6.003 Quiz 1 Fall 2021

Name: Solutions

Kerberos (Athena) username:

Please WAIT until we tell you to begin.

This quiz is closed book, but you may use one 8.5×11 sheet of paper (two sides).

You may NOT use any electronic devices (such as calculators and phones).

If you have questions, please **come to us at the front** to ask them.

Please enter all solutions in the boxes provided.

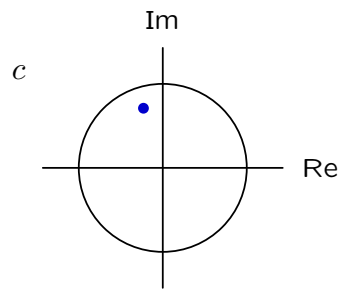
Work on other pages with QR codes will be considered for partial credit.
Please provide a note if you continue work on worksheets at the end of the exam (pages 13 and higher).

Trigonometric Identities Reference

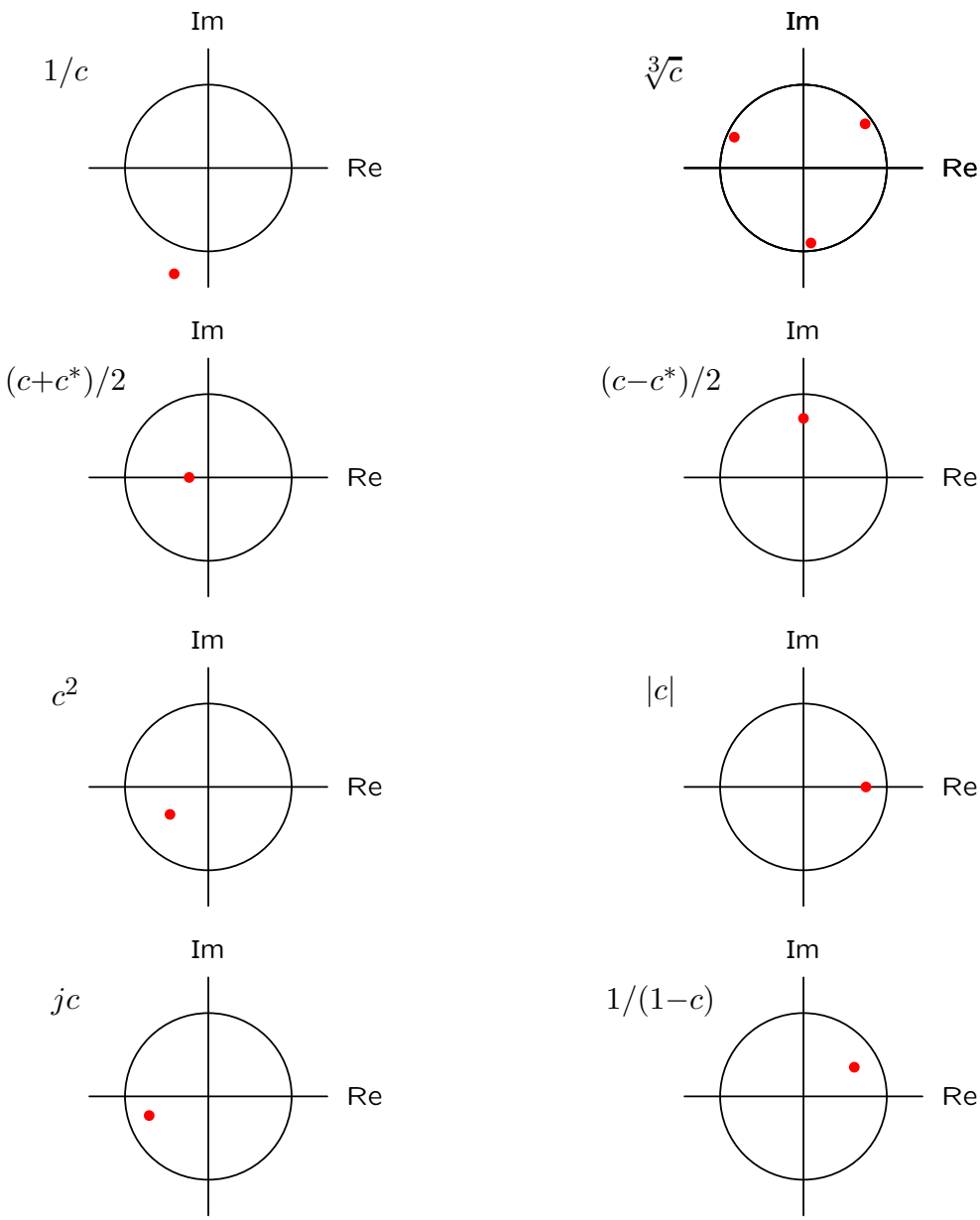
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$
$$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$
$$\cos(a+b)+\cos(a-b) = 2\cos(a)\cos(b)$$
$$\cos(a+b)-\cos(a-b) = -2\sin(a)\sin(b)$$
$$\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$$
$$\sin(a+b)-\sin(a-b) = 2\cos(a)\sin(b)$$
$$2\cos(a)\cos(b) = \cos(a-b)+\cos(a+b)$$
$$2\sin(a)\sin(b) = \cos(a-b)-\cos(a+b)$$
$$2\sin(a)\cos(b) = \sin(a+b)+\sin(a-b)$$
$$2\cos(a)\sin(b) = \sin(a+b)-\sin(a-b)$$

