6.003 Signal Processing

Week 14, Lecture A: Wrapping Up
Wrapping up

• What we have learned

• What is next
  ➢ Where you could be using what we learned
  ➢ Future classes to take (Recitation)
CT signals, DT signals, sampling

A CT signal $x(t) = \cos(\omega t)$ sampled at $t = n\Delta T$, the resulting DT signal $x[n] = \cos(\Omega n)$ with $\Omega = \omega \Delta T$

$$x(t) = \cos(\omega t) \quad \xrightarrow{\Omega = \omega/f_s} \quad x[n] = \cos(\Omega n)$$

### Aliasing and Nyquist frequency:

$$x[n] = \cos(\Omega n) = \cos((\Omega + 2\pi)n) = \cos((\Omega + 2k\pi)n)$$

Nyquist frequency: $\frac{1}{2} f_s$

- when the highest frequency of a signal is less than the Nyquist frequency, the resulting DT signal is free of aliasing.
- Or, the sampling rate need to be larger than twice the highest frequency in the signal to prevent aliasing

What we have learned
**Fourier Representations**

Signals: periodic vs aperiodic  
continuous vs discrete

<table>
<thead>
<tr>
<th>Time domain</th>
<th>Frequency domain</th>
<th>CTFS</th>
<th>DTFS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodic,</td>
<td>Discrete</td>
<td>$x(t) = x(t + T) = \sum_{k=\infty}^{\infty} X[k] e^{j2\pi k t / T}$</td>
<td>$x[n] = x[n + N] = \sum_{k=\infty}^{\infty} X[k] e^{j\Omega_0 k n}$</td>
</tr>
<tr>
<td>Discrete</td>
<td></td>
<td>$X[k] = \frac{1}{T} \int_T x(t) e^{-j2\pi k t / T} dt$</td>
<td>$X[k] = X[k + N] = \frac{1}{N} \sum_{n=\infty}^{\infty} x[n] e^{-j\Omega_0 k n}$</td>
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</tbody>
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<tr>
<td>Aperiodic,</td>
<td>Continuous</td>
<td>$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$</td>
<td>$x[n] = \frac{1}{2\pi} \int_{2\pi}^{\infty} X(\Omega) \cdot e^{j\Omega n} d\Omega$</td>
</tr>
<tr>
<td>Continuous</td>
<td></td>
<td>$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$</td>
<td>$X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$</td>
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**Synthesis Equation**: reconstruct signal from Fourier components  
**Analysis Equation**: Finding the Fourier components

What we have learned
# Properties of Fourier Transforms

## Continuous-Time Fourier Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>$y(t)$</th>
<th>$Y(\omega)$</th>
</tr>
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<tbody>
<tr>
<td><strong>Linearity</strong></td>
<td>$ax_1(t) + bx_2(t)$</td>
<td>$aX_1(\omega) + bX_2(\omega)$</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x(-t)$</td>
<td>$X(-\omega)$</td>
</tr>
<tr>
<td>Time delay</td>
<td>$x(t - t_0)$</td>
<td>$e^{-j\omega t_0}X(\omega)$</td>
</tr>
<tr>
<td>Conjugation</td>
<td>$x^*(t)$</td>
<td>$X^*(-\omega)$</td>
</tr>
<tr>
<td>Scaling time</td>
<td>$x(at)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>Time derivative</td>
<td>$\frac{dx(t)}{dt}$</td>
<td>$j\omega X(\omega)$</td>
</tr>
<tr>
<td>Frequency derivative</td>
<td>$tx(t)$</td>
<td>$j\frac{d}{d\omega}X(\omega)$</td>
</tr>
</tbody>
</table>

