6.003: Signal Processing

Using the Discrete Fourier Transform

We will start @ 8:05am Eastern

1 October 2020
Single Sinusoid

\[ x_1[n] = \cos \left( \frac{8\pi}{100} n \right) \]

\[ X_1[k] = \frac{1}{2}DFT\{x_1[0 : 100]\} \]

\[ X_1[k] = X_1[k+100] \]

Which values of \( k \) are non-zero?

\[ k = \pm 4 \]

\[ e^{j\frac{2\pi k n}{N}} \]
Next, consider a signal $x_5[\cdot]$, described by:

$$x_5[n] = \cos\left(\frac{8\pi}{100}n\right) + \cos\left(\frac{9\pi}{100}n\right)$$
Analyzing Signals with Multiple Frequencies

Consider analyzing with $N = 100$ to find the frequencies:

$$x_5[n] = \cos(8\pi n/100) + \cos(9\pi n/100)$$

$$|X_5[k]|$$
Analyzing Signals with Multiple Frequencies

$N = 100$ zoomed

$$x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)$$

Really hard to tell if this is one peak or two!

How can we do better?
What is the minimum window size $N$ needed to resolve $\Omega = \frac{8\pi}{100}$ from $\frac{9\pi}{100}$?

$$x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)$$
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\( N = 100 \)

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

$N = 100$ zoomed

$$x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)$$

| $X_5[k]$ |
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 200 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 200 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 400 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 400 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

These frequencies are barely resolved with \( N = 200 \).
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians in DT, which is equivalent to $(f_s/N)$ Hz in CT.

\[
\Omega = \frac{2\pi}{N} [\text{rad/sample}]
\]

\[
\Omega = \frac{f_s}{N} [\text{Hz}]
\]
Analyzing Signals with Multiple Frequencies

Two frequencies are resolved if they are separated by more than \( \frac{2\pi}{N} \).

\[ \Omega_1 = \frac{8\pi}{100} \quad \text{and} \quad \Omega_2 = \frac{9\pi}{100} \quad \text{will be resolved if} \]

\[ \Delta \Omega = \Omega_2 - \Omega_1 = \frac{9\pi}{100} - \frac{8\pi}{100} = \frac{\pi}{100} > \frac{2\pi}{N} \]

That is, if \( N > 200 \).

We can think of \( \frac{2\pi}{N} \) as the frequency resolution of the DFT.

Notice 8 full cycles of \( \Omega_1 \) and 9 full cycles of \( \Omega_2 \) fit in \( N = 200 \).
Frequency Resolution

How many Fourier components are in a signal?
What are their amplitudes and frequencies?

Example: cello
Listen to the wav file cello3.wav (fs = 44,100 Hz).

How can we characterize the frequency components?
Frequency Resolution

Extract 1024 samples and calculate DFT (using FFT algorithm).

Does it matter which part of the wav file we take the samples from?

Why use 1024 samples?

Would more be better? Would fewer be better?
Frequency Resolution

Most of structure is in low frequencies. Zoom in on that part.

![Waveform and Spectrum](image)

What value of $k$ corresponds to the big peak?

What is corresponding value of frequency (in Hz)?

$$f = \frac{k}{N} \cdot f_s = \frac{7}{1024} \cdot 44100 \text{ Hz}$$

$\approx 308$ Hz
Frequency Resolution

Most of structure is in low frequencies. Zoom in on that part.

What value of $k$ corresponds to the big peak? 7
What is corresponding value of frequency (in Hz)?

$$\frac{k_o}{N/2} = \frac{f_o}{f_s/2}$$

$$f_o = \frac{k_o}{N} f_s = \frac{7}{1024} \times 44100 \approx 301.46 \text{ Hz}$$

between D (293.66 Hz) and E flat (311.13 Hz)
Improving Frequency Resolution

The DFT provides integer resolution in $k$. Therefore, the peak at $k = 7$ could be off by as much as $\pm \frac{1}{2}$.