1 Complex Numbers (24 points)

Let c represent the complex number shown by a filled dot in the following diagram, where the real and imaginary parts of c are shown on the horizontal and vertical axes, respectively, and the circle has a radius of 1.



Below are eight complex-valued functions of c, each paired with a depiction of the complex plane demarked by the unit circle. Evaluate each expression and mark its value on the complex plane with a dot. If the expression can represent multiple complex numbers, mark at least two of them.



Worksheet (intentionally blank)

Part a. $\frac{1}{c}$**:**

Represent c in polar form: $c = re^{j\theta}$. Then $\frac{1}{c} = \frac{1}{r}e^{-j\theta}$.

Part b. $\sqrt[3]{c}$**:**

Represent c in polar form: $c = re^{j\theta}$. Then $\sqrt[3]{c} = \sqrt[3]{r} \times e^{j\theta/3}$ or $\sqrt[3]{r} \times e^{j(\theta+2\pi)/3}$ or $\sqrt[3]{r} \times e^{j(\theta+4\pi)/3}$.

Part c. $(c + c^*)/2$**:**

Represent c in Cartesian form: $c = a + jb$. Then $(c + c^*)/2 = (a + jb + a - jb)/2 = a = \text{Re}\{c\}$.

Part d. $(c - c^*)/2$**:**

Represent c in Cartesian form: $c = a + jb$. Then $(c - c^*)/2 = (a + jb - a + jb)/2 = jb = j\text{Im}\{c\}$.

Part e. c^2**:**

Represent c in polar form: $c = re^{j\theta}$. Then $c^2 = r^2e^{j2\theta}$.

Part f. $|c|$**:**

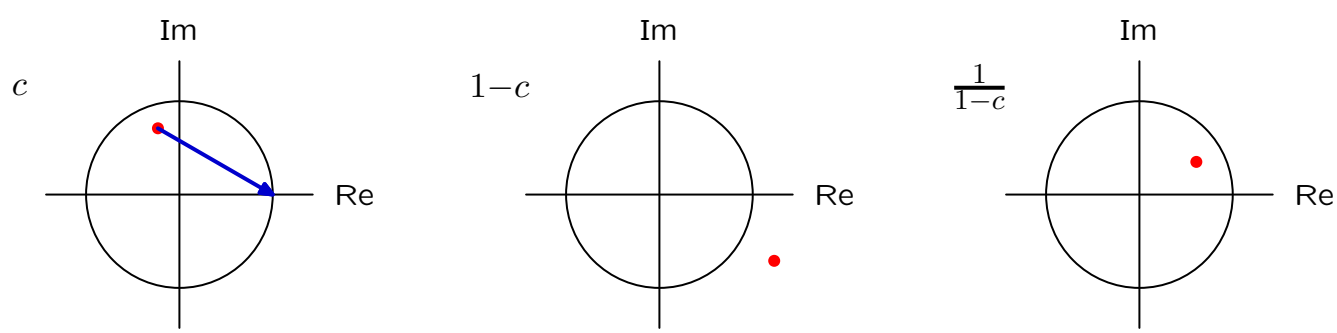
Represent c in polar form: $c = re^{j\theta}$. Then $|c| = r$.

