6.003: Signal Processing

we will start @ 10:05am EasteRN

Convolution and Filtering

$x \rightarrow \boxed{LTI} \rightarrow y$

$h[n]$

time domain

$$y(t) = (h * x)(t) = \int h(\tau)x(t-\tau)\,d\tau$$

$$y[n] = (h * x)[n] = \sum_m h[m]x[n-m]$$

frequency domain

$$Y(\omega) = H(\omega)X(\omega)$$

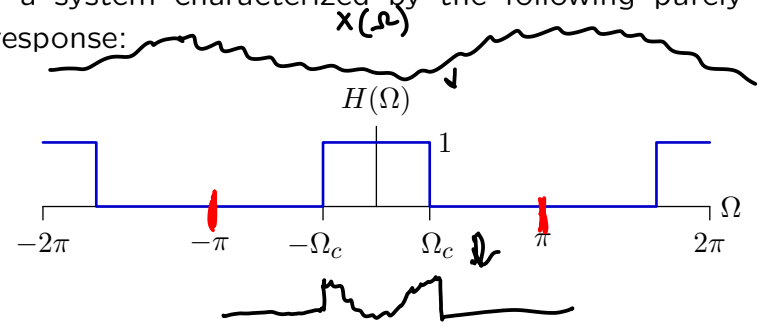
$$Y(\Omega) = H(\Omega)X(\Omega)$$

$x \rightarrow \boxed{H(e)} \rightarrow y$

20 October 2020

The "Ideal" Low-Pass Filter

Consider a system characterized by the following purely real frequency response:



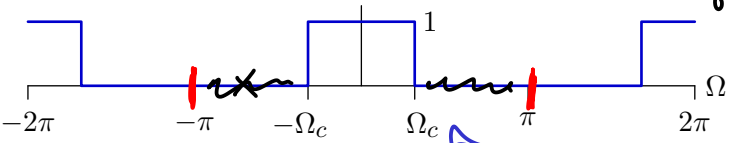
Such a system is called a **low-pass filter**, because it allows low frequencies to pass through unmodified, while attenuating high frequencies.

We could apply this filter to a signal by multiplying the DTFT of that signal by the values above. But we could also apply the filter by operating in the time domain.

The "Ideal" Low-Pass Filter

$h[n] \overset{DTFT}{\longleftrightarrow} H(\Omega)$

USR Freq Resp



We can apply this filter to a signal by convolving with its unit sample response. What is the unit sample response of the system whose frequency response is shown above?

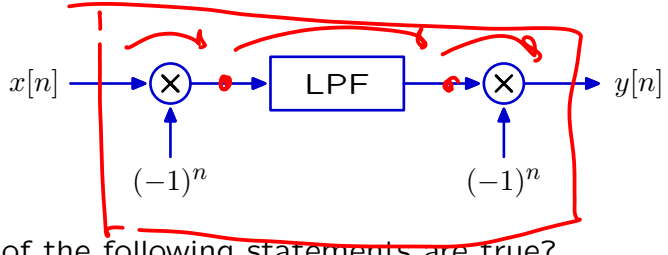
$$h[n] = \frac{1}{2\pi}\int_{2\pi} H(\Omega)e^{j\Omega n}\,d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n}\,d\Omega$$

$$= \frac{1}{2\pi}\frac{1}{jn}e^{j\Omega n}\Big|_{\Omega=-\Omega_c}^{\Omega_c} = \frac{1}{\cancel{2\pi}\,\cancel{j}n}\overbrace{\left(e^{j\Omega_c n} - e^{-j\Omega_c n}\right)}^{2j\sin(\Omega_c n)} = \frac{\sin(\Omega_c n)}{\pi n}$$

sinc $\rightarrow \dfrac{\sin(x)}{x}$

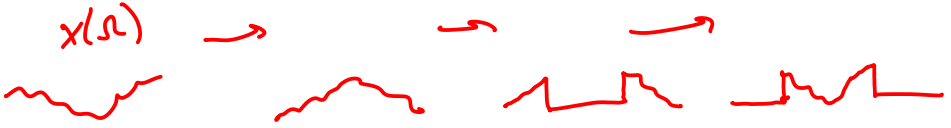
Cascaded System

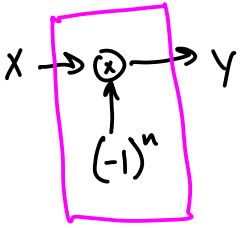
Consider this system, where LPF represents a lowpass filter of the form discussed on the previous slides.



How many of the following statements are true?