## Discrete-Time Fourier Transform

<table>
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<tr>
<th>Property</th>
<th>$y[n]$</th>
<th>$Y(\Omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Linearity</strong></td>
<td>$ax_1[n] + bx_2[n]$</td>
<td>$aX_1(\Omega) + bX_2(\Omega)$</td>
</tr>
<tr>
<td>Time reversal</td>
<td>$x[-n]$</td>
<td>$X(-\Omega)$</td>
</tr>
<tr>
<td>Time delay</td>
<td>$x[n-n_0]$</td>
<td>$e^{-j\Omega n_0}X(\Omega)$</td>
</tr>
<tr>
<td>Conjugation</td>
<td>$x^*[n]$</td>
<td>$X^*(-\Omega)$</td>
</tr>
<tr>
<td>Frequency derivative</td>
<td>$nx[n]$</td>
<td>$j\frac{d}{d\Omega}X(\Omega)$</td>
</tr>
</tbody>
</table>

What we have learned
Useful Signals and Their Fourier Transforms

\[ X(\Omega) = 2 \frac{\sin(\omega S)}{\omega} \]

\[ P_S(\Omega) = \frac{\sin(\Omega(S + \frac{1}{2}))}{\sin(\frac{\Omega}{2})} \]

\[ x(t) = e^{j\omega_0 t} \quad \text{CTFT} \quad X(\omega) = 2\pi \delta(\omega - \omega_0) \]

\[ x[n] = e^{j\Omega_0 n} \quad \text{DTFT} \quad X(\Omega) = \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_0 + 2\pi m) \]

\[ e^{j\Omega_0 n} \quad \text{DTFT} \quad X(\Omega) \]

\[ \delta(t) \quad \text{CTFT} \quad 1 \quad X(\omega) \]

\[ \delta[n] \quad \text{DTFT} \quad X(\Omega) = 1 \]

\[ \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - 2\pi m) \]

Duality

If \( x(t) \quad \text{CTFT} \quad X(\omega) \) then \( X(t) \quad \text{CTFT} \quad 2\pi x(-\omega) \)
Discrete Fourier Transform (DFT)

\[ x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n} \]
\[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n} \]

DFT (especially efficient FFT algorithm) significantly facilitate computation.

### Frequency Resolution in DFT:

\[ \Omega = \frac{2\pi k}{N} \]
\[ \Omega = \frac{2\pi f}{f_s} \]

- \(-N/2\)
- \(-\pi\)
- \(-f_s/2\)
- \(0\)
- \(N/2\)
- \(\pi\)
- \(f_s/2\)

\(k: \) integer (frequency)
\(\Omega: \) rad/sample
\(f: \) cycles/second (Hz)

Connection with DTFS and DTFT:

- with DTFS: DFT is the DTFS of periodically extended version, \(x_p[n]\)
- with DTFT:

\[ X[k] = \frac{1}{N} X_w(\frac{2\pi k}{N}) \]
\[ X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]w[n] \cdot e^{-j\Omega n} \]
Three different representations for **Linear, Time-Invariant (LTI) systems**: 

**Difference/Differential Equation:**
\[
y[n] - \alpha y[n-1] = x[n], \quad 0 < \alpha < 1
\]
\[
y(t) + \alpha \frac{dy(t)}{dt} = x(t), \quad \alpha > 0
\]

**Convolution:**
\[
y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]
\]
\[
y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)d\tau
\]

**Filter:**
\[
e^{j\Omega n} \rightarrow \text{LTI} \rightarrow H(\Omega) e^{j\Omega n}
\]
\[
Y(\Omega) = H(\Omega)X(\Omega)
\]

We can carry out filtering either in time domain or frequency domain. 
Freq. domain multiplication correspond to time domain (circular) convolution:

\[
(x * h)[n] \overset{DTFT}{\leftrightarrow} H(\Omega)X(\Omega)
\]
\[
\frac{1}{N} \left( x \ast h \right)[n] \overset{DFT}{\leftrightarrow} H[k]X[k]
\]

What we have learned
2D Signal Processing (I)

