

# 6.003: Signal Processing

## Sinusoids and Series

- Relations between time and frequency.
- Fourier series for discontinuous functions.
- Fourier analysis of a vibrating string.

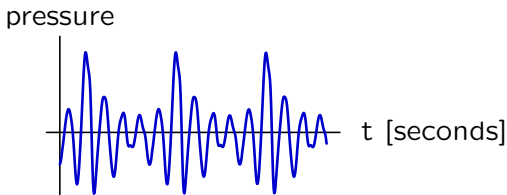
*September 14, 2021*

## Last Time

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Signals are functions that are used to convey information.

Example: a musical sound can be represented as a function of time.

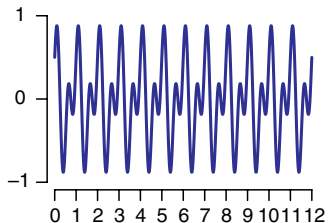


Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

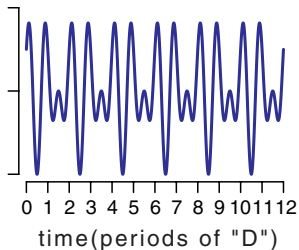
## Last Time

Time functions do a poor job of conveying consonance and dissonance.

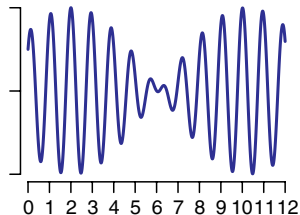
octave (D+D')



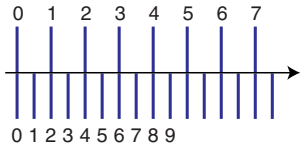
fifth (D+A)



D+E<sub>b</sub>

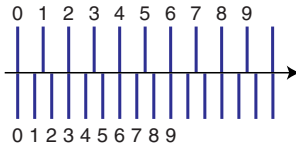


D'



D

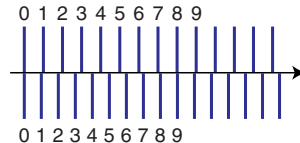
A



D

harmonics

E<sub>b</sub>



D

Harmonic structure conveys consonance and dissonance better.

## Last Time

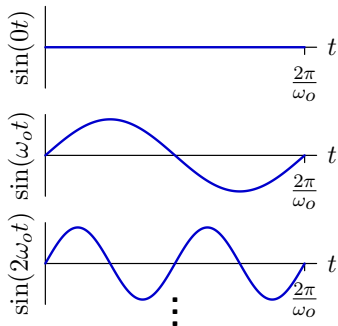
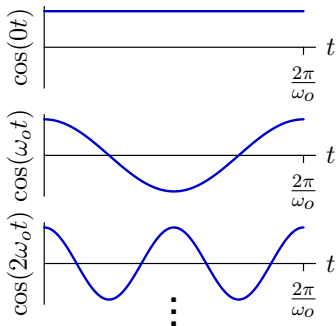
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**Fourier series** are sums of harmonically related sinusoids.

$$f(t) = \sum_{k=0}^{\infty} (c_k \cos(k\omega_o t) + d_k \sin(k\omega_o t))$$

where  $\omega_o = 2\pi/T$  represents the fundamental frequency.

Basis functions:



## Last Time

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How do we find the coefficients  $c_k$  and  $d_k$ ?

Key idea: Sift out the component of interest by

- multiplying by the corresponding basis function, and then
- integrating over a period.

This results in the following expressions for the Fourier series coefficients:

$$c_0 = \frac{1}{T} \int_T f(t) dt$$

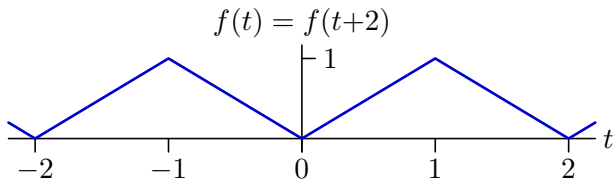
$$c_k = \frac{2}{T} \int_T f(t) \cos(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

$$d_k = \frac{2}{T} \int_T f(t) \sin(k\omega_0 t) dt; \quad k = 1, 2, 3, \dots$$

## Example of Analysis

---

Find the Fourier series coefficients for the following triangle wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \int_0^2 f(t) dt = \frac{1}{2}$$

$$c_k = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2\pi kt}{T} dt = 2 \int_0^1 t \cos(\pi kt) dt = \begin{cases} -\frac{4}{\pi^2 k^2} & k \text{ odd} \\ 0 & k = 2, 4, 6, \dots \end{cases}$$

$$d_k = 0 \quad (\text{by symmetry})$$

## Example of Synthesis

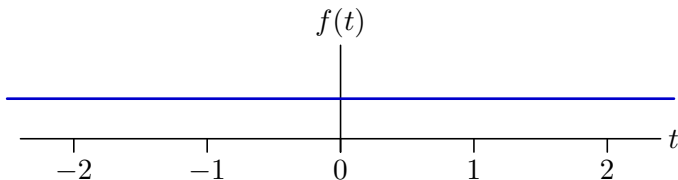
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Generate  $f(t)$  from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$

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## Example of Synthesis

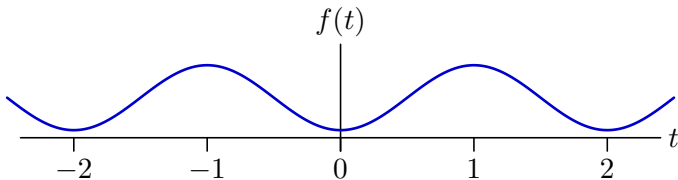
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## Example of Synthesis

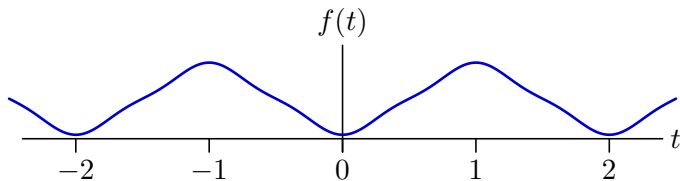
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## Example of Synthesis

