Week 5, Lecture A:
Discrete Fourier Transform (I)
### Review

So far, we have learned four transforms, useful for analyzing different kinds of signals:

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<tr>
<th>Transform</th>
<th>Time Domain</th>
<th>Frequency Domain</th>
<th>Formula</th>
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<tbody>
<tr>
<td>CTFS</td>
<td>Periodic, Aperiodic</td>
<td>Continuous, Discrete</td>
<td>( x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} X[k] e^{\frac{j2\pi k t}{T}} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Continuous</td>
<td>( X[k] = \frac{1}{T} \int_{T} x(t) e^{-\frac{j2\pi k t}{T}} dt )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Discrete</td>
<td>( X[k] = X[k + N] = \frac{1}{N} \sum_{n=-N}^{N-1} x[n] e^{-j\Omega_0 kn} )</td>
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<tr>
<td>CTFT</td>
<td>Aperiodic, Continuous</td>
<td>Continuous, Discrete</td>
<td>( x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Continuous</td>
<td>( X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt )</td>
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<tr>
<td></td>
<td></td>
<td>Discrete</td>
<td>( X(\Omega) = X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} )</td>
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Today: Learn another transform: **DFT (Discrete Fourier Transform)**
Toward DFT: why?

Let us look at the DTFT analysis equation:

\[ X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n} \]

If we want to compute this using computers (not analytical solution by hand), two things stand in the way:

• infinite sum
• continuous function of frequency

Solutions:
• only consider a finite number of samples in time, and
• only consider a finite number of frequencies.
Toward DFT:

Start with the DTFT analysis equation:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\Omega n}$$

• only consider N samples
• take N uniformly spaced frequencies from the range $0 \leq \Omega < 2\pi$

The DFT:

$$X[k] = \frac{1}{N} X \left( \frac{2\pi k}{N} \right) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi k}{N}n}$$
DFT: Definition

\[ X[k] = \frac{1}{N} X \left( \frac{2\pi k}{N} \right) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n} \]

\[ x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n} \]

DFT (Discrete Fourier Transform) is discrete in both domains. Computationally feasible (opens doors to analyzing complicated signals).

Most modern signal processing is based on the DFT, and we’ll use the DFT almost exclusively moving forward in 6.003.

The FFT (Fast Fourier Transform) is an algorithm for computing the DFT efficiently.
# DFT: Comparison to Other Fourier Representations

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<tr>
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<th>Analysis</th>
<th>Synthesis</th>
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<tr>
<td><strong>DFT</strong></td>
<td>[X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n}]</td>
<td>[x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n}]</td>
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<tr>
<td><strong>DTFS</strong></td>
<td>[X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi k}{N} n}]</td>
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</tr>
<tr>
<td><strong>DTFT</strong></td>
<td>[X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j \Omega n}]</td>
<td>[x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) \cdot e^{j \Omega n} , d\Omega]</td>
</tr>
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</table>

DTFS: \(x[\cdot]\) is periodic in \(N\)
DTFT: \(x[\cdot]\) is arbitrary
DFT: only a portion of an arbitrary \(x[\cdot]\) is considered, \(N\) is only the analysis window
Relation Between DFT and DTFS

If a signal is periodic in the DFT analysis window $N$, then the DFT coefficients are equal to the DTFS coefficients.

Consider $x_1[n] = \cos \left( \frac{2\pi}{64} n \right)$, when analyzed with $N = 64$, the DFT coefficients are:

$$X_1[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] \cdot e^{-j \frac{2\pi k}{N} n} = \frac{1}{64} \sum_{n=0}^{63} \frac{1}{2} (e^{j \frac{2\pi}{64} n} + e^{-j \frac{2\pi}{64} n}) \cdot e^{-j \frac{2\pi k}{64} n}$$

$$= \frac{1}{2} \cdot \frac{1}{64} \sum_{n=0}^{63} e^{-j \frac{2\pi}{64} (k-1)n} + \frac{1}{2} \cdot \frac{1}{64} \sum_{n=0}^{63} e^{-j \frac{2\pi}{64} (k+1)n} = \frac{1}{2} \delta[k - 1] + \frac{1}{2} \delta[k + 1]$$

The DFT coefficients are the same as the Fourier series coefficients.
Relation Between DFT and DTFS

If a signal is not periodic in the DFT analysis window $N$, then there are no DTFS coefficients to compare.

Consider $x_2[n] = \cos \left( \frac{3\pi}{64} n \right)$, when analyzed with $N = 64$, the DFT coefficients are:

$$X_2[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n}$$

Even though $x_2[n]$ contains a single frequency $\Omega = 3\pi/64$, the DFT coefficients are non-zero at almost every component $k$. 

The reason is that $x_2[n]$ is not periodic with period $N = 64$. 

![Graph showing DFT coefficients](image)
Relation Between DFT and DTFS

Although $x_2[n] = \cos\left(\frac{3\pi}{64}n\right)$ is not periodic in $N=64$, we can define a signal $x_3[n]$ that is equal to $x_2[n]$ for $0 \leq n < 64$ and that is periodic with $N=64$.

The DTFS coefficients of $x_3[n]$ equal the DFT coefficients of $x_2[n]$. The large number of non-zero coefficients are necessary to produce the step discontinuity at $n = 64$.

