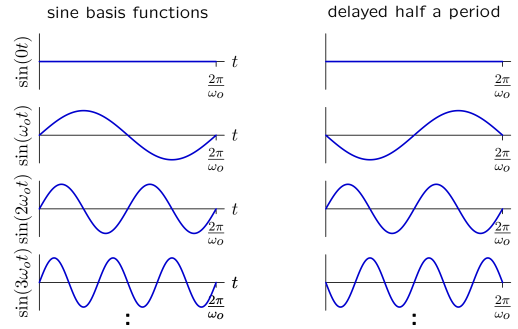


Half-period shift inverts c_k terms if k is odd. It has no effect if k is even.

Half-Period Shift

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/2)=\sum_{k=0}^{\infty}c_k\cos\left(k\omega_o(t-T/2)\right)+\sum_{k=1}^{\infty}d_k\sin\left(k\omega_o(t-T/2)\right)$$

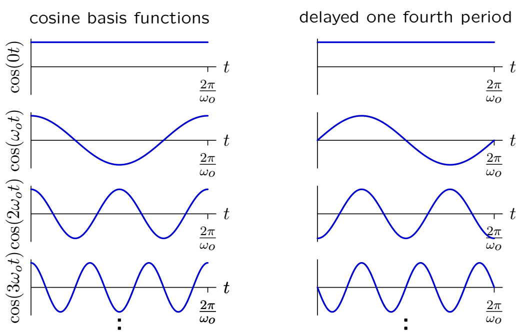


Half-period shift inverts d_k terms if and only if k is odd.

Quarter-Period Shift

Shifting by $T/4$ is **even more complicated**.

$$f(t-T/4)=\sum_{k=0}^{\infty}c_k\cos\left(k\omega_o(t-T/4)\right)+\sum_{k=1}^{\infty}d_k\sin\left(k\omega_o(t-T/4)\right)$$

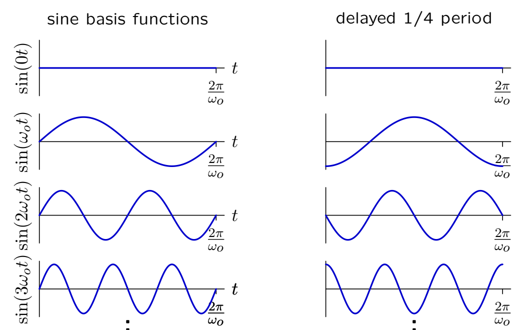


$\cos(\omega_o t)\to\sin(\omega_o t)$; $\cos(2\omega_o t)\to-\cos(2\omega_o t)$; $\cos(3\omega_o t)\to-\sin(3\omega_o t)$

Check Yourself: Alternative (more intuitive) Approach

Shifting $f(t)$ shifts the underlying basis functions of it Fourier expansion.

$$f(t-T/4)=\sum_{k=0}^{\infty}c_k\cos\left(k\omega_o(t-T/4)\right)+\sum_{k=1}^{\infty}d_k\sin\left(k\omega_o(t-T/4)\right)$$



$\sin(\omega_o t)\to-\cos(\omega_o t)$; $\sin(2\omega_o t)\to-\sin(2\omega_o t)$; $\sin(3\omega_o t)\to\cos(3\omega_o t)$

Summary of Shift Results

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t)=f(t+T)=c_0+\sum_{k=1}^{\infty}c_k\cos(k\omega_o t)+\sum_{k=1}^{\infty}d_k\sin(k\omega_o t)$$

and c_k' and d_k' represent those for a half-period delay.

$$g(t)=f(t-T/2)=c_0+\sum_{k=1}^{\infty}c_k'\cos(k\omega_o t)+\sum_{k=1}^{\infty}d_k'\sin(k\omega_o t)$$

Then $c_k'=(-1)^k c_k$ and $d_k'=(-1)^k d_k$.

Let c_k'' and d_k'' represent those for a quarter-period delay.

$$g(t)=f(t-T/2)=c_0+\sum_{k=1}^{\infty}c_k'\cos(k\omega_o t)+\sum_{k=1}^{\infty}d_k'\sin(k\omega_o t)$$

Then

$$c_k''=\begin{cases}c_k & \text{if } k=0,4,8,12,\ldots\\ d_k & \text{if } k=1,5,9,13,\ldots\\ -c_k & \text{if } k=2,6,10,14,\ldots\\ -d_k & \text{if } k=3,7,11,15,\ldots\end{cases}\qquad d_k''=\begin{cases}d_k & \text{if } k=0,4,8,12,\ldots\\ -c_k & \text{if } k=1,5,9,13,\ldots\\ -d_k & \text{if } k=2,6,10,14,\ldots\\ c_k & \text{if } k=3,7,11,15,\ldots\end{cases}$$

Other Shifts Yield Even More Complicated Results

Let c_k and d_k represent the Fourier series coefficients for $f(t)$

$$f(t)=f(t+T)=c_0+\sum_{k=1}^{\infty}c_k\cos(k\omega_o t)+\sum_{k=1}^{\infty}d_k\sin(k\omega_o t)$$

and c_k''' and d_k''' represent those for n eighth-period delay.

$$g(t)=f(t-T/8)=c_0+\sum_{k=1}^{\infty}c_k'\cos(k\omega_o t)+\sum_{k=1}^{\infty}d_k'\sin(k\omega_o t)$$

$$c_k'''=\begin{cases}c_k & \text{if } k=0,8,16,24,\ldots\\ \frac{\sqrt{2}}{2}(c_k+d_k) & \text{if } k=1,9,17,25,\ldots\\ d_k & \text{if } k=2,10,18,26,\ldots\\ \frac{\sqrt{2}}{2}(-c_k+d_k) & \text{if } k=3,11,19,27,\ldots\\ -c_k & \text{if } k=4,12,20,28,\ldots\\ \frac{\sqrt{2}}{2}(-c_k-d_k) & \text{if } k=5,13,21,29,\ldots\\ -d_k & \text{if } k=6,14,22,30,\ldots\\ \frac{\sqrt{2}}{2}(c_k-d_k) & \text{if } k=7,15,23,31,\ldots\end{cases}\qquad d_k'''=\ldots$$

Effects of Time Shifts on Complex Exponential Series

Delaying time by τ multiplies the complex exponential coefficients of a Fourier series by a constant $e^{-jk\omega_o\tau}$.

Let a_k represent the complex exponential series coefficients of $f(t)$ and a'_k represent the complex exponential series coefficents of $g(t) = f(t-\tau)$.

$$\begin{aligned}
a'_k &= \frac{1}{T}\int_T g(t)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(t-\tau)e^{-jk\omega_o t}dt \\
&= \frac{1}{T}\int_T f(s)e^{-jk\omega_o(s+\tau)}ds \\
&= e^{-jk\omega_o\tau}\frac{1}{T}\int_T f(s)e^{-jk\omega_o s}ds \\
&= e^{-jk\omega_o\tau}a_k
\end{aligned}$$

Each coefficient a'_k in the series for $g(t)$ is a constant $e^{-jk\omega_o t}$ times the corresponding coefficient a_k in the series for $f(t)$.

Summary

We introduced the complex exponential form of Fourier series.

- complex numbers
- complex exponentials and their relation to sinusoids
- analysis and synthesis with complex exponentials
- delay property: much simpler with complex exponentials