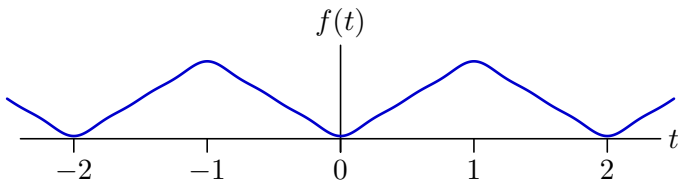
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## Example of Synthesis

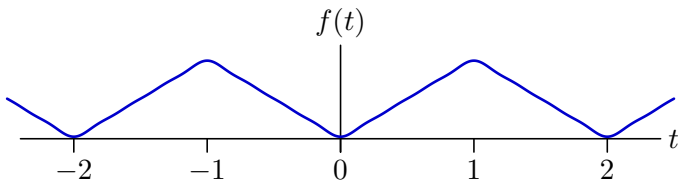
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## Example of Synthesis

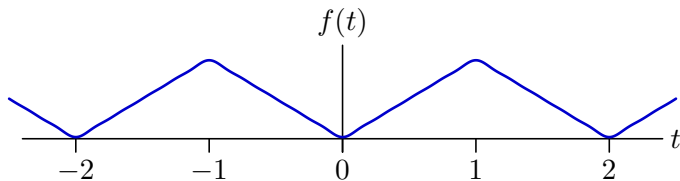
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## Example of Synthesis

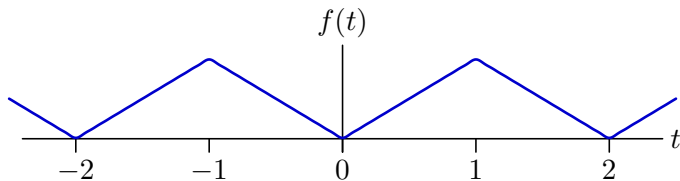
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## Example of Synthesis

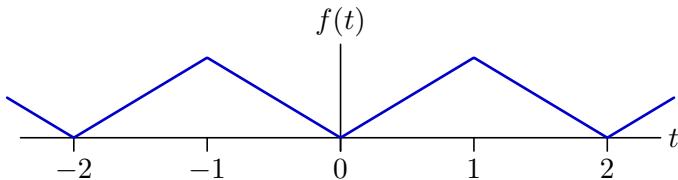
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$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2} \cos(k\pi t)$$



The synthesized function approaches original as number of terms increases.

## Fourier Synthesis

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The previous example illustrates that the sum of an infinite number of sinusoids can generate a piecewise linear function **with discontinuous slope!**

Note that **none** of the Fourier components had a slope discontinuity.

What if the function of interest has **discontinuous values?**

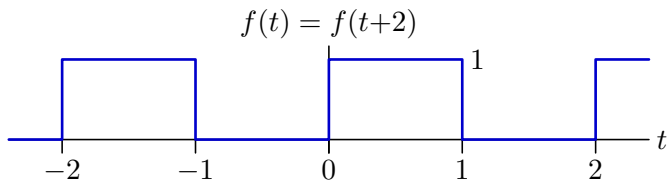
The question was put to Fourier, who claimed that discontinuities were not a problem.

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

## Fourier Analysis of a Square Wave

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Find the Fourier series coefficients for the following square wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2} \int_0^2 f(t) dt = \frac{1}{2}$$

$$c_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega_o t) dt = \int_0^1 \cos(k\pi t) dt = \left. \frac{\sin(k\pi t)}{k\pi} \right|_0^1 = 0 \text{ for } k = 1, 2, 3, \dots$$

$$d_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega_o t) dt = \int_0^1 \sin(k\pi t) dt = - \left. \frac{\cos(k\pi t)}{k\pi} \right|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \dots \\ 0 & \text{otherwise} \end{cases}$$



## Fourier Synthesis of a Square Wave

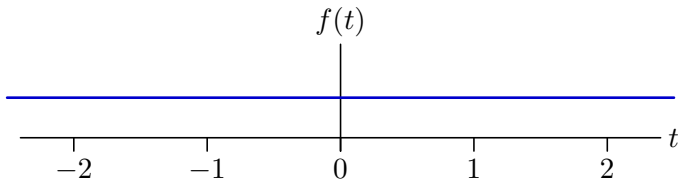
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$$f(t) = c_0 + \sum_{k=1}^{\infty} (c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t)) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

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## Fourier Synthesis of a Square Wave

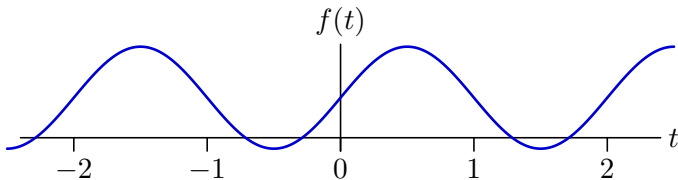
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## Fourier Synthesis of a Square Wave

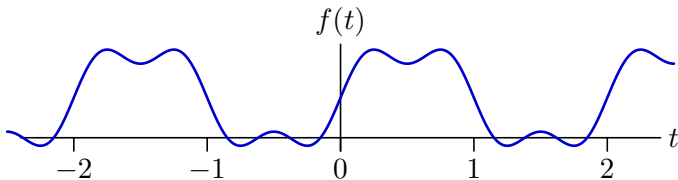
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## Fourier Synthesis of a Square Wave

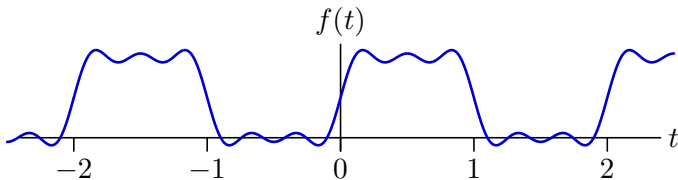
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## Fourier Synthesis of a Square Wave