Part g. jc**:**

Represent c in polar form: $c = re^{j\theta}$. Then $jc = re^{j(\theta+\pi/2)}$.

Part h. $1/(1-c)$**:**

Find the value of $1-c$ on the complex plane by constructing the vector representation shown by the blue arrow in the left panel below and by the red dot in the center panel below.



The reciprocal of $1-c$ has a magnitude of $\frac{1}{|1-c|}$ and an angle that is the negative of that of $1-c$, as shown in the right panel above.

2 Sampling Sinuoids (25 points)

Let $f(t)$ represent the following continuous-time signal:

$$f(t) = 4\cos(300\pi t) + 2\sin(400\pi t) + \cos(600\pi t)$$

Part a. Let $f_1[n]$ represent a discrete-time signal that is obtained by samping $f(t)$ with sampling frequency $f_s = 100\,\text{Hz}$, so that

$$f_1[n] = f(n/f_s)$$

Determine the fundamental (shortest) period of $f_1[n]$. Your answer should be a number or a numeric expression that can include common constants (such as π).

Enter your answer in the box below. Enter **none** if $f_1[n]$ is not periodic.

fundamental period: 2

$$f_1[n] = f(n/100) = 4\cos(3\pi n) + 2\sin(4\pi n) + \cos(6\pi n) = 4(-1)^n + 0 + 1$$

Part b. Let $f_2[n]$ represent a discrete-time signal that is obtained by samping $f(t)$ with sampling frequency $f_s = 200\,\text{Hz}$, so that

$$f_2[n] = f(n/f_s)$$

Determine the fundamental (shortest) period of $f_2[n]$. Your answer should be a number or a numeric expression that can include common constants (such as π).

Enter your answer in the box below. Enter **none** if $f_2[n]$ is not periodic.

fundamental period: 4

$$f_2[n] = f(n/200) = 4\cos(3\pi n/2) + 2\sin(2\pi n) + \cos(3\pi n) = 4\cos(3\pi n/2) + 0 + (-1)^n$$

Part c. Let $f_3[n]$ represent a discrete-time signal that is obtained by samping $f(t)$ with sampling frequency $f_s = 300\,\text{Hz}$, so that

$$f_3[n] = f(n/f_s)$$

Determine the fundamental (shortest) period of $f_3[n]$. Your answer should be a number or a numeric expression that can include common constants (such as π).

Enter your answer in the box below. Enter **none** if $f_3[n]$ is not periodic.

fundamental period: 6

$$f_3[n] = f(n/300) = 4\cos(pin) + 2\sin(4\pi n/3) + \cos(2\pi n) = 4(-1)^n + 2\sin(4\pi n/3) + 1$$

Part d. Let $f_4[n]$ represent a discrete-time signal that is obtained by samping $f(t)$ with sampling frequency $f_s = 1200\,\text{Hz}$, so that

$$f_4[n] = f(n/f_s)$$

Determine the fundamental (shortest) period of $f_4[n]$. Your answer should be a number or a numeric expression that can include common constants (such as π).

Enter your answer in the box below. Enter **none** if $f_4[n]$ is not periodic.

fundamental period: 24

$$f_4[n] = f(n/1200) = 4\cos(\pi n/4) + 2\sin(4\pi n/3) + \cos(\pi n/2)$$

Part e. Let $f_5[n]$ represent a discrete-time signal that is obtained by samping $f(t)$ with sampling frequency $f_s = 2400$ Hz, so that

$$f_5[n] = f(n/f_s)$$

Determine the fundamental (shortest) period of $f_5[n]$. Your answer should be a number or a numeric expression that can include common constants (such as π).

Enter your answer in the box below. Enter **none** if $f_5[n]$ is not periodic.

fundamental period: 48

$$f_5[n] = f(n/2400) = 4\cos(\pi n/8) + 2\sin(4\pi n/6) + \cos(\pi n/4)$$

3 Find All (24 points)

Consider the following signals:

$$f_1[n] = \frac{1}{2} + 6\cos(\pi n/2) + 4\cos(\pi n/5 - \pi/2)$$

$$f_2[n] = \cos(1.8\pi n) + 2\sin(2.7\pi n)$$

$$f_3[n] = \left|\sin(\pi n/10)\right| \qquad \text{; where } |x| \text{ represents the magnitude of } x$$

$$f_4[n] = \mathrm{Im}\left\{e^{j\left(2\pi n/20+\pi/2\right)}\right\} \qquad \text{; where } \mathrm{Im}\{x\} \text{ represents the imaginary part } x$$

Part a. Determine which (if any) of signals $f_1[n]$ through $f_4[n]$ are **symmetric** about n=0, and circle **all** of those signals below. Circle **None** if no signal is symmetric about n=0.

