6.003: Signal Processing

Discrete Fourier Transform (DFT)

Definition and comparison to other Fourier representations.

**analysis**

\[
DFT: \quad X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}
\]

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} kn}
\]

**synthesis**

\[
DTFS: \quad X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi}{N} kn}
\]

\[
x[n] = \sum_{k=\langle N \rangle} X[k] e^{j \frac{2\pi}{N} kn}
\]

\[
DTFT: \quad X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \Omega n}
\]

\[
x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j \Omega n} d\Omega
\]

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Analyzing Frequency Content of Arbitrary Signals

Why use a DFT?

Fourier Series: conceptually simple, but limited to periodic signals.

Fourier Transforms: arbitrary signals, but continuous domain \((\omega, \Omega)\).

Discrete Fourier Transform: arbitrary DT signals (finite length)
  - Discrete in both domains: nice for computation
  - FFT: Efficient means of computation

Today: using the DFT to analyze frequency content of a signal.
Create three signals of the following form:

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]
\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
\[ x_3[n] = \cos\left(\frac{9\pi n}{100}\right) \]
\[ x_4[n] = \cos\left(\frac{9\pi n}{100} - \frac{\pi}{2}\right) \]

Each should have a duration of 1 second and should use a sample frequency of 8kHz.

Compare the DFTs of the first 100 samples of each of these signals.
Single Sinusoid

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

What is the frequency of this tone if the sample rate is 8kHz?
**Single Sinusoid**

Write a program to calculate the DFT of an input sequence.

Use that program to calculate $X_1[k]$, which is the DFT of $x_1[n]$. 
Single Sinusoid

Plot the magnitude of $X_1[\cdot]$. 
Frequency Scales

We can think of the DFT as having spectral resolution of \((2\pi/N)\) radians in DT, which is equivalent to \((f_s/N)\) Hz in CT.
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

\[ x_1[n] = \cos\left(\frac{8\pi n}{100}\right) \]

\[ x_2[n] = \cos\left(\frac{8\pi n}{100} - \frac{\pi}{4}\right) \]
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

$$x_1[n] = \cos\left(\frac{8\pi n}{100}\right)$$

$$x_3[n] = \cos\left(\frac{9\pi n}{100}\right)$$
Compare Two Signals

How will plots of DFT magnitudes differ for the following signals?

- $x_3[n] = \cos(9\pi n/100)$
- $x_4[n] = \cos(9\pi n/100 - \pi/2)$
What is the minimum window size $N$ needed to resolve $\Omega = \frac{8\pi}{100}$ from $\frac{9\pi}{100}$?

$$x_5[n] = \cos \left( \frac{8\pi n}{100} \right) + \cos \left( \frac{9\pi n}{100} \right)$$
Frequency Scales

We can think of the DFT as having spectral resolution of $(2\pi/N)$ radians in DT, which is equivalent to $(f_s/N)$ Hz in CT.

$$\begin{align*}
-\pi & \quad 0 & \quad \pi & \quad \Omega \quad \text{[rad/sample]} \\
-N/2 & \quad N/2 & \quad k \\
-fs/2 & \quad fs/2 & \quad f \quad \text{[Hz]}
\end{align*}$$
Check Yourself!

For a portion of the Chopin song containing only one chord, 250332 samples long, and recorded with a sampling rate of 48kHz, the DFT magnitudes look like:

Peaks in magnitude around $k \approx 1021, 1282, 1715, 2037, 2576, 3062, \ldots$

What are the frequencies (in Hz) of the notes being played?
What chord does this correspond to?