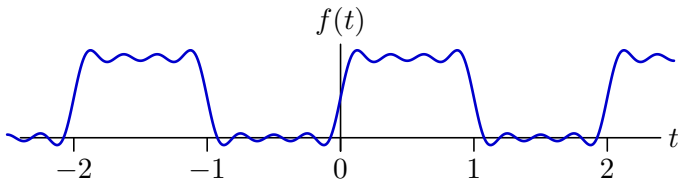
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## Fourier Synthesis of a Square Wave

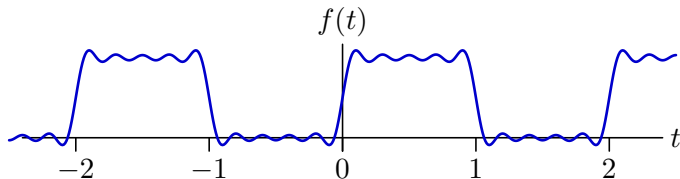
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## Fourier Synthesis of a Square Wave

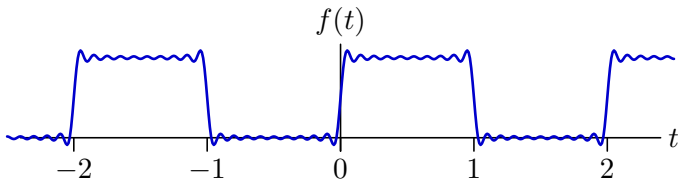
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## Fourier Synthesis of a Square Wave

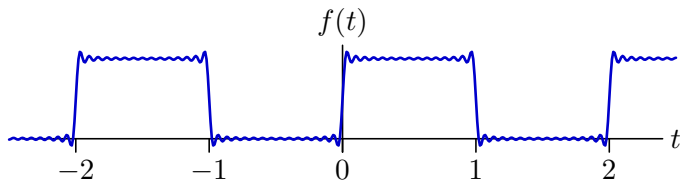
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## Fourier Synthesis of a Square Wave

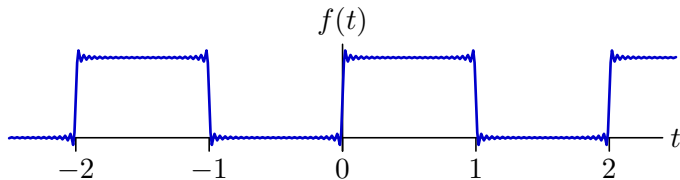
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## Fourier Synthesis of a Square Wave

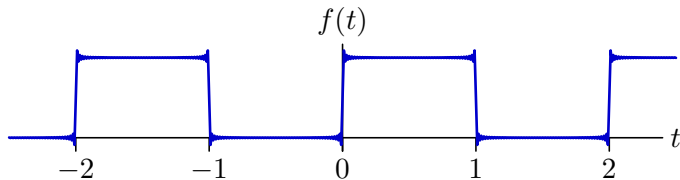
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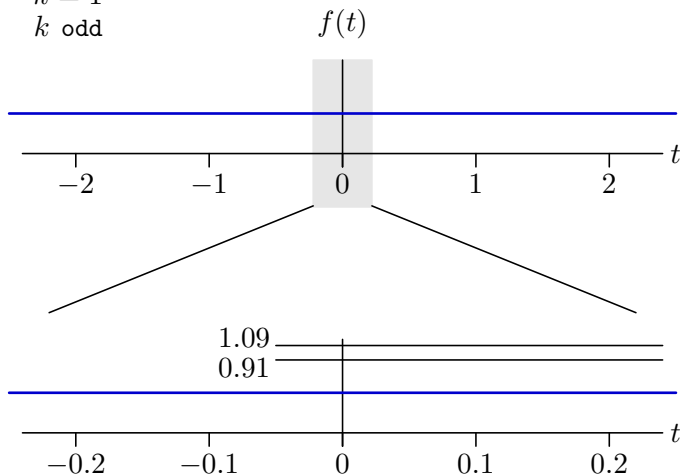
The synthesized function approaches original as number of terms increases.

## Fourier Synthesis of a Square Wave

---

Zoom in on the step discontinuity at  $t = 0$ .

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi} \sin(k\pi t)$$

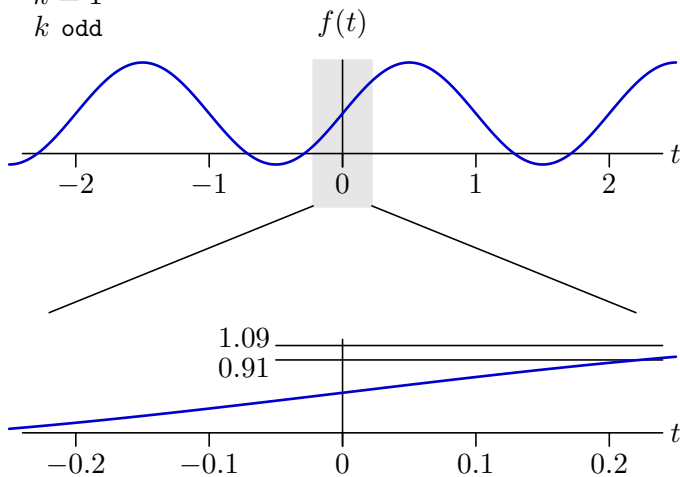


## Fourier Synthesis of a Square Wave

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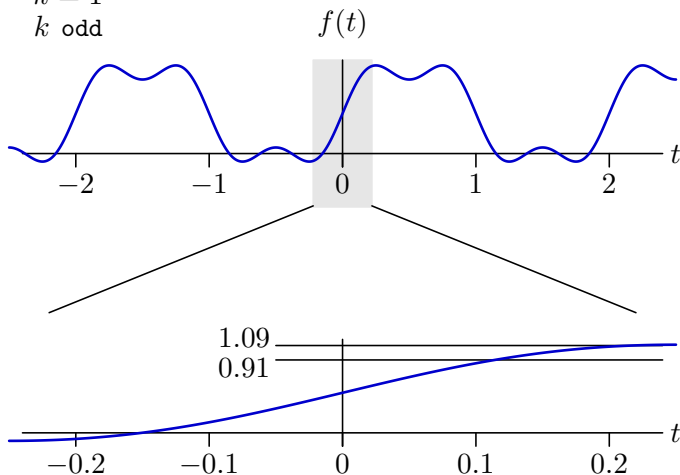
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## Fourier Synthesis of a Square Wave