2D DFT:

\[
F[k_r, k_c] = \frac{1}{RC} \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} f[r, c] \cdot e^{-j(\frac{2\pi k_r r}{R} + \frac{2\pi k_c c}{C})}
\]

\[
f[r, c] = \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} F[k_r, k_c] \cdot e^{-j(\frac{2\pi k_r r}{R} + \frac{2\pi k_c c}{C})}
\]

Multiple 2D Fourier Transform pairs:
- 2D unit sample => 2D constant
- 2D constant => 2D unit sample
- vertical line => horizontal line
- horizontal line => vertical line

The concept of circular convolution:

\[(f_b \circledast f_a)[n] = (f_b \ast f_{ap})[n], f_{ap}[n] = \sum_{m=-\infty}^{\infty} f_a[n - mN]\]

2D convolution:
- Direct extension from 1D convolution.
- When DFT is used to implement convolution, the result is circular convolution.

Rotating an image rotates its Fourier transform by the same angle.
2D vs. 1D: In many aspects the signal processing in 2D is direct extension from 1D scenario. But there are important new learnings:

- The phase of the Fourier transform is crucial to representing sharp edges in an image.
- Low frequency vs high frequency: human perception.

Inverse Filtering: Important for many applications. Challenges:
- We don’t know what $H[k_r, k_c]$ is! Need to have a good guess
- Noise could be introduced in the process
- $H[k_r, k_c] = 0$ for some of the $k_r, k_c$ values

What we have learned
Applications (I)

Short-time Fourier Transform:
- Streaming applications
- Spectrogram

Speech
- Source-Filter model

What we have learned
Applications (II)

**Fourier-based Compression:**

Discrete Cosine Transform (DCT)

The DFT of the folded, stretched, and doubled signal is the DCT of the original function.

JPEG Encoding

**Magnetic Resonance Imaging:**

Gradient Fields for Frequency Encoding of Spatial Information:

\[
F_c[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right)
\]

\[
f[n] = F_c[0] + 2 \sum_{k=1}^{N-1} F_c[k] \cos \left( \frac{\pi k}{N} \left( n + \frac{1}{2} \right) \right)
\]

What we have learned
Modulation is an effective way to use frequencies in one band to communicate messages in a different band.

Communications:

**Modulation** is an effective way to use frequencies in one band to communicate messages in a different band.

\[ x(t) \quad \times \quad y(t) \]

\[ \cos(\omega_c t) \]

\[ X_1(\omega) \]

\[ X_2(\omega) \]

\[ X_3(\omega) \]

\[ Z(\omega) \]

\[ \omega_1, \omega_2, \omega_3 \]

\[ \omega \]

\[ \omega_c \]

\[ -\omega_c \]

\[ \pi \]

\[ C(\omega) = \pi \delta(\omega - \omega_c) + \pi \delta(\omega + \omega_c) \]

\[ \omega \]

\[ \omega_c \]

\[ -\omega_c \]

\[ \pi \]

\[ Y(\omega) = \frac{1}{2\pi} (X*C)(\omega) \]

\[ \omega \]

\[ \omega_c \]

\[ -\omega_c \]

\[ |Z(\omega)| \]

\[ -2\omega_c \]

\[ 2\omega_c \]

What we have learned
The image shows a bright ring formed as light bends in the intense gravity around a black hole that is 6.5 billion times more massive than the Sun. Image credit: Event Horizon Telescope Collaboration

Fourier Optics

Blurring is inversely related to the diameter of the lens.

Numerical Aperture: $NA = n \sin \theta$. $n$ is refractive index.
Fourier Optics

Considering solving the homogeneous, scalar wave equation in source-free regions:

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) u(r, t) = 0$$

where $u(r, t)$ is a real valued Cartesian component of an electromagnetic wave propagating through free space.

