6.003 Signal Processing

Week 5, Lecture B:
Discrete Fourier Transform (II)
Discrete Fourier Transform

A new Fourier representation for DT signals:

\[ x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n} \quad \text{Synthesis equation} \]

\[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n} \quad \text{Analysis equation} \]

The DFT has a number of features that make it particular convenient

- It is not limited to periodic signals.
- It is discrete in both domains, making it computationally feasible

The FFT (Fast Fourier Transform) is an algorithm for computing the DFT efficiently.
Two Ways to Think About DFT

We can think about the DFT in two different ways:

1. Think about DFT as **Fourier series** of N samples of the signal, periodically extended.

   \[ X_0[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \]

   We can see why DFT of a single sinusoid is not concentrated in a single k component.
Two Ways to Think About DFT

We can think about the DFT in two different ways:

1. Think about DFT as **Fourier series** of N samples of the signal, periodically extended.

2. Think about DFT as the scaled **Fourier transform** of a “windowed” version of the original signal.
DFT: Relation to DTFT

\[ x[n] \]

\[
x_w[n] = x[n]w[n]
\]

Window \( w[n] = \begin{cases} 
1 & 0 \leq n < N \\
0 & \text{otherwise}
\end{cases} \)

\[
X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\Omega n}
\]

\[ X[k] = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right) \]

Sample: \( \Omega \rightarrow \frac{2\pi k}{N} \)

Scale: \( 1/N \)
Effect of Windowing on Fourier Representations

Example: complex exponential signal

\[ x[n] = e^{j\Omega_0 n} \]

Compute the DTFT:

We need to find \( X(\Omega) \) such that:

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega = e^{j\Omega_0 n}
\]

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega n} \, d\Omega = e^{j\Omega_0 n}
\]

\[
X(\Omega) = \sum_{m=-\infty}^{\infty} 2\pi \delta(\Omega - \Omega_0 + 2\pi m)
\]
Effect of Windowing on Fourier Representations

Apply a rectangular window \( w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \)

to the complex exponential signal \( x[n] = e^{j\Omega_0 n} \)

such that \( x_w[n] = x[n] \cdot w[n] = e^{j\Omega_0 n}w[n] \)

What the Fourier Transform \( X_w(\Omega) \) look like?

\[
X_w(\Omega) = \sum_{n=-\infty}^{\infty} x_w[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} e^{j\Omega_0 n}w[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} w[n]e^{-j(\Omega - \Omega_0)n} = W(\Omega - \Omega_0)
\]

\( e^{j\Omega_0 n}w[n] \quad \overset{DTFT}{\longleftrightarrow} \quad W(\Omega - \Omega_0) \)
Effect of Windowing on Fourier Representations

Let \( w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \)

As shown below for N=15

What is the DTFT of \( w[n] \)?

\[
W(\Omega) = \sum_{n=-\infty}^{\infty} w[n] e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n}
\]

If \( \Omega = 0 \), \( W(\Omega = 0) = N \)

If \( \Omega \neq 0 \):

\[
W(\Omega) = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = \frac{e^{-j\Omega \frac{N}{2}}(e^{j\Omega \frac{N}{2}} - e^{-j\Omega \frac{N}{2}})}{e^{-j\Omega \frac{1}{2}}(e^{j\Omega \frac{1}{2}} - e^{-j\Omega \frac{1}{2}})} = \frac{\sin(\Omega \frac{N}{2})}{\sin(\Omega \frac{1}{2})} e^{-j\Omega \frac{N-1}{2}}
\]
From Lecture 04B:

\[ P_S(\Omega) = \frac{\sin(\Omega(S + \frac{1}{2})}{\sin(\frac{\Omega}{2})} \]
Effect of Windowing on Fourier Representations

Let \( w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \) As shown below for \( N=15 \)

What is the DTFT of \( w[n] \)?

\[
W(\Omega) = \sum_{n=-\infty}^{\infty} w[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} e^{-j\Omega n} = \begin{cases} N & \Omega = 0 \\ \frac{\sin(\Omega \frac{N}{2})}{\sin(\Omega \frac{1}{2})} e^{-j\Omega \frac{N-1}{2}} & \Omega \neq 0 \end{cases}
\]
Effect of Windowing on Fourier Representations

The effect of windowing can be seen from this example of complex exponential signal:

\[ x_w[n] = e^{j\Omega_0 n} w[n] \quad \text{DTFT} \quad X_w(\Omega) = W(\Omega - \Omega_0) \]

The frequency content of \( X(\Omega) \) is at discrete frequencies
\[ \Omega = \Omega_0 + 2\pi m \]

The frequency content of \( X_w(\Omega) \) is most dense at these same frequencies, but is spread out over almost all other frequencies as well.