- The transformation from $x[n]$ to $y[n]$ is linear.
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.





linear? ✓

time invariant? ✗

$$x[n] = x_1[n-n_0]$$

$$y[n] = (-1)^n x[n]$$

$$= (-1)^n x_1[n-n_0]$$

$$= (-1)^{n_0} \boxed{(-1)^{n-n_0} x_1[n-n_0]}$$

$$= (-1)^{n_0} y_1[n-n_0]$$

$$y[n] = (-1)^{n-n_0} x_1[n-n_0]$$

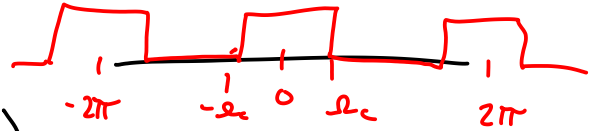
$$x[n] = a x_1[n] + b x_2[n]$$

$$y[n] \stackrel{?}{=} a y_1[n] + b y_2[n] \checkmark$$

$$y[n] = (-1)^n x[n] = (-1)^n (a x_1[n] + b x_2[n])$$

$$= a \underbrace{(-1)^n x_1[n]}_{y_1[n]} + b \underbrace{(-1)^n x_2[n]}_{y_2[n]}$$

$$x \rightarrow \boxed{\text{LPF}} \rightarrow y$$



$$Y(\Omega) = X(\Omega)H(\Omega)$$

linear? time invariant?

$$y[n] = (x * h)[n]$$

convolution is linear, time-invariant

$\Rightarrow$ LPF is LTI ✓

Convolution linearity ✓

$$x[n] = a\,x_1[n] + b\,x_1[n]$$

$$(x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]\,h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \left(a x_1[m] + b x_2[m]\right) h[n-m]$$

$$= \sum_{m=-\infty}^{\infty} \left(a x_1[m] h[n-m] + b\, x_2[m] h[n-m]\right)$$

$$= a \boxed{\sum_{m=-\infty}^{\infty} x_1[m]\,h[n-m]} + b \boxed{\sum_{m=-\infty}^{\infty} x_2[m]\,h[n-m]} = a(x_1 * h)[n] + b(x_2 * h)[n]$$

Convolution time invariance ✓

$$x[n] = x_1[n-n_0]$$

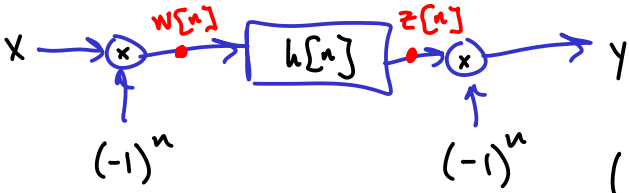
$$(x * h)[n] = \sum_{m=-\infty}^{\infty} x[m]\, h[n-m] = \sum_{m=-\infty}^{\infty} x_1[m-n_0]\, h[n-m]$$

let $l = m - n_0$

$$= \sum_{l=-\infty}^{\infty} x_1[l]\, h[n-(l+n_0)] = \sum_{l=-\infty}^{\infty} x_1[l]\, h[(n-n_0)-l]$$

$$= (x_1 * h)[n-n_0]$$

$$\boxed{(f*g)[n] = \sum_{m=-\infty}^{\infty} f[m]\, g[n-m]}$$



$(x * g)[n]$, where $g[n] = (-1)^n h[n]$

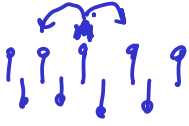
$$w[n] = (-1)^n x[n]$$

$$z[n] = (w * h)[n] = \sum_{m=-\infty}^{\infty} (-1)^m x[m]\, h[n-m]$$

$$y[n] = (-1)^n z[n] = \boxed{(-1)^n \sum_{m=-\infty}^{\infty} (-1)^m x[m]\, h[n-m]}$$

$$= \sum_{m=-\infty}^{\infty} (-1)^{\overset{n-m}{\cancel{n+m}}} x[m]\, h[n-m] = \sum_{m=-\infty}^{\infty} x[m]\, (-1)^{n-m} h[n-m]$$

$$(-1)^{n+m} = (-1)^{n-m} \quad \forall n, m$$



$$g[n] = (-1)^n h[n] = \underbrace{e^{j\pi n} h[n]}$$

frequency shift!

$H(\Omega)$ low-pass filter

$$G(\Omega) = H(\Omega - \pi)$$

$G(\Omega)$ high-pass filter

$$g[n] = \underbrace{(-1)^n}\,\underbrace{h[n]}$$

$G(\Omega)$

$$x[n] \overset{DTFT}{\longleftrightarrow} X(\Omega)$$

$$x[n-n_0] \longleftrightarrow \underline{e^{-j\Omega n_0}}\,\underline{X(\Omega)}$$

TIME SHIFT

$$x[n] \longleftrightarrow X(\Omega)$$

$$\underline{e^{j\Omega_0 n}x[n]} \longleftrightarrow X(\Omega-\Omega_0)$$

$$x[n] \leftrightarrow X(\Omega)$$

$$y[n] = \boxed{e^{j\Omega_0 n}}\, x[n]$$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n}\, x[n]\, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\Omega-\Omega_0)n}$$

$$= X(\Omega-\Omega_0)$$

$$(-1)^n x[n] = a\,\overbrace{\boxed{x_1[n]}}^{(-1)^n} + b\,\overbrace{\boxed{x_2[n]}}^{(-1)^n}$$