6.003: Signal Processing

Sinusoids and Series

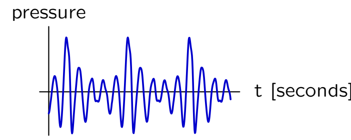
- Relations between time and frequency.
- Fourier series for discontinuous functions.
- Fourier analysis of a vibrating string.

September 14, 2021

Last Time

Signals are functions that are used to convey information.

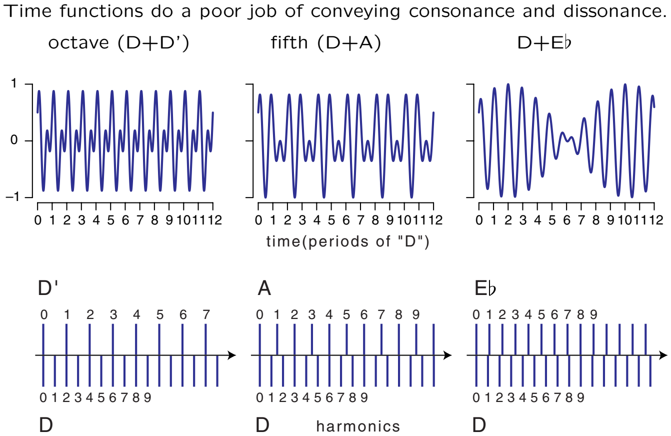
Example: a musical sound can be represented as a function of time.



Although this time function is a complete description of the sound, it does not expose many of the important properties of the sound.

Last Time

Time functions do a poor job of conveying consonance and dissonance.



Harmonic structure conveys consonance and dissonance better.

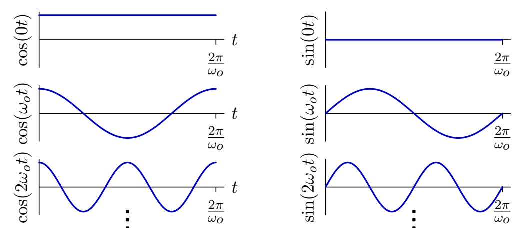
Last Time

Fourier series are sums of harmonically related sinusoids.

$$f(t)=\sum_{k=0}^{\infty}\left(c_k\cos(k\omega_o t)+d_k\sin(k\omega_o t)\right)$$

where $\omega_o=2\pi/T$ represents the fundamental frequency.

Basis functions:



Last Time

How do we find the coefficients c_k and d_k?

Key idea: Sift out the component of interest by

- multiplying by the corresponding basis function, and then
- integrating over a period.

This results in the following expressions for the Fourier series coefficients:

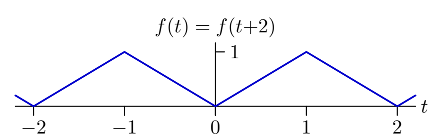
$$c_0=\frac{1}{T}\int_T f(t)\,dt$$

$$c_k=\frac{2}{T}\int_T f(t)\cos(k\omega_o t)\,dt;\;\; k=1,2,3,\ldots$$

$$d_k=\frac{2}{T}\int_T f(t)\sin(k\omega_o t)\,dt;\;\; k=1,2,3,\ldots$$

Example of Analysis

Find the Fourier series coefficients for the following triangle wave:



$$T=2$$

$$\omega_o=\frac{2\pi}{T}=\pi$$

$$c_0=\frac{1}{T}\int_0^T f(t)\,dt=\frac{1}{2}\int_0^2 f(t)dt=\frac{1}{2}$$

$$c_k=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}\,dt=2\int_0^1 t\cos(\pi kt)\,dt=\begin{cases}-\frac{4}{\pi^2k^2} & k \text{ odd}\\ 0 & k=2,4,6,\ldots\end{cases}$$

$d_k=0$ (by symmetry)

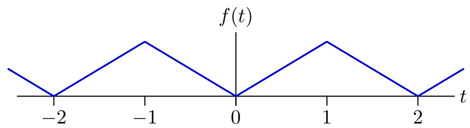
Example of Synthesis

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\right) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{\pi^2 k^2}\cos(k\pi t)$$

$$f(t) = \frac{1}{2} - \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{4}{\pi^2 k^2}\cos(k\pi t)$$



The synthesized function approaches original as number of terms increases.