If considering a single frequency, the time-harmonic form of the optical field can be written as:

$$u(r, t) = \text{Re}\{\psi(r)e^{j\omega t}\}$$

Let $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ and solve the wave equation giving rise to:

$$\psi(r) = \psi(x, y, z) = e^{j(k_xx + k_yy + k_zz)} = e^{j(k_xx + k_yy)}e^{jk_zz} = e^{j(k_xx + k_yy)}e^{\pm jz\sqrt{k^2-k_x^2-k_y^2}}$$

this represents a propagating or exponentially decaying uniform plane wave solution to the wave equation.

A general solution: a weighted superposition of all possible elementary plane wave solutions:

$$\psi(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_0(k_x, k_y) e^{j(k_xx + k_yy)} e^{\pm jz\sqrt{k^2-k_x^2-k_y^2}} dk_x dk_y$$

$$\psi_0(x, y) = \psi(x, y, z) \big|_{z=0} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_0(k_x, k_y) e^{j(k_xx + k_yy)} dk_x dk_y$$

← basic foundation of Fourier optics
Diffraction

Graph and image of single-slit diffraction.

Infinitely many points (three shown) along length $d$ project phase contributions from the wavefront, producing a continuously varying intensity $\theta$ on the registering plate.

A diffraction pattern of a red laser beam projected onto a plate after passing through a small circular aperture in another plate.

Computer generated intensity pattern formed on a screen by diffraction from a square aperture.

Electron diffraction patterns of the Al$_{65}$Cu$_{15}$Co$_{20}$ alloy; a) ten-fold, b) two-fold and c) pseudo two-fold axes, d) [111] zone axes from the cubic phase, e) as-cast Al$_{65}$Cu$_{15}$Co$_{20}$ alloy showing characteristic decaprism of the decagonal phase.


DOI: 10.1557/PROC-1242-S4-P116

Fourier Optics

If a target is located in the focal plane of a lens, light from a point on the target forms a plane wave as it passes through the lens.

If the target point lies on the axis of the lens, then the plane wave is perpendicular to the imaging plane.

If the target point lies off the axis of the lens, then the plane wave is no longer perpendicular to the image plane. The light striking the image plane has linearly increasing phase delay with distance.
Fourier Optics

Light from the point $x = 0$ generates a plane wave, that is everywhere in phase at the imaging plane.

Light from $x = x_0$ generates a plane wave with linearly increasing phase lag.
Fourier Optics

The target can be described as a collection of point sources of light: \[ f(x) = \int f(x_0) \delta(x - x_0) \, dx_0 \]

And the result in the image plane is a superposition of plane waves, one for each point in the target:

\[ F(\omega) = \int f(x) e^{-j\omega x} \, dx \]

\( F(\omega) \) is the Fourier transform of \( f(x) \).
Fourier Optics

If an object is more than one focal distance from the lens, then the light converges to create an image of the object in the image plane.

This is equivalent to two lenses: one located a focal distance from the object and one located a focal distance from the image.
Fourier Optics

Now the Fourier transform relation holds for both halves of the system.

\[
F(\omega) = \int f(x) e^{-j\omega x} dx
\]

\[
f'(x') = \frac{1}{2\pi} \int F(\omega) e^{j\omega x'} d\omega
\]

Ideally, both limits of integration would be infinite. However the finite diameter of the lens limits the highest frequencies $|\omega|$.

Light emanating from the target at large angles is not captured by the lens.

As a result, the image at $x'$ is a lowpass version of the target at $x$. 
6.003 Approach to Increased Resolution

Phase modulated Microscopy:

Poster: \( \cos(\omega_c y + f(x,y)) \)

Projector: \( \cos(\omega_c y) \)

Poster with Projector: \( \cos(\omega_c y) \cos(\omega_c y + f(x,y)) \)

Standing wave illumination Spectrum:

Slide courtesy of Prof. Denny Freeman and his research group.
Summary

Fourier transforms are important in many science and engineering branches: physics, mathematics, material science, chemistry, biology, clinical imaging, electrical engineering, computer science....

In this course we’ve barely scratched the surface. There are many interesting applications that built upon what we have learned here.

Enjoy your future explorations!