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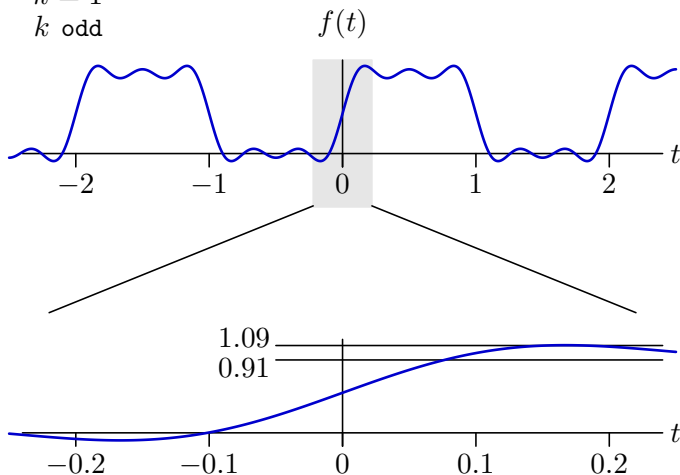
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## Fourier Synthesis of a Square Wave

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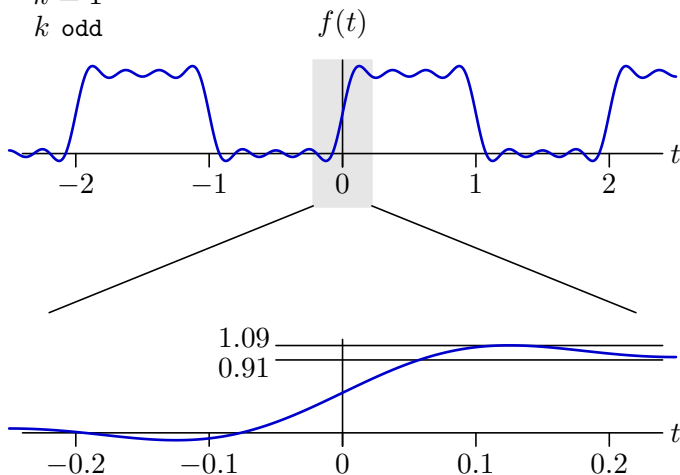
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## Fourier Synthesis of a Square Wave

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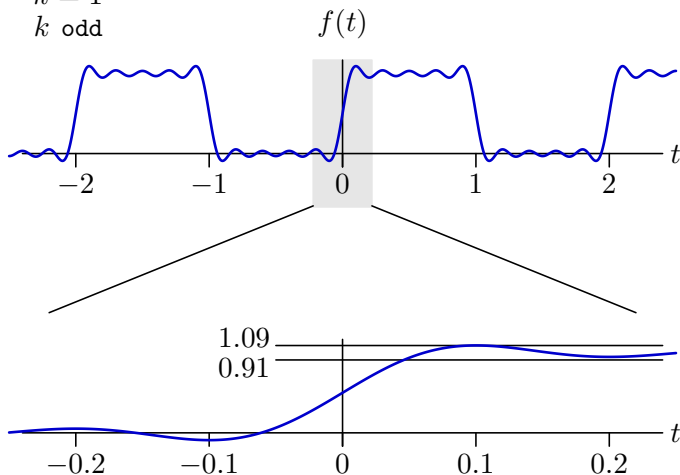
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## Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at  $t = 0$ .

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^9 \frac{2}{k\pi} \sin(k\pi t)$$

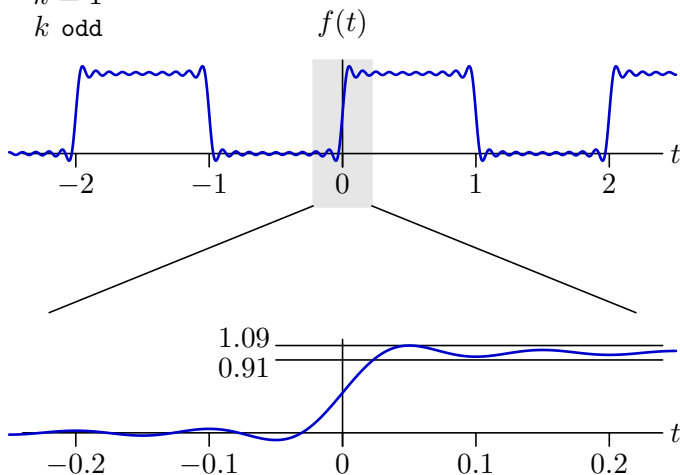




## Fourier Synthesis of a Square Wave

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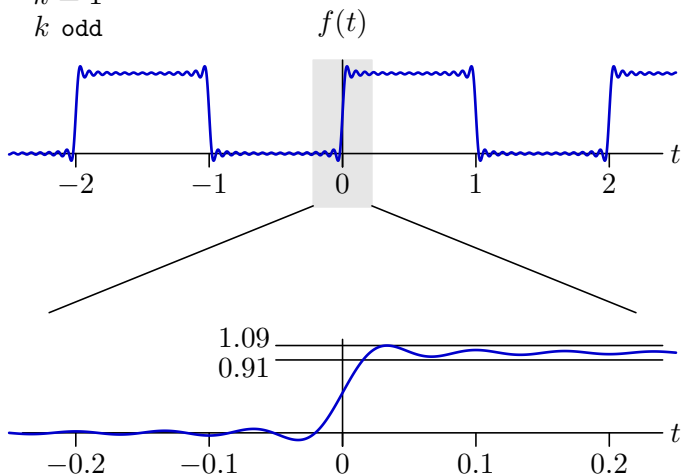
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## Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at  $t = 0$ .

