Sinusoidal Signals

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Signals

6.003 is about **signal processing**.

Abstractly, a *signal* is a function that conveys information.

Examples:

- medical (EKG, EEG, MRI, …)
- speech signals
- music
- seismic signals
- images
- video
Signal processing is about extracting meaningful information from signals, and/or manipulating information in signals to produce new signals.
Signals are functions that convey information. Sometimes, though, the straightforward way of representing a signal may not expose important properties of the signal.

It is often useful to have multiple different ways of looking at a signal.
For example, consider the following two representations of a speech signal:
We call such an alternative view of a signal a transform.

In 6.003, we will focus primarily on a family of transforms referred to as Fourier transforms, where signals are represented as sums of sinusoids, for example:

\[ f(t) = \sum_{k=0}^{\infty} (c_k \cos k\omega_0 t + d_k \sin k\omega_0 t) \]

Sinusoids are interesting because they are prevalent in nature (and human perception), and because they have some nice mathematical properties.

As such, much of our focus throughout 6.003 will be on sinusoidal functions (and, eventually, complex exponentials).
A periodic signal is one that repeats after some amount of time $T$, such that, for all times $t$,

$$\forall t, x(t) = x(t + T)$$

A function that is periodic in $T$ is also periodic in $2T$, $3T$, $4T$, .... For a given signal, the smallest value $T$ for which the above holds is called that signal’s fundamental period.

The associated frequency, $\omega_0 = \frac{2\pi}{T}$, is the signal’s fundamental frequency.

Frequencies are harmonically related if they are integer multiples of some fundamental frequency.

$\cos (2\omega_0 t)$  $\sin (3\omega_0 t)$
In general, a CT sinusoid has the form:

$$x(t) = A \cos(\omega t + \phi)$$

- $A$ is referred to as the amplitude
- $\omega$ is referred to as the radian frequency
- $\phi$ is referred to as a phase offset

units of $\omega$: radians/second
units of $A$: ?
units of $x(t)$: ?
$\omega$: radians/second
$\phi$: radians

$2\pi$ radians/cycle
$f = \frac{\text{cycles}}{\text{second}}$
Check Yourself: Frequency

Consider \( f(t) = \cos(\omega_1 t) \), shown below:

Clicker: What is the value of \( \omega_1 \)?

A. \( \frac{1}{2} \)  
B. 1  
C. 2  
D. \( \frac{\pi}{2} \)  
E. \( \pi \)  
F. \( 2\pi \)

In the chat: What are the units of \( \omega_1 \)?
Check Yourself: Phase

Consider a small, positive value \( a \), where \( 0 \leq a \leq \pi \). Does the following represent \( \cos(\omega_1 t + a) \) or \( \cos(\omega_1 t - a) \)?

**Clicker:**

A. \( \cos(\omega_1 t + a) \)

B. \( \cos(\omega_1 t - a) \)
Check Yourself: Sine Waves

In general, a CT sinusoid has the form:

\[ x(t) = A \cos(\omega t + \phi) = A \cos\left(\frac{2\pi}{T} t + \phi\right) \]

Can we represent \( \sin(\omega t) \) in this form?

\[ \cos(\omega t + \frac{\pi}{2}) \]

Clicker:

A. Yes
B. No

\[ A = 1 \]
\[ \omega = \omega \]
\[ \phi = \frac{-\pi}{2} \]

If so, how? If not, why not?
Consider the following signal, consisting of the sum of two sinusoids:

\[ f(t) = \cos \left( \frac{2\pi}{3} t + \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{2} t \right) \]

Is \( f(\cdot) \) periodic?

\[ f(t) = f(t+T) \]

**Clicker:**

A. Yes  
B. No

\[ \frac{\omega_1}{\omega_2} \text{ must be rational} \]

\[ 12 \text{ sec} \]

\[ w_1 = 2, \quad w_2 = 52 \]

\[ g(t) = \cos(w_1 t + \phi) + \cos(w_2 t + \phi) \]

Is \( g(\cdot) \) necessarily periodic?
Summing Sinusoids

What about the following signal?

\[ f_2(t) = \cos(2\pi t) + 0.5 \cos(2\pi t + \pi/4) \]

- \( f_2 \) periodic?  what period?  \( T = 1 \)
- shape of \( f_2(t) \)?
What is the result of evaluating the following definite integral?

\[ \int_{t_0}^{t_0+T} \cos \left( \frac{2\pi}{T} t \right) \, dt = ? \]
Product of Harmonically-related sinusoids

What is the result of evaluating the following definite integral, where \( k \) and \( m \) are positive integers?

(Hint: it might help to look up some trig identities!)

\[
\int_{t_0}^{t_0+T} \left( \sin \left( \frac{2\pi k}{T} t \right) \cos \left( \frac{2\pi m}{T} t \right) \right) \, dt = ?
\]

\[
\sin(v) \cos(v) = \frac{1}{2} \left( \sin(v+v) + \sin(v-v) \right)
\]

\[
\int_{t_0}^{t_0+T} \frac{1}{2} \left( \sin \left( \frac{2\pi (k+m)}{T} t \right) \right) \, dt + \int_{t_0}^{t_0+T} \frac{1}{2} \left( \sin \left( \frac{2\pi (k-m)}{T} t \right) \right) \, dt
\]
Product of Harmonically-related sinusoids

What is the result of evaluating the following definite integral, where \( k \) and \( m \) are positive integers?

\[
\int_{t_0}^{t_0+T} \left( \cos \left( \frac{2\pi k}{T} t \right) \cos \left( \frac{2\pi m}{T} t \right) \right) \, dt = ?
\]

\[
= \begin{cases} 
T/2 & \text{if } k=m \\
0 & \text{otherwise}
\end{cases}
\]
DT Sinusoids

In general, a DT sinusoid has the form:

$$x[n] = A \cos(\Omega n + \phi)$$

- $A$ is referred to as the amplitude
- $\Omega$ is referred to as the discrete frequency
- $\phi$ is referred to as a phase offset

Importantly, $n$ is always an integer!

(this has some really interesting consequences, which we’ll explore in more detail in lecture 1B)

$$f = \frac{cycles}{second} \quad f_s = \frac{samples}{seconds}$$