$f_1[n]$ $f_2[n]$ $f_3[n]$ $f_4[n]$ **None**

$f_1[n]$ is not symmetric about n=0 because of the non-zero phase of the last term.
$f_2[n]$ is not symmetric about n=0 because $\sin$ is antisymmetric.
$f_3[n]$ is symmetric about n=0 because $\sin$ is purely antisymmetric, and the magnitude of a purely antisymmetric function is purely symmetric.
$f_4[n]$ is symmetric about n=0 because $f_4[n] = \mathrm{Im}\{je^{j2\pi n/10}\} = \cos(2\pi n/10)$

Part b. Determine which (if any) of signals $f_1[n]$ through $f_4[n]$ are **periodic** with a fundamental (smallest) period N=20, and circle **all** of those signals below. Circle **None** if no signal is periodic with fundamental period N=20.

$f_1[n]$ $f_2[n]$ $f_3[n]$ $f_4[n]$ **None**

$f_1[n]$: $\cos(\pi n/2)$ is periodic with $N = 4 = 2 \times 2$. $\sin(\pi n/5 - \pi/2)$ is periodic with $N = 10 = 2 \times 5$. Their sum is periodic with $N = 2 \times 2 \times 5 = 20$.

$f_2[n]$: $\cos(1.8\pi n)$ is periodic with N=10. $\sin(2.7\pi n)$ is periodic with N=20. Their sum is periodic with N=20.

$\sin(\pi n/10)$ is periodic in N=20, and the magnitude halves the period. So $f_3[n]$ is periodic with N=10.

$f_4[n]$ is periodic with N=20 because the complex exponential is periodic with N=20.

Part c. Determine which (if any) of signals $f_1[n]$ through $f_4[n]$ can be represented by Fourier series with **purely imaginary** coefficients, and circle **all** of those entries below. Circle **None** if none of these signals can be represented by Fourier series with purely imaginary coefficients.

$f_1[n]$ $f_2[n]$ $f_3[n]$ $f_4[n]$ **None**

All of the time functions $f_1[n]$ through $f_4[n]$ are real-valued. Therefore the Fourier series coefficients

$$F[k] = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=\langle N\rangle} f[n]\Big(\cos(2\pi kn/N) + j\sin(2\pi kn/N)\Big)$$

will have purely imaginary values iff $f[n]$ is antisymmetric about n=0, so that the cosine terms sum to zero and the sine terms do not.

$f_1[n]$ is not antisymmetric about n=0 because both the constant term and the cosine term are symmetric.
Similarly, $f_2[n]$ is not antisymmetric about n=0 because the cosine term is symmetric.
$f_3[n]$ is symmetric about n=0 because $\sin$ is purely antisymmetric, and the magnitude of a purely antisymmetric function is purely symmetric.
$f_4[n]$ is symmetric about n=0 because $f_4[n] = \mathrm{Im}\{je^{j2\pi n/10}\} = \cos(2\pi n/10)$

Part d. Determine which (if any) of signals $f_i[n]$ can be represented by Fourier series coefficients $F_i[k]$ that are symmetric functions of k (i.e., $F_i[k] = F_i[-k]$) and circle **all** of those entries below. Circle **None** if none of the Fourier series coefficients are symmetric functions of k.

$f_1[n]$ $f_2[n]$ $f_3[n]$ $f_4[n]$ **None**

All of the time functions $f_i[n]$ are real-valued. Therefore the Fourier series coefficients

$$X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\Big(\cos(2\pi kn/N) + j\sin(2\pi kn/N)\Big)$$

will equal

$$X[-k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\Big(\cos(2\pi kn/N) - j\sin(2\pi kn/N)\Big)$$

iff $x[n]$ is a symmetric function of n, so that the sine terms integrate to zero.