Basically DFT of a signal with analysis window $N$ is equivalent to take those $N$ samples and generate periodic extensions and do DTFS.
DFT: Relation to DTFT

\[ X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\frac{2\pi k}{N} n} \]

DTFT

\[ X_w(\Omega) = \sum_{n=0}^{N-1} x_w[n] \cdot e^{-j\Omega n} \]

Note: here we only have for DFT because:

\[ x[n] \xrightarrow{DFT} X[k] \xrightarrow{iDFT} x_p[n], \text{ with } x_p[n] \text{ being the periodically extended version of } x[n] \text{ with } 0 \leq n \leq N - 1 \]
Two Ways to Think About DFT

We can think about the DFT in two different ways:

1. The DFT is equal to the DTFS of the periodic extension of the first N samples of a signal.

2. The DFT is equal to (scaled) samples of the DTFT of a “windowed” version of the original signal.

These views are equivalent, but they highlight different phenomena.
Check yourself!

Consider a waveform containing a single, pure sinusoid. This waveform was recorded with a sampling rate of 8kHz, and we have 60 samples of the waveform. Computing the DFT magnitudes, we find:

![DFT Magnitudes](image)

What note is being played? How accurately can we tell?
We only have N distinct samples of the DTFT.

We’re uniformly breaking up a range of $2\pi$ into N discrete samples: the spacing between samples is $2\pi/N$. The $k^{th}$ coefficient is associated with $\Omega = 2\pi k / N$

In Hz, the spacing between samples is $f_s/N$. Thus, the $k^{th}$ coefficient is associated with a frequency of $f = k f_s / N$.

Trade-off: increasing frequency resolution necessarily requires considering more samples of the signal (i.e., increasing N)
Check yourself!

Consider a waveform containing a single, pure sinusoid. This waveform was recorded with a sampling rate of 8kHz, and we have 60 samples of the waveform. Computing the DFT magnitudes, we find:

\[ f = \frac{f_s k}{N} = \frac{8000 \times 2}{60} = 266.7 \, \text{Hz} \]

\[ \text{peak @ } k = 2, -2 \]

Middle C?

Frequency resolution:

\[ f_s = \frac{8000}{60} = 133.3 \]

What note is being played? How accurately can we tell?

How many samples do we need to consider in order to be able to determine the frequency of the tone to within 1Hz? Within 0.1Hz?
DFT: Properties

Very similar in form to the other transforms we’ve seen (particularly the DTFS).

Note: $x[n]$ and $y[n]$ are only defined in the range $0 \leq n \leq N - 1$, but $x_p[n]$ is the periodically extended version using samples of $x[n]$ with $0 \leq n \leq N - 1$.

<table>
<thead>
<tr>
<th>Property</th>
<th>$y[n]$</th>
<th>$Y[k]$</th>
</tr>
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<tbody>
<tr>
<td>Linearity</td>
<td>$A x_1[n] + B x_2[n]$</td>
<td>$A X_1[k] + B X_2[k]$</td>
</tr>
<tr>
<td>Time Reversal</td>
<td>$x_p[N - n]$</td>
<td>$X[-k]$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$x_p[n - n_0]$, or $x[(n - n_0) \mod N]$</td>
<td>$e^{-j\frac{2\pi k_0}{N}n} \cdot X[k]$</td>
</tr>
<tr>
<td>Frequency Shift</td>
<td>$e^{-j\frac{2\pi k_0}{N}n} \cdot x[n]$</td>
<td>$X[k - k_0]$</td>
</tr>
<tr>
<td>Conjugation</td>
<td>$x^*[n]$</td>
<td>$X^*[-k]$</td>
</tr>
</tbody>
</table>
DFT Properties: Time Reversal

If: \[ y[n] = x_p[N - n] \]
then: \[ Y[k] = X[-k] \]

\[
Y[k] = \frac{1}{N} \sum_{n=0}^{N-1} y[n] \cdot e^{-j \frac{2\pi k}{N} n} = \frac{1}{N} \sum_{n=0}^{N-1} x_p[N - n] \cdot e^{-j \frac{2\pi k}{N} n} \quad \text{let } m = N - n, \text{then } n = N - m
\]

\[
Y[k] = \frac{1}{N} \sum_{m=N}^{1} x_p[m] \cdot e^{-j \frac{2\pi k}{N}(N-m)} = \frac{1}{N} \sum_{m=N}^{1} x_p[m] \cdot e^{-j \frac{2\pi k}{N}(-m)}
\]

Since \( x_p[N] = x_p[0] \), the summation of \( m \) goes from \( N \) to \( 1 \) is the same as \( m \) goes from \( 0 \) to \( N - 1 \)

\[
Y[k] = \frac{1}{N} \sum_{m=0}^{N-1} x_p[m] \cdot e^{-j \frac{2\pi k}{N}(-m)} = \frac{1}{N} \sum_{m=0}^{N-1} x_p[m] \cdot e^{-j \frac{2\pi (-k)}{N} m} = X[-k]
\]
Summary

Today we introduced a new Fourier representation for DT signals: the Discrete Fourier Transform (DFT).

\[
x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi k}{N} n}
\]

**Synthesis equation**

\[
X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \cdot e^{-j \frac{2\pi k}{N} n}
\]

**Analysis equation**

The DFT has a number of features that make it particular convenient

- It is not limited to periodic signals.
- It has discrete frequency (k instead of Ω) and finite length: convenient for numerical computation.

The analysis window size N also determines the resolution in frequency.