Fourier Synthesis

The previous example illustrates that the sum of an infinite number of sinusoids can generate a piecewise linear function **with discontinuous slope**!

Note that **none** of the Fourier components had a slope discontinuity.

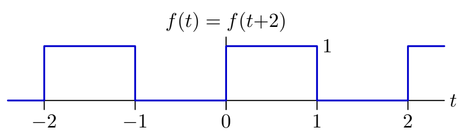
What if the function of interest has **discontinuous values**?

The question was put to Fourier, who claimed that discontinuities were not a problem.

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

Fourier Analysis of a Square Wave

Find the Fourier series coefficients for the following square wave:



$$T = 2$$

$$\omega_o = \frac{2\pi}{T} = \pi$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}$$

$$c_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_o t)\,dt = \int_0^1 \cos(k\pi t)dt = \frac{\sin(k\pi t)}{k\pi}\bigg|_0^1 = 0 \text{ for } k = 1, 2, 3, \ldots$$

$$d_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_o t)\,dt = \int_0^1 \sin(k\pi t)dt = -\frac{\cos(k\pi t)}{k\pi}\bigg|_0^1 = \begin{cases} \frac{2}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

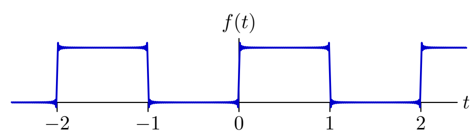
Fourier Synthesis of a Square Wave

Generate $f(t)$ from the Fourier coefficients in the previous slide.

Start with the Fourier coefficients

$$f(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\right) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{2}{k\pi}\sin(k\pi t)$$

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{99} \frac{2}{k\pi}\sin(k\pi t)$$

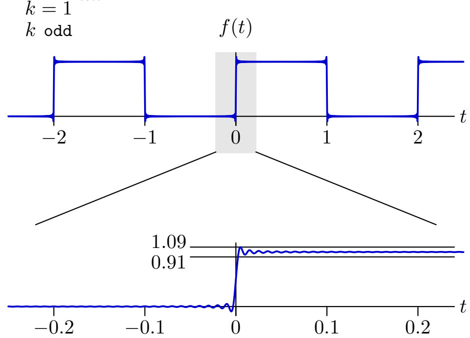


The synthesized function approaches original as number of terms increases.

Fourier Synthesis of a Square Wave

Zoom in on the step discontinuity at $t = 0$.

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{199} \frac{2}{k\pi}\sin(k\pi t)$$



Increasing the number of terms does not decrease the peak overshoot, but it does shrink the region of time that is occupied by the overshoot.

Convergence of Fourier Series

If there is a step discontinuity in $f(t)$ at $t = t_0$, then the Fourier series for $f(t_0)$ converges to the average of the limits of $f(t)$ as t approaches t_0 from the left and from the right.

Let $f_K(t)$ represent the partial sum of the Fourier series using just N terms:

$$f_K(t) = a_0 + \sum_{k=0}^{K}\Big(c_k\cos(k\omega_o t) + d_k\sin(k\omega_o t)\Big)$$

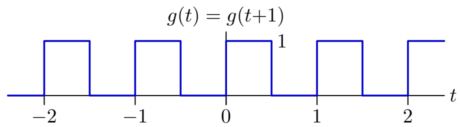
As $K \to \infty$,

- the maximum difference between $f(t)$ and $f_K(t)$ converges to $\approx 9\%$ of $|f(t_0^+) - f(t_0^-)|$ and
- the region over which the absolute value of the difference exceeds any small number ϵ shrinks to zero.

We refer to this characteristic type of overshoot as **Gibb's Phenomenon**.

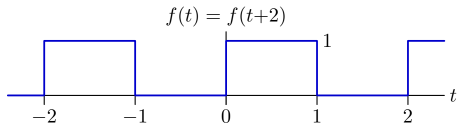
Properties of Fourier Series: Scaling Time

Find the Fourier series coefficients for the following square wave:



We could repeat the process used to find the Fourier coefficients for $f(t)$. Alternatively, we can take advantage of the relation between $f(t)$ and $g(t)$:

$g(t) = f(2t)$



Scaling Time

We already know the Fourier series expansion of $f(t)$:

$$f(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi}\sin(k\pi t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi}\sin(k\omega_o t)$$

Thus $c_0 = \frac{1}{2}$, $d_k = 0$, and

$$c_k = \begin{cases} \frac{1}{k\pi} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$.

