Convolution and Filtering

\[ y(t) = (h * x)(t) = \int h(\tau)x(t - \tau) \, d\tau \]

\[ y[n] = (h * x)[n] = \sum_m h[m]x[n - m] \]

\[ Y(\omega) = H(\omega)X(\omega) \]

\[ Y(\Omega) = \tilde{H}(\Omega)\tilde{X}(\Omega) \]

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The “Ideal” Low-Pass Filter

Consider a system characterized by the following purely real frequency response:

$$H(\Omega) = \frac{\Omega - \Omega_c}{\pi} - \frac{\Omega - \Omega_c}{2\pi}$$

Such a system is called a **low-pass filter**, because it allows low frequencies to pass through unmodified, while attenuating high frequencies.

We could apply this filter to a signal by multiplying the DTFT of that signal by the values above. But we could also apply the filter by operating in the time domain.
The “Ideal” Low-Pass Filter

We can apply this filter to a signal by convolving with its unit sample response. What is the unit sample response of the system whose frequency response is shown above?

\[ h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\Omega) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j2\pi c n} d\Omega \]

\[ = \frac{1}{2\pi} \frac{1}{j\Omega} e^{j2\pi c n} \bigg|_{-\pi}^{\pi} = \frac{1}{2\pi j\Omega} \left( e^{j2\pi c n} - e^{-j2\pi c n} \right) \]

\[ \text{Sinc} \rightarrow \frac{\sin(x)}{x} \]

\[ \text{Freq Resp} \]

\[ \text{USR} \]

\[ H(\Omega) \]
Consider this system, where LPF represents a lowpass filter of the form discussed on the previous slides.

How many of the following statements are true?

- The transformation from $x[n]$ to $y[n]$ is linear.
- The transformation from $x[n]$ to $y[n]$ is time invariant.
- The transformation from $x[n]$ to $y[n]$ is a high-pass filter.
$x \rightarrow (\cdot) \rightarrow y$

linear?

$X[n] = a x_1[n] + b x_2[n]$

$Y[n] = a Y_1[n] + b Y_2[n]$

$Y[n] = (-1)^n X[n] = (-1)^n (a x_1[n] + b x_2[n])$

$= a (-1)^n x_1[n] + b (-1)^n x_2[n]$

$Y_1[n]$  $Y_2[n]$

$Y[n] = (-1)^n x_1[n-n_0]$

$= (-1)^{n_0} (-1)^{n-n_0} x_1[n-n_0]$

$Y[n] = (-1)^{n-n_0} x_1[n-n_0]$
$x \rightarrow \text{LPF} \rightarrow y$

$Y(\omega) = X(\omega)H(\omega)$

Linear? \quad \text{Time invariant?}

$y[n] = (x * h)[n]$

Convolution is linear, time-invariant

$\Rightarrow$ LPF is LTI \quad \checkmark$
Convolution linearity \( \checkmark \)

\[ x[n] = a x_1[n]+ b x_2[n] \]

\[ (x * h)[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] \]

\[ = \sum_{m=-\infty}^{\infty} (a x_1[m] + b x_2[m]) h[n-m] \]

\[ = a \sum_{m=-\infty}^{\infty} x_1[m] h[n-m] + b \sum_{m=-\infty}^{\infty} x_2[m] h[n-m] \]

\[ = a (x_1 * h)[n] + b (x_2 * h)[n] \]
Convolution time invariance

\[ x[n] = x_1[n-n_0] \]

\[ (x \ast h)[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] = \sum_{m=-\infty}^{\infty} x_1[m-n_0] h[n-m] \]

\[ \text{let } l = m-n_0 \]

\[ = \sum_{l=-\infty}^{\infty} x_1[l] h[n-(l+n_0)] = \sum_{l=-\infty}^{\infty} x_1[l] h[(n-n_0)-l] \]

\[ = (x_1 \ast h)[n-n_0] \]

\[ (f \ast g)[n] = \sum_{m=-\infty}^{\infty} f[m] g[n-m] \]
\[ w[n] = (-1)^n x[n] \]

\[ z[n] = (w * h)[n] = \sum_{m=-\infty}^{\infty} (-1)^m x[m] h[n-m] \]

\[ y[n] = (-1)^n z[n] = \sum_{m=-\infty}^{\infty} (-1)^m x[m] h[n-m] \]

\[ = \sum_{m=-\infty}^{\infty} (-1)^{n+m} x[m] h[n-m] = \sum_{m=-\infty}^{\infty} x[m] (-1)^{n-m} h[n-m] \]

\[ (-1)^{n+m} = (-1)^{n-m} \quad \forall n, m \]
\[ g[n] = (-1)^n h[n] = e^{j\pi n} h[n] \]

frequency shift!

\[ G(\omega) = H(\omega - \pi) \]

\[ H(\omega) \quad \text{low-pass filter} \]

\[ G(\omega) \quad \text{high-pass filter} \]
\[ g[n] = (-1)^n h[n] \]

\[ g(t) \]
\[ x[n] \overset{\text{DFT}}{\leftrightarrow} X(\omega) \]

\[ x[n-n_0] \leftrightarrow e^{j2\pi n_0} \overline{X(\omega)} \]

TIME SHIFT

\[ x[n] \leftrightarrow X(\omega) \]

\[ e^{j\omega n_0} x[n] \leftrightarrow X(\omega - \omega_0) \]
$$x[n] \leftrightarrow X(e^{j\omega})$$

$$y[n] = e^{j\omega_0 n} x[n]$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{j\omega_2 n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n}$$

$$= X(e^{-j(\omega-\omega_0)})$$
\((-1)^n x[n]\) = \alpha x_1[n] + b \delta[n] + c \delta[n+1] + d \delta[n+2] + e \delta[n+3] + f \delta[n+4]