$f_1[n]$ is not symmetric about n=0 because of the phase shift in the its term.
$f_2[n]$ is not symmetric about n=0 because of the sine term.
$f_3[n]$ is symmetric about n=0 because $\sin$ is purely antisymmetric, and the magnitude of a purely antisymmetric function is purely symmetric.
$f_4[n]$ is symmetric about n=0 because $f_4[n] = \mathrm{Im}\{je^{j2\pi n/10}\} = \cos(2\pi n/10)$

4 Alternative Representations (27 points)

Let $f_1(t)$ represent a function of continuous time t that is represented by a trigonometric Fourier series:

$$f_1(t) = \sum_{k=1}^{\infty} \frac{\sin(kt)}{k}$$

Part a. What is the average value of $f_1(t)$? Your answer should be a number or numeric expression that can include common constants (such as π). Your answer should not include $f_1(t)$ or any integrals or infinite sums.

Enter your answer here: 0

The general form for a Fourier series in trig form is

$$f_1(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d_k \sin(2\pi kt/T)$$

where c_0 is the average value. Since $c_0 = 0$, the average value of $f_1(t)$ is 0.

Part b. Determine the numerical value of the following integral:

$$\int_0^{2\pi} f_1(t)\cos(3t)dt$$

Your answer should be a number or numeric expression that can include common constants (such as π). Your answer should not include $f_1(t)$ or any integrals or infinite sums.

Enter your answer here: 0

The sum that defines $f_1(t)$ contains trigonometric basis functions of the form 1 (for c_0), $\cos(kt)$ (for c_k) and $\sin(kt)$ (for d_k) where the period T is 2π. Since the basis functions are orthogonal to each other and there are not $\cos(3t)$ terms in $f_1(t)$,

$$\int_0^{2\pi} f_1(t)\cos(3t)dt = 0$$

Part c. Determine the numerical value of the following integral:

$$\int_0^{2\pi} f_1(t)\sin(5t)dt$$

Your answer should be a number or numeric expression that can include common constants (such as π). Your answer should not include $f_1(t)$ or any integrals or infinite sums.

Enter your answer here: $\pi/5$

$$\int_{2\pi} f_1(t)\sin(5t)dt = \int_{2\pi} \sum_{k=1}^{\infty} \frac{1}{k}\sin(kt)\sin(5t)dt$$

Since the basis functions are orthogonal, this integral "sifts out" the $\sin(5t)$ component of $f_1(t)$:

$$\int_{2\pi} \sum_{k=1}^{\infty} \frac{1}{k}\sin(kt)\sin(5t)dt = \int_{2\pi} \frac{1}{5}\sin^2(5t)dt = \int_{2\pi} \frac{1}{5}\left(\frac{1}{2} - \frac{1}{2}\cos(10t)\right)dt = \int_{2\pi} \frac{1}{5}\frac{1}{2}dt = \frac{\pi}{5}$$

Part d. The function $f_1(t)$ can also be expressed as a complex exponential Fourier series as follows:

$$f_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Determine the numerical values of ω_o, a_{-2}, a_{-1}, a_0, a_1, a_2 and enter those values in the table below. Each answer should be a number or numeric expression that can include common constants (such as π). Your answers should not include $f_1(t)$ or any integrals or infinite sums.

ω_o:	1
a_{-2}:	$j/4$
a_{-1}:	$j/2$
a_0:	0
a_1:	$-j/2$
a_2:	$-j/4$

$$f_1(t) = \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} = \sum_{k=1}^{\infty} \frac{e^{jkt} - e^{-jkt}}{j2k} = \sum_{k=-\infty}^{-1} \frac{-j}{2k}e^{jkt} + \sum_{k=1}^{\infty} \frac{-j}{2k}e^{jkt} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

where $\omega_o = 1$ and

$$a_k = \begin{cases} -j/2k & k \neq 0 \\ 0 & k = 0 \end{cases}$$

Part e. Let c_k and d_k represent the trigonometric Fourier series coefficients for a periodic function $f_2(t)$ of continuous time t with period T:

$$f_2(t) = f_2(t+T) = c_0 + \sum_{k=1}^{\infty} c_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d_k \sin(2\pi kt/T)$$

Determine expressions for the trigonometric Fourier series coefficients c'_k and d'_k for $f_3(t) = \frac{d}{dt} f_2(t)$:

$$f_3(t) = \frac{d}{dt} f_2(t) = c'_0 + \sum_{k=1}^{\infty} c'_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d'_k \sin(2\pi kt/T)$$

as functions of $\ldots c_0, c_1, c_2, \ldots$ and/or $\ldots d_0, d_1, d_2, \ldots$.