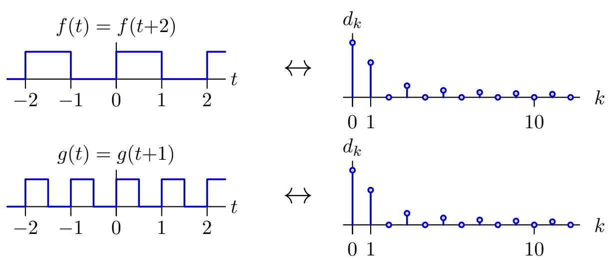
Since $g(t) = f(2t)$ it follows that

$$g(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi}\sin(k\pi 2t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi}\sin(k\omega_1 t)$$

The Fourier series coefficients for $g(t)$ are thus identical to those of $f(t)$. Only the fundamental frequency has changed, from $\omega_o = \pi$ to $\omega_1 = 2\pi$.

Scaling Time

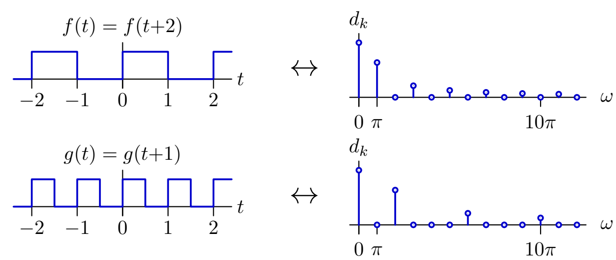
Compare the Fourier series coefficients of $f(t)$ and $g(t) = f(2t)$.



Only the fundamental frequency has changed.

Scaling Time

Plot the Fourier series coefficients on a frequency scale.



Compressing the time axis has stretched the frequency axis.

Check Yourself

Assume that $f(t)$ is periodic in time with period T:

$$f(t) = f(t+T).$$

Let $g(t)$ represent a version of $f(t)$ shifted by half a period:

$$g(t) = f(t - T/2).$$

How many of the following statements correctly describe the effect of this shift on the Fourier series coefficients.

- cosine coefficients c_k are negated
- sine coefficients d_k are negated
- odd-numbered coefficients $c_1, d_1, c_3, d_3, \ldots$ are negated
- sine and cosine coefficients are swapped: $c_k \to d_k$ and $d_k \to c_k$

Why Focus on Fourier Series?

What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
$\rightarrow$ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Orthogonality: $f(t)$ and $g(t)$ are orthogonal over $0 \le t \le T$ if

$$\int_T f(t)g(t)\,dt = 0$$

Example: Calculate this integral for the k^{th} and l^{th} harmonics of $\cos(\omega_o t)$.

$$\int_T \cos(k\omega_o t)\cos(l\omega_o t)\,dt$$

We can use trigonometry to express the product of the two cosines as the sum of cosines at the sum and difference frequencies:

$$\int_T \left(\frac{1}{2}\cos((k+l)\omega_o t) + \frac{1}{2}\cos((k-l)\omega_o t)\right) dt$$

The sum and difference frequencies are also harmonics of ω_o, so their integral over $2\pi/T$ is zero (provided $k \ne l$).

Why Focus on Fourier Series?

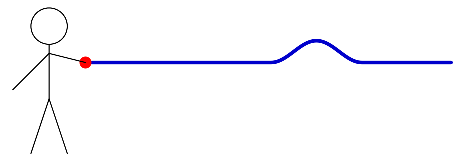
What's so special about sines and cosines?

Sinusoidal functions have interesting **mathematical properties**.
→ harmonically related sinusoids are **orthogonal** to each other over $[0, T]$.

Sines and cosines also play important roles in **physics** – especially the physics of waves.

Physical Example: Vibrating String

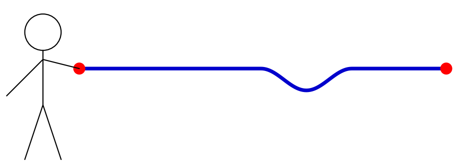
A taut string supports wave motion.