Effect of windowing: spectrum smear
DFT: Relation to DTFT

\[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n} \]

\[ X[k] = \frac{1}{N} X_w\left(\frac{2\pi k}{N}\right) \]

Window:
\[ w[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases} \]

\[ x[n] = x[n]w[n] \]

DTFT:
\[ X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\Omega n} \]

Sample:
\[ \Omega \rightarrow \frac{2\pi k}{N} \]

Scale:
\[ 1/N \]
Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n}w[n]$ with $\Omega_0 = \frac{2\pi}{15}$.

One sample is taken at the peak, and the others fall on zeros. Because the signal is periodic within the analysis window $N$. 

$$\Omega = \frac{2\pi k}{N}$$
Effect of Windowing on Fourier Representations

Considering the example of \( x[n] = e^{j\Omega_0 n} \) and \( x_w[n] = e^{j\Omega_0 n} w[n] \) with \( \Omega_0 = \frac{4\pi}{15} \):

One sample is taken at the peak, and the others fall on zeros. Because the signal is periodic within the analysis window \( N \).

\[ \Omega = \frac{2\pi k}{N} \]
Effect of Windowing on Fourier Representations

Considering the example of $x[n] = e^{j\Omega_0 n}$ and $x_w[n] = e^{j\Omega_0 n}w[n]$ with $\Omega_0 = \frac{3\pi}{15}$.

Now none of the samples fall on zeros.
Effect of Windowing on Fourier Representations

Considering the example of \( x[n] = e^{j\Omega_0 n} \) and \( x_w[n] = e^{j\Omega_0 n} w[n] \) with \( \Omega_0 = \frac{2\pi}{15} \):

**original signal**

\[
x[n] = e^{j\Omega_0 n}
\]

**windowed**

\[
x_w[n] = x[n]w[n]
\]

**sampled and scaled**

\[
x_w[n] = x[n]w[n]
\]

Generally, the relation between the samples is complicated.
Spectral Blurring & Frequency Resolution

Longer windows provide finer frequency resolution.

\[ W(\Omega) = \frac{\sin(\Omega \frac{N}{2})}{\sin(\frac{\Omega}{2})} e^{-j\Omega \frac{N-1}{2}} \]

The width of the central lobe is inversely related to window length.
Spectral Blurring & Time/Frequency Tradeoff

However, longer windows provide less temporal resolution.

\[ w[n] = \begin{cases} 
1 & 0 \leq n < N \\
0 & \text{otherwise} 
\end{cases} \]

→ fundamental tradeoff between resolution in frequency and time.
Other types of Windows

Rectangular window

\[ w_1[n] = u[n] - u[n-N] \]

triangular window

\[ w_2[n] = \left( \frac{N/2 - |N/2 - n|}{N/2} \right) w_1[n] \]

Hann (or “Hanning”) window

\[ w_3[n] = \left( 1 - \cos \left( \frac{2\pi n}{N-1} \right) \right) w_1[n] \]
Effects of Different Types of Windows

The DFT coefficients of $x_2[n] = \cos\left(\frac{3\pi}{64} n\right)$ analyzed with $N = 64$:

- Triangular window
- Hann (or “Hanning”) window:
Summary

The Discrete Fourier Transform (DFT)

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n}
\]

**Synthesis equation**

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n}
\]

**Analysis equation**

The two different perspectives looking at DFT and insights gained

The effect of windowing and different types of windows

A central issue in using the DFT is understanding the tradeoff between time and frequency

- Long analysis windows N provide high resolution in frequency but poor resolution in time.
- Short analysis windows N provide high resolution in time but poor resolution in frequency.