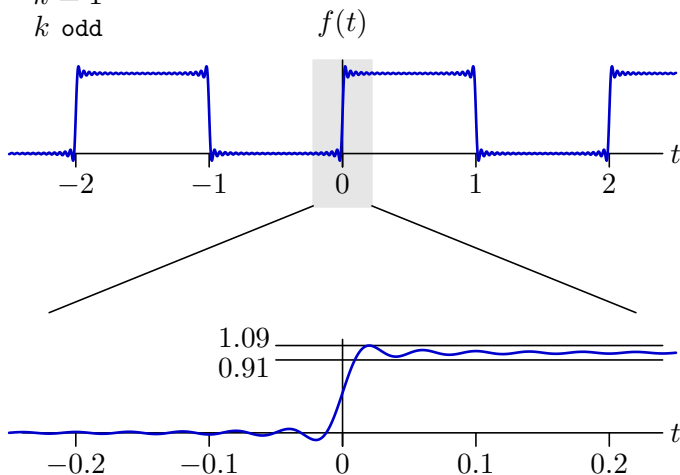
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{29} \frac{2}{k\pi} \sin(k\pi t)$$



## Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at  $t = 0$ .

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{49} \frac{2}{k\pi} \sin(k\pi t)$$

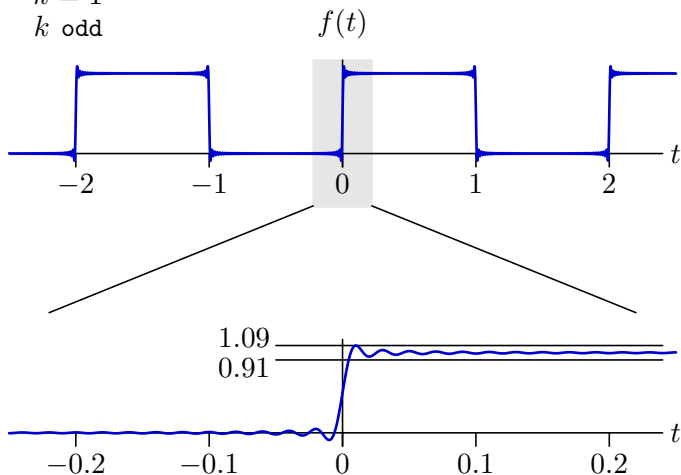


## Fourier Synthesis of a Square Wave

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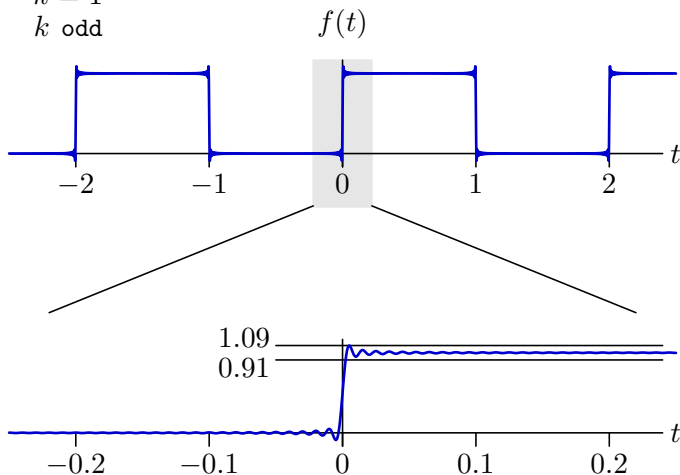
$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi} \sin(k\pi t)$$



## Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at  $t = 0$ .

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi} \sin(k\pi t)$$



Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

## Convergence of Fourier Series

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If there is a step discontinuity in  $f(t)$  at  $t = t_0$ , then the Fourier series for  $f(t_0)$  converges to the average of the limits of  $f(t)$  as  $t$  approaches  $t_0$  from the left and from the right.

Let  $f_K(t)$  represent the partial sum of the Fourier series using just  $N$  terms:

$$f_K(t) = a_0 + \sum_{k=0}^K \left( c_k \cos(k\omega_0 t) + d_k \sin(k\omega_0 t) \right)$$

As  $K \rightarrow \infty$ ,

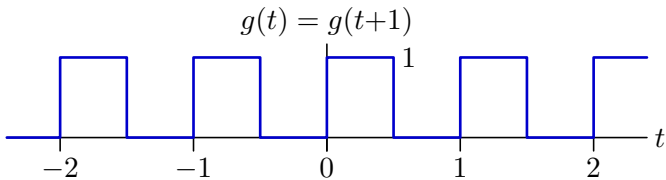
- the maximum difference between  $f(t)$  and  $f_K(t)$  converges to  $\approx 9\%$  of  $|f(t_0^+) - f(t_0^-)|$  and
- the region over which the absolute value of the difference exceeds any small number  $\epsilon$  shrinks to zero.

We refer to this characteristic type of overshoot as **Gibb's Phenomenon**.

## Properties of Fourier Series: Scaling Time

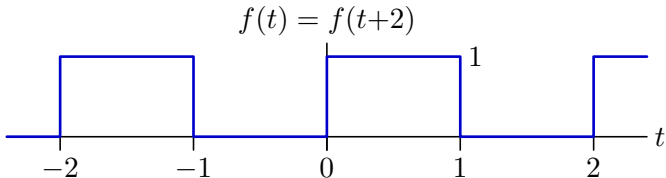
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Find the Fourier series coefficients for the following square wave:



We could repeat the process used to find the Fourier coefficients for  $f(t)$ . Alternatively, we can take advantage of the relation between  $f(t)$  and  $g(t)$ :

$$g(t) = f(2t)$$



## Scaling Time

---

We already know the Fourier series expansion of  $f(t)$ :

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi} \sin(k\pi t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi} \sin(k\omega_0 t)$$

Thus  $c_0 = \frac{1}{2}$ ,  $d_k = 0$ , and

$$c_k = \begin{cases} \frac{1}{k\pi} & k = 1, 3, 5, \dots \\ 0 & \text{otherwise} \end{cases}$$

where  $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$ .

