Discrete Fourier Transform (DFT)

Definition and comparison to other Fourier representations.

**DFT:**
\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} kn}
\]

**DTFS:**
\[
X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi}{N} kn}
\]

**DTFT:**
\[
X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}
\]

\[x[n] = \sum_{k=\langle N \rangle} X[k] e^{j\frac{2\pi}{N} kn}\]

\[x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega\]
Analyzing Frequency Content of Arbitrary Signals

Why use a DFT?

- Fourier Series: conceptually simple, but limited to periodic signals.
- Fourier Transforms: arbitrary signals, but continuous domain \((\omega, \Omega)\).
- Discrete Fourier Transform: arbitrary DT signals (finite length)
  - Discrete in both domains: nice for computation
  - FFT: Efficient means of computation

Today: using the DFT to analyze frequency content of a signal.
Create three signals of the following form:

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]
\[ x_4[n] = \cos\left(\frac{9\pi n}{100} - \frac{\pi}{2}\right) \]

Each should have a duration of 1 second and should use a sample frequency of 8kHz.

Compare the DFTs of the first 100 samples of each of these signals.
What is the frequency of this tone if the sample rate is 8kHz?

\[ f \text{ (cycles/second)} = \frac{\Omega \text{ (radians/sample)} \times \frac{1}{2\pi}}{\text{cycles/radian} \times f_s \text{ (samples/second)}} \]
Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$. 

\[ \begin{array}{c}
\{x[0], x[1], x[2], \ldots, x[N-1]\} \\
\downarrow \\
\{x[0], x[1], x[2], \ldots, x[N-1]\}
\end{array} \]
Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$.

def dft(x):
    N = len(x)
    return [sum(x[n] * e**(-2j*pi*k*n/N) for n in range(N)) / N
            for k in range(N)]

X1 = dft(x[0:100])
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$. 
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$.

$$X_1[k] = X_1[k+N]$$

$$X_1[k] = DFT\{x_1[0 : 100]\}$$

Which values of $k$ are non-zero?
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians in DT, which is equivalent to $(f_s/N)$ Hz in CT.

\[ \omega_0 = \frac{2\pi}{2N} \]

\[ \frac{k}{N/2} = \frac{\omega}{\pi} \]

\[ \frac{K}{N} = \frac{\omega}{2\pi} \]

\[ \omega = \frac{2\pi K}{N} \]
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
Compare Two Signals

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]

No difference in magnitudes.
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

- $x_1[n] = \cos(8\pi n/100)$
- $x_3[n] = \cos(9\pi n/100)$

$k = \pm 4$

$k = \pm 4.5$
**Compare Two Signals**

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]

**Why are these DFTs so different?**
Compare Two Signals

\[ x_1[n] = \cos \left( \frac{8\pi n}{100} \right) \]

\[ x_3[n] = \cos \left( \frac{9\pi n}{100} \right) \]

\[ |X_1[k]| \]

\[ |X_3[k]| \]

\[ \Omega_1 \neq \Omega_3. \] Even more importantly, \( x_3[n] \) is not periodic in \( N = 100 \)!
**Single Sinusoid**

This blurring occurs because the signal is not periodic in the analysis window \((N = 100)\).

\[
x_3[n] = \cos(\frac{9\pi n}{100})
\]

What value of \(k\) corresponds to \(\Omega = \frac{9\pi}{100}\)?
This blurring occurs because the signal is not periodic in the analysis window \((N = 100)\).

\[
x_3[n] = \cos\left(\frac{9\pi n}{100}\right)
\]

What value of \(k\) corresponds to \(\Omega = \frac{9\pi}{100}\)?

\[
\Omega = \frac{9\pi}{100} = \frac{2\pi k}{N}
\]

\(k = 4.5\)

The signal frequency fell between the analysis frequencies.
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

- $x_3[n] = \cos(9\pi n/100)$
- $x_4[n] = \cos(9\pi n/100 - \pi/2)$
Compare Two Signals

\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]

\[ x_4[n] = \cos\left(\frac{9\pi n}{100} - \frac{\pi}{2}\right) \]

\[ \Omega_3 = \Omega_4. \text{ But DC bigger. Higher frequencies smaller. Why?} \]
Analyzing Signals with Multiple Frequencies

What is the minimum window size $N$ needed to resolve $\Omega = \frac{8\pi}{100}$ from $\frac{9\pi}{100}$?

$$x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right)$$
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 100 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 100 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 200 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 200 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

\[ |X_5[k]| \]
Two frequencies can look like one if analysis window is too small.

\[ N = 400 \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]
Analyzing Signals with Multiple Frequencies

Two frequencies can look like one if analysis window is too small.

\[ N = 400 \text{ zoomed} \]

\[ x_5[n] = \cos\left(\frac{8\pi n}{100}\right) + \cos\left(\frac{9\pi n}{100}\right) \]

These frequencies are barely resolved with \( N = 200 \).
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians in DT, which is equivalent to $(f_s/N)$ Hz in CT.

\[
\begin{array}{cccc}
-\pi & 0 & \pi & \Omega \text{ [rad/sample]} \\
-N/2 & N/2 & k & \\
-fs/2 & fs/2 & f & \text{[Hz]} \\
\end{array}
\]
Two frequencies are resolved if they are separated by more than $\frac{2\pi}{N}$.

$\Omega_1 = \frac{8\pi}{100}$ and $\Omega_2 = \frac{9\pi}{100}$ will be resolved if

$$\Delta\Omega = \Omega_2 - \Omega_1 = \frac{9\pi}{100} - \frac{8\pi}{100} = \frac{\pi}{100} > \frac{2\pi}{N}$$

That is, if $N > 200$.

We can think of $\frac{2\pi}{N}$ as the frequency resolution of the DFT.

Notice 8 full cycles of $\Omega_1$ and 9 full cycles of $\Omega_2$ fit in $N = 200$. 
Check Yourself!

For a portion of the Chopin song containing only one chord, 250332 samples long, and recorded with a sampling rate of 48kHz, the DFT magnitudes look like:

Peaks in magnitude around $k \approx 1021, 1282, 1715, 2037, 2576, 3062, \ldots$

What are the frequencies (in Hz) of the notes being played?

What chord does this correspond to?