Enter expressions for $c'_0, c'_1, d'_1, c'_2, d'_2, c'_3$, and d'_3 in the table below. Your expressions may contain any combination of the original (unprimed) coefficients ($\ldots c_0, c_1, c_2, \ldots$ and $\ldots d_0, d_1, d_2, \ldots$) but must not include $f_2(t)$ or $f_3(t)$ or any integrals or infinite sums.

Notice that d'_0 is not defined, and it's box is x'd out of the table.

	c'_k	d'_k
0:	0	
1:	$2\pi k d_1/T$	$-2\pi k c_1/T$
2:	$2\pi k d_2/T$	$-2\pi k c_2/T$
3:	$2\pi k d_3/T$	$-2\pi k c_3/T$

$$f_2(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d_k \sin(2\pi kt/T)$$

$$\begin{aligned} f_3(t) = \frac{d}{dt} f_2(t) &= \frac{d}{dt}\left(c_0 + \sum_{k=1}^{\infty} c_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d_k \sin(2\pi kt/T)\right) \\ &= \sum_{k=1}^{\infty} -\frac{2\pi k c_k}{T} \sin(2\pi kt/T) + \sum_{k=1}^{\infty} \frac{2\pi k d_k}{T} \cos(2\pi kt/T) \\ &= c'_0 + \sum_{k=1}^{\infty} c'_k \cos(2\pi kt/T) + \sum_{k=1}^{\infty} d'_k \sin(2\pi kt/T) \end{aligned}$$

Therefore

$$c'_0 = 0$$

$$c'_k = \frac{2\pi k d_k}{T}$$

$$d'_k = -\frac{2\pi k d_k}{T}$$

Part f. Let a_k represent the complex exponential Fourier series coefficients for a periodic function $f_4(t)$ of continuous time t with period T:

$$f_4(t) = f_4(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

Determine expressions for the complex exponential Fourier series coefficients a'_k for $f_5(t) = f_4(t)\cos(2\pi t/T)$

$$f_5(t) = f_4(t)\cos(2\pi t/T) = \sum_{k=-\infty}^{\infty} a'_k e^{j2\pi kt/T}$$

as functions of the original (unprimed) coefficients (a_k).

Enter expressions for a'_{-2}, a'_{-1}, a'_0, a'_1, and a'_2 in the table below. Your expressions may contain any combination of the original (unprimed) coefficients a_k but must not include $f_4(t)$ or $f_5(t)$ or any integrals or infinite sums.

a'_{-2}:	$(a_{-3}+a_{-1})/2$
a'_{-1}:	$(a_{-2}+a_0)/2$
a'_0:	$(a_{-1}+a_1)/2$
a'_1:	$(a_0+a_2)/2$
a'_2:	$(a_1+a_3)/2$

$$\begin{aligned}
f_5(t) &= f_4(t)\cos(2\pi t/T) \\
&= \sum_k a_k e^{j\frac{2\pi}{T}kt}\cos(2\pi t/T) \\
&= \sum_k \frac{a_k}{2} e^{j\frac{2\pi}{T}kt} e^{j\frac{2\pi}{T}t} + \sum_k \frac{a_k}{2} e^{j\frac{2\pi}{T}kt} e^{-j\frac{2\pi}{T}kt} e^{j\frac{2\pi}{T}t} \\
&= \sum_k \frac{a_k}{2} e^{j\frac{2\pi}{T}(k+1)t} + \sum_k \frac{a_k}{2} e^{j\frac{2\pi}{T}(k-1)t} \\
&= \sum_l \frac{a_{l-1}}{2} e^{j\frac{2\pi}{T}lt} + \sum_m \frac{a_{m+1}}{2} e^{j\frac{2\pi}{T}mt} \\
&= \sum_k \frac{a_{k-1}+a_{k+1}}{2} e^{j\frac{2\pi}{T}kt}
\end{aligned}$$

$$a'_k = \frac{a_{k-1}+a_{k+1}}{2}$$

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