Since  $g(t) = f(2t)$  it follows that

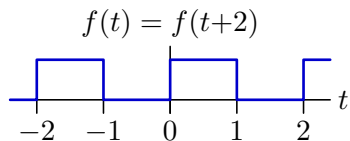
$$g(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi} \sin(k\pi 2t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi} \sin(k\omega_1 t)$$

The Fourier series coefficients for  $g(t)$  are thus identical to those of  $f(t)$ . Only the fundamental frequency has changed, from  $\omega_0 = \pi$  to  $\omega_1 = 2\pi$ .

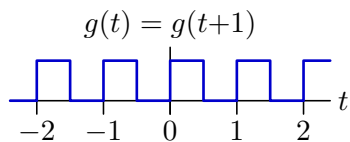
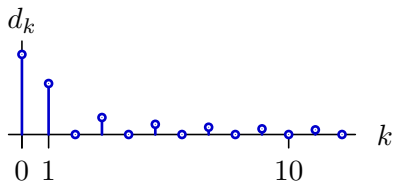


## Scaling Time

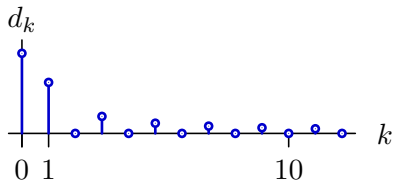
Compare the Fourier series coefficients of  $f(t)$  and  $g(t) = f(2t)$ .



$\leftrightarrow$



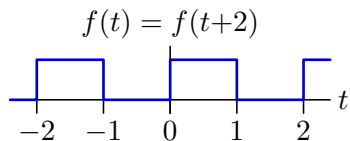
$\leftrightarrow$



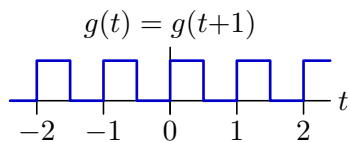
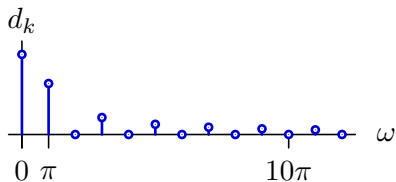
Only the fundamental frequency has changed.

## Scaling Time

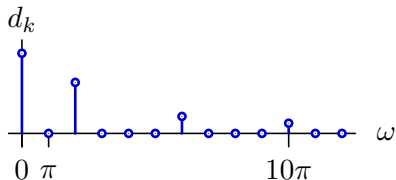
Plot the Fourier series coefficients on a frequency scale.



$\leftrightarrow$



$\leftrightarrow$



Compressing the time axis has stretched the frequency axis.

## Check Yourself

---

Assume that  $f(t)$  is periodic in time with period  $T$ :

$$f(t) = f(t + T).$$

Let  $g(t)$  represent a version of  $f(t)$  shifted by half a period:

$$g(t) = f(t - T/2).$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients  $c_k$  are negated
- sine coefficients  $d_k$  are negated
- odd-numbered coefficients  $c_1, d_1, c_3, d_3, \dots$  are negated
- sine and cosine coefficients are swapped:  $c_k \rightarrow d_k$  and  $d_k \rightarrow c_k$

## Check Yourself

---

Let  $c_k$  and  $c'_k$  represent the cosine coefficients of  $f(t)$  and  $g(t)$  respectively.

$$c_k = \frac{2}{T} \int_0^T f(t) \cos(k\omega_o t) dt$$

$$c'_k = \frac{2}{T} \int_0^T g(t) \cos(k\omega_o t) dt$$

$$= \frac{2}{T} \int_0^T f(t - T/2) \cos(k\omega_o t) dt \quad | \quad g(t) = f(t - T/2)$$

$$= \frac{2}{T} \int_0^T f(s) \cos(k\omega_o(s + T/2)) ds \quad | \quad s = t - T/2$$

$$= \frac{2}{T} \int_0^T f(s) \cos(k\omega_o s + k\omega_o T/2) ds \quad | \quad \text{distribute } k\omega_o \text{ over sum}$$

$$= \frac{2}{T} \int_0^T f(s) \cos(k\omega_o s + k\pi) ds \quad | \quad \omega_o = 2\pi/T$$

$$= \frac{2}{T} \int_0^T f(s) \cos(k\omega_o s) (-1)^k ds \quad | \quad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$= (-1)^k c_k \quad | \quad \text{pull } (-1)^k \text{ outside integral}$$

## Check Yourself

---

Let  $d_k$  and  $d'_k$  represent the sine coefficients of  $f(t)$  and  $g(t)$  respectively.

$$d_k = \frac{2}{T} \int_0^T f(t) \sin(k\omega_0 t) dt$$

$$d'_k = \frac{2}{T} \int_0^T g(t) \sin(k\omega_0 t) dt$$

$$= \frac{2}{T} \int_0^T f(t - T/2) \sin(k\omega_0 t) dt \quad | \quad g(t) = f(t - T/2)$$

$$= \frac{2}{T} \int_0^T f(s) \sin(k\omega_0(s + T/2)) ds \quad | \quad s = t - T/2$$

$$= \frac{2}{T} \int_0^T f(s) \sin(k\omega_0 s + k\omega_0 T/2) ds \quad | \quad \text{distribute } k\omega_0 \text{ over sum}$$