The speed of the wave depends on the tension on and mass of the string.

Physical Example: Vibrating String

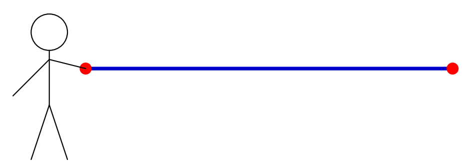
The wave will reflect off a rigid boundary.



The amplitude of the reflected wave is opposite that of the incident wave.

Physical Example: Vibrating String

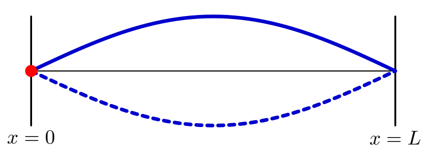
Reflections can interfere with excitations.



The interference can be constructive or destructive depending on the frequency of the excitation.

Physical Example: Vibrating String

We get constructive interference if round-trip travel time equals the period.

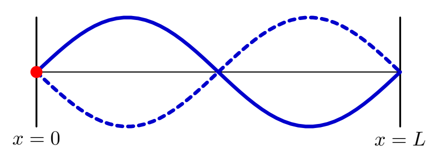


Round-trip travel time $= \frac{2L}{v} = T$

$$\omega_o = \frac{2\pi}{T} = \frac{2\pi}{2L/v} = \frac{\pi v}{L}$$

Physical Example: Vibrating String

We also get constructive interference if round-trip travel time is $2T$.

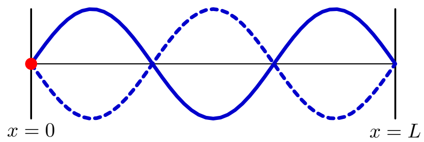


Round-trip travel time $= \frac{2L}{v} = 2T$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{L/v} = \frac{2\pi v}{L} = 2\omega_o$$

Physical Example: Vibrating String

In fact, we also get constructive interference if round-trip travel time is kT.



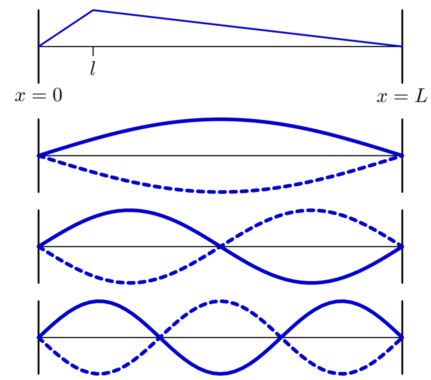
Round-trip travel time $= \frac{2L}{v} = kT$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L/kv} = \frac{k\pi v}{L} = k\omega_o$$

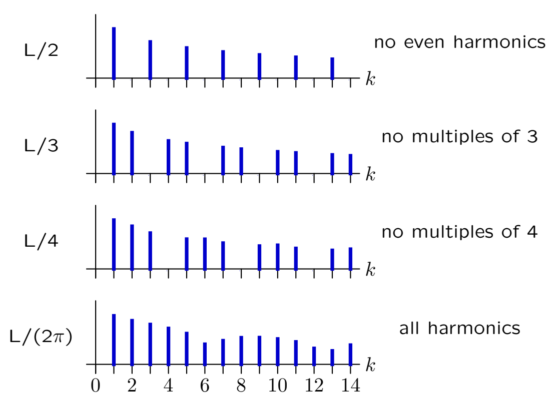
Only certain frequencies (harmonics of $\omega_o = \pi v/L$) persist.
This is the basis of stringed instruments.

Physical Example: Vibrating String

More complicated motions can be expressed as a sum of normal modes using Fourier series. Here the string is "plucked" at $x = l$.



Physical Example: Vibrating String



Differences in harmonic structure generate differences in timbre.

Summary

- We examined the convergence of Fourier series.
  - Functions with discontinuous slopes well represented.
  - Functions with discontinuous values generate ripples
    → Gibb's phenomenon.
- We investigated several **properties** of Fourier series.
  - scaling time
  - shifting time
  - We will find that there are **many** others
- We saw how Fourier series are useful for modeling a vibrating string.