To improve frequency resolution, increase length of analysis window.

Two methods

- Zero padding
- Longer sequence
Zero Padding

Original (N=1024).

$x_1[n]$

$|X_1[k]|$

→ N = 2048
Zero Padding

Lengthen by a factor of 2 (N=2048).

\[ X_2[k] = \frac{1}{2} X_1[k] \]

What (if any) relations exist between \( |X_2[k]| \) (this slide) and \( |X_1[k]| \) (previous slide)?

\[ X_1[k] = \frac{1}{1024} X_{1,\omega} \left( \Omega = \frac{2\pi k}{1024} \right) \]

\[ X_2[k] = \frac{1}{2048} X_{1,\omega} \left( \Omega = \frac{2\pi k}{2048} \right) \]
Zero Padding

Lengthen by a factor of 2 (N=2048).

What (if any) relations exist between $|X_2[k]|$ (this slide) and $|X_1[k]|$ (previous slide)?

$$X_2[2k] = \frac{1}{2} X_1[k]$$
Zero Padding

Lengthen by a factor of 4 (N=4096).

\[ x_3[n] \quad \rightarrow \quad X_3[k] \]
Zero Padding

Lengthen by a factor of 8 (N=8192).

The stem plots can be distracting when they are close together. (They also take a long time to compute!)

Replot using lines (but remember that the signals are DT).
Zero Padding

Original (N=1024).
Zero Padding

Lengthen by a factor of 2 (N=2048).
Zero Padding

Lengthen by a factor of 4 (N=4096).
Zero Padding

Lengthen by a factor of 8 (N=8192).

Peak is now at $k = 55$ suggesting that analysis for $N = 1024$ (i.e., $k = 7$) was only approximate. Here we would get $k_{1024} = \frac{55}{8} = 6.875$. 
Conclusion

Padding with zeros allowed us to compute more samples in frequency.

The greater number of samples better illustrate the shape of the window function that governs the between-frequency interpolation. Regardless of the analysis width, resolution of the signal is determined by a square window of length 1024, since there are only 1024 samples of data available in any of the preceding analyses.

In order to increase **frequency resolution**, we need to include more data.
Longer Sequence

Original (N=1024).

$x_5[n]$  $|X_5[k]|$
Longer Sequence

Lengthen by a factor of 2 ($N=2048$).

\[ x_6[n] \]

\[ |X_6[k]| \]
Longer Sequence

Lengthen by a factor of 4 (N=4096).
Longer Sequence

Lengthen by a factor of 8 (N=8192).

Switching again to line plots ...
Longer Sequence

Original (N=1024).
Longer Sequence

Lengthen by a factor of 2 (N=2048).
Longer Sequence

Lengthen by a factor of 4 ($N=4096$).

\[ |X_7[k]| \]
Longer Sequence

Lengthen by a factor of 8 (N=8192).

\[ |X_8[k]| \]
Longer Sequence

Lengthen by a factor of 16 (N=16,384).

\[ |X_9[k]| \]
Longer Sequence

Lengthen by a factor of 32 (N=32,768).

Clear peaks at $k = 217$ and $k = 228$ ($f = 292.04$ Hz and $f = 306.85$ Hz).

Notice that these are the second harmonics of lower frequencies.

The fundamental components were not clearly resolved with N=1024 but are clear with N=32,768.
Increasing the length of the analysis increases the number of frequencies that result.

Increasing the length of the analysis by zero padding can better illustrate the shape of the window function that governs the between-frequency interpolation.

However, it does not increase frequency resolution.

To increase frequency resolution we must include more data.
Improving Frequency Resolution

The DFT provides integer resolution in $k$.

To improve frequency resolution, increase length of analysis window.

Two methods

- Zero padding
  - better assessment of peak frequency.
  - does not improve resolution of multiple components
- Longer sequence
  - improves resolution of multiple components
  - averages frequency content over longer period (bad if signal changes with time)