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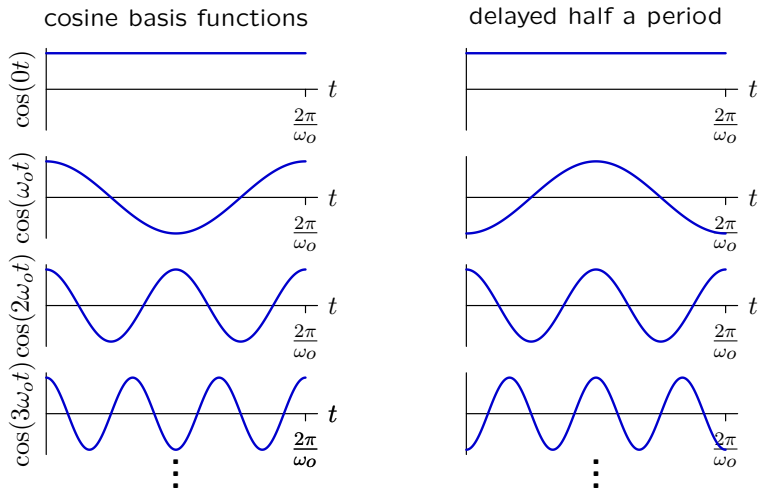
$$= \frac{2}{T} \int_0^T f(s) \sin(k\omega_0 s) (-1)^k ds \quad | \quad \sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$= (-1)^k d_k \quad | \quad \text{pull } (-1)^k \text{ outside integral}$$

## Check Yourself: Alternative (more intuitive) Approach

Shifting  $f(t)$  shifts the underlying basis functions of its Fourier expansion.

$$f(t - T/2) = \sum_{k=0}^{\infty} c_k \cos(k\omega_0(t - T/2)) + \sum_{k=1}^{\infty} d_k \sin(k\omega_0(t - T/2))$$

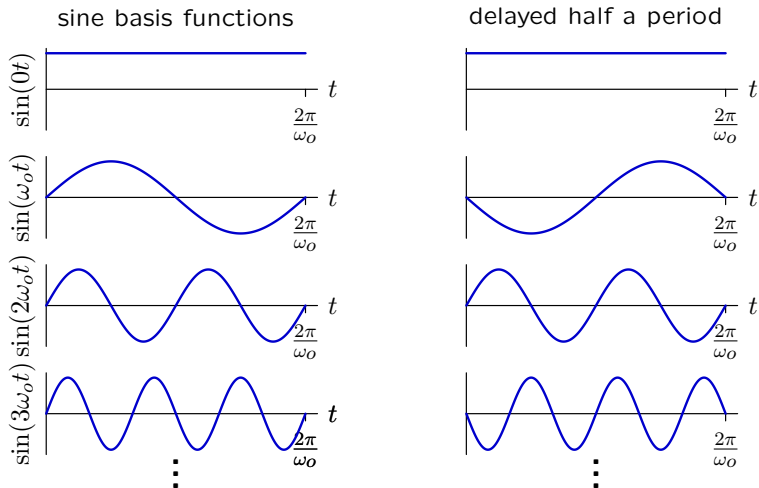


Half-period shift inverts odd harmonics. No effect on even harmonics.

## Check Yourself: Alternative (more intuitive) Approach

Shifting  $f(t)$  shifts the underlying basis functions of its Fourier expansion.

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Half-period shift inverts odd harmonics. No effect on even harmonics.

## Check Yourself

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Assume that  $f(t)$  is periodic in time with period  $T$ :

$$f(t) = f(t + T).$$

Let  $g(t)$  represent a version of  $f(t)$  shifted by half a period:

$$g(t) = f(t - T/2).$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients? **1**

- cosine coefficients  $c_k$  are negated ✗
- sine coefficients  $d_k$  are negated ✗
- odd-numbered coefficients  $c_1, d_1, c_3, d_3, \dots$  are negated ✓
- sine and cosine coefficients are swapped:  $c_k \rightarrow d_k$  and  $d_k \rightarrow c_k$  ✗



## Why Focus on Fourier Series?

---

What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.

→ harmonically related sinusoids are **orthogonal** to each other over  $[0, T]$ .

**Orthogonality:**  $f(t)$  and  $g(t)$  are orthogonal over  $0 \leq t \leq T$  if

$$\int_T f(t)g(t) dt = 0$$

Example: Calculate this integral for the  $k^{\text{th}}$  and  $l^{\text{th}}$  harmonics of  $\cos(\omega_o t)$ .

$$\int_T \cos(k\omega_o t) \cos(l\omega_o t) dt$$

We can use trigonometry to express the product of the two cosines as the sum of cosines at the sum and difference frequencies:

$$\int_T \left( \frac{1}{2} \cos((k+l)\omega_o t) + \frac{1}{2} \cos((k-l)\omega_o t) \right) dt$$

The sum and difference frequencies are also harmonics of  $\omega_o$ , so their integral over  $2\pi/T$  is zero (provided  $k \neq l$ ).

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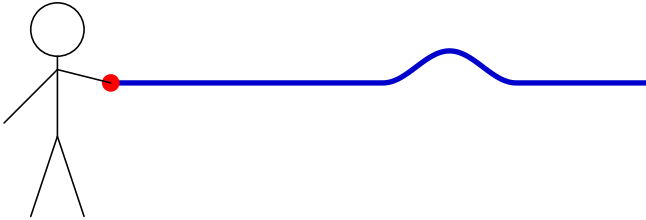
→ harmonically related sinusoids are **orthogonal** to each other over  $[0, T]$ .

Sines and cosines also play important roles in **physics** – especially the physics of waves.

## Physical Example: Vibrating String

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A taut string supports wave motion.

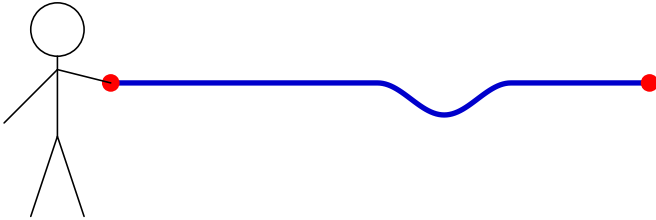


The speed of the wave depends on the tension on and mass of the string.

## Physical Example: Vibrating String

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The wave will reflect off a rigid boundary.

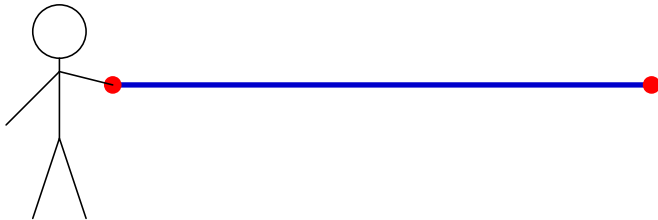


The amplitude of the reflected wave is opposite that of the incident wave.

## Physical Example: Vibrating String

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Reflections can interfere with excitations.

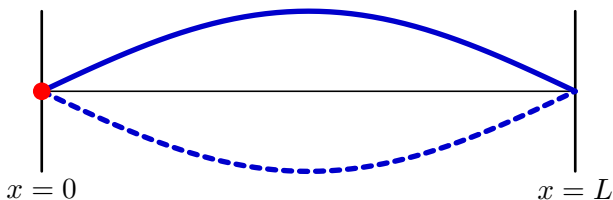


The interference can be constructive or destructive depending on the frequency of the excitation.

## Physical Example: Vibrating String

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We get constructive interference if round-trip travel time equals the period.



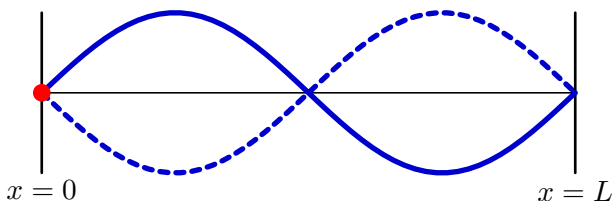
$$\text{Round-trip travel time} = \frac{2L}{v} = T$$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

## Physical Example: Vibrating String

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We also get constructive interference if round-trip travel time is  $2T$ .



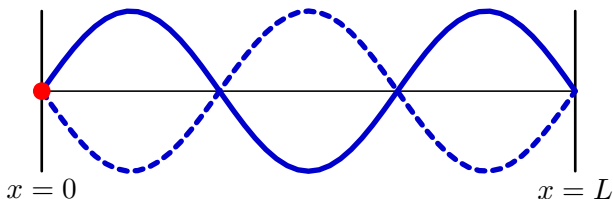
$$\text{Round-trip travel time} = \frac{2L}{v} = 2T$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

## Physical Example: Vibrating String

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In fact, we also get constructive interference if round-trip travel time is  $kT$ .



$$\text{Round-trip travel time} = \frac{2L}{v} = kT$$

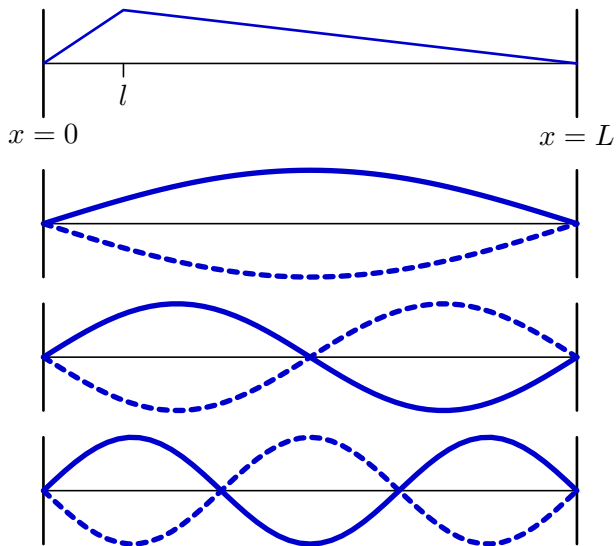
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

Only certain frequencies (harmonics of  $\omega_o = \pi v/L$ ) persist.  
This is the basis of stringed instruments.



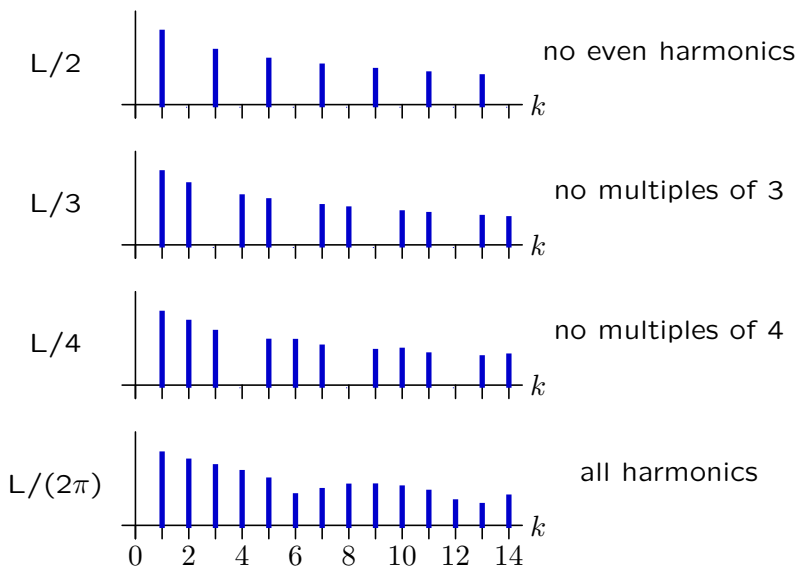
## Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is “plucked” at  $x = l$ .



## Physical Example: Vibrating String

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Differences in harmonic structure generate differences in timbre.

## Summary

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- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples  
→ Gibb's phenomenon.
- We investigated several **properties** of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are **many** others
- We saw how Fourier series are useful